



## XLV. On the resolving power of telescopes and spectroscopes for lines of finite width

F. L. O. Wadsworth

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XLV. *On the Resolving Power of Telescopes and Spectroscopes for Lines of Finite Width.* By F. L. O. WADSWORTH\*.

THE question of the theoretical resolving power of optical instruments has been discussed by a number of writers—most fully and comprehensively by Rayleigh †, who has shown that the theoretical angular resolving power of any instrument having an aperture of width  $b$  is  $\alpha = m \frac{\lambda}{b}$ , where  $\alpha$  is the angle between two fine lines or points which can just be separated (two stars for example),  $\lambda$  is the mean wave-length of light,  $b$  the linear aperture of the instrument, and  $m$  a constant varying from unity for a rectangular aperture to about 1.1 for a circular aperture. It is possible to determine at once from this expression the spectral resolution or separation of a spectroscope, by remembering that the function of the dispersing train, which may consist either of prisms or of a transmission or reflection grating, is simply to form a series of spectral images of a single source,—the slit of the spectroscope. Of these images only those will be resolved or separated for which the difference in angular dispersion is equal to or exceeds the angular resolution  $\alpha$  of the spectroscope aperture. In the case of the spectral images

\* Communicated by Lord Rayleigh, F.R.S.

† “Resolving or Separating Power of Optical Instruments,” *Phil. Mag.* Oct. 1879, p. 261. “Resolving Power of Telescopes,” *Phil. Mag.* Aug. 1880. “The Manufacture and Theory of Diffraction Gratings,” *Phil. Mag.* 1874, p. 5. Also articles on Optics, vol. xvii., and Wave Theory, vol. xxiv. *Enc. Brit.*

*Phil. Mag.* S. 5. Vol. 43. No. 264. May 1897. 2 C

of a slit this resolving power is in general less than the theoretical resolving power for infinitely narrow lines: (1) because of the finite angular width of the slit; (2) because of the dispersion of the spectroscope train, which for radiations which are not monochromatic produces the same effect as a widening of the slit. Theoretically we shall distinguish between four cases:—

1. The resolving power (theoretical) of a spectroscope train for an infinitely narrow slit and monochromatic radiations, *i.e.*, infinitely narrow spectral lines. This is the quantity usually denoted by  $r$ .

2. The resolving power (also theoretical) for a wide slit and monochromatic radiations. Usually denoted by  $p$ , the "purity" of the spectrum.

3. The resolving power (limiting) for an infinitely narrow slit, but for lines of finite width  $\Delta\lambda$ . This quantity we will denote by  $R$ .

4. The resolving power (practical) for a wide slit and non-monochromatic radiations ranging for each line over a small value  $\Delta\lambda$  as in (3). This quantity, which represents the practical resolving power or purity of the spectrum, will be denoted by  $P$ .

Let  $D = \frac{d\theta}{d\lambda}$  be the angular dispersion of the spectroscope train. The spectroscopic resolution for any case is defined by the ratio  $\frac{\lambda}{d\lambda}$ , where  $d\lambda$  is the difference in wave-length of two lines of mean wave-length,  $\lambda$ , that are just resolved. Therefore for the first case

$$\frac{d\theta}{d\lambda}(d\lambda)_1 = \alpha = m \frac{\lambda}{b}$$

or

$$r = \frac{Db}{m}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

a perfectly general relation which holds good whatever may be the nature, form, or arrangement of the spectroscope train.

Introducing the values of  $D = \frac{d\theta}{d\lambda} = \frac{d\theta}{dn} \cdot \frac{dn}{d\lambda}$ , we obtain at once the usual expressions

$$\left. \begin{aligned} r &= 2Nb \frac{\sin \frac{\phi}{2}}{\sqrt{1 - n^2 \sin^2 \frac{\phi}{2}}} \cdot \frac{dn}{d\lambda} \\ &= (t_1 - t_2) \frac{dn}{d\lambda} \end{aligned} \right\} \quad . \quad . \quad . \quad (2)$$

for a train of  $N$  prisms of refracting angle  $\phi$  through which the rays pass at minimum deviation, and

$$r = \frac{mn}{b'' \cos \theta} \cdot b = mn \quad . \quad . \quad . \quad . \quad (3)$$

for a grating of  $n$  lines.

In the case of the grating the expression for the resolving power may be put into a form which will bring out more clearly one fact that is not generally emphasized in the theory of the gratings, *i. e.*, that for a given position of the grating, the resolving power is independent of the number of lines  $n$  and is determined, as in any optical instrument, simply by the linear aperture  $b$ . This proposition may be very simply proved from the fundamental equation of the diffraction-grating,

$$m\lambda = s(\sin i + \sin \theta).$$

Multiplying both sides by  $n$  we get

$$mn\lambda = ns(\sin i + \sin \theta). \quad . \quad . \quad . \quad . \quad (4)$$

But  $mn = r$  and  $ns = b$  the linear aperture of the grating. Hence

$$r = \frac{b}{\lambda} (\sin i + \sin \theta), \quad . \quad . \quad . \quad . \quad (5)$$

an expression which is independent of  $n$  and depends only on  $b$  and on the position into which the grating is turned.

The maximum value of  $r$  is that for which  $i = \theta = 90^\circ$ . Then we have

$$r_{\max.} = 2 \frac{b}{\lambda},$$

which shows that the resolving power of a grating is an expression of the same form as the corresponding expression for a microscope, telescope, and reflecting mirror. The maximum resolving power is the same (though expressed in different units) as that for a mirror of the same horizontal aperture.

This theoretical maximum, however, can never be realized, because for large angles of incidence and diffraction the angular aperture of the grating becomes very small, and the light consequently excessively faint. In practice the angle of incidence  $i$  never exceeds  $60^\circ$  for an angle of diffraction  $\theta = 0$ , nor more than  $45^\circ$ – $50^\circ$  when the angles of incidence and diffraction are equal (Littrow type). Hence maximum practical resolving power, which we will call  $r_0$ , varies from

$$r_0 = \frac{7}{8} \frac{b}{\lambda} \text{ to } r_0 = \frac{3}{2} \frac{b}{\lambda}.$$

If we take the higher limit we find that the limit of resolving power of the best and largest gratings now in use (ruled surface  $5\frac{1}{2}$  inches) is for the middle part of the spectrum ( $\lambda = .00055$  mm.) about 375000 units, just sufficient to "resolve" a double line whose components are about .016 tenth-metres apart. The view at one time held that higher resolving powers than this were unnecessary because of the discontinuities in a train of light-waves is now known to be erroneous. Michelson's recent work has shown that some of the spectral lines which appear single in the most powerful spectroscopes yet constructed, are in reality very complex, consisting of three, four, or even more components whose distance apart in some cases is probably not much more than 0.006 tenth-metres. To resolve these by means of a grating, we need, therefore, instruments having at least three times the aperture of those now in use. Were the interferometer or wave-comparer universally applicable in spectroscopic analysis, there would be little occasion to attempt to rival its performance by gratings, but it is unfortunately only applicable to the more intense of the *bright* lines of a spectrum. For the more detailed study of faint lines, and absorption-lines, gratings of larger resolving power than have yet been constructed would seem to be the first essential. The mechanical difficulties to be overcome are very formidable. The chief difficulty does not seem to lie in the production of a screw of sufficient accuracy, since by Rowland's method we are enabled to produce a screw of the required length in which the errors of run, periodicity, &c. are less than those unavoidably introduced by eccentricities in the mountings and divided head\*; but in avoiding the errors of spacing caused by unequal wear of the ways on which the ruling-point carriage moves, and in maintaining sufficiently constant temperature conditions during the ruling.

How great these difficulties actually are may be better appreciated when it is remembered that to rule a 15-inch grating (of 20000 lines per inch) the ruling-engine would have to run continuously for nearly two weeks (a 6-inch grating requires five days and nights), that in such a grating a displacement of one five-hundred-thousandth part of an inch in the position of the lines in any part of the grating would greatly impair the definition and resolution in any order higher than the second, and that such a change would be brought about by the smallest amount of unequal wear, or even by a slight change in thickness of the film of oil on one of the ways of the ruling-carriage, or by an unsymmetrical

\* See Rowland's article on the Screw, *Enc. Brit.* vol. xxi.

change of temperature of the grating or of parts of the ruling-engine of less than  $\frac{1}{3}^{\circ}$  C. But the immense value that such large gratings would have in rendering possible a more detailed study of the complex character of spectral lines, and a more exact determination of their wave-length under varying conditions of production, would seem sufficient to amply justify any expenditure of time and money necessary to make their production possible and practicable\*.

It is worth while remarking that the independence of resolving power of the fineness of ruling, already pointed out, makes it possible to considerably reduce the time and difficulties of ruling large gratings by very considerably increasing the grating space, provided only that ruling-points can be found (by trial) which will produce gratings sufficiently bright in the higher orders. The two objections usually urged to coarse-ruled gratings are the increased overlapping of the different orders of spectra, and the increased accuracy of spacing required. I have recently shown how the first objection may be overcome by a very simple and efficient optical device placed in front of the spectroscope slit†. The second objection is not a valid one. It has been shown (Rayleigh) that in a given grating the allowable error in the spacing  $s$  is  $\frac{1}{4}s$  in the first order,  $\frac{1}{8}s$  in the second,  $\frac{1}{12}s$  in the third, or in general  $\frac{1}{4m}s$ . But for a given resolving power, *i. e.*, for a given aperture and given position of grating, we have from (4)  $\frac{m}{s} = \text{constant}$ , or for two gratings of the same aperture but of different spacing,  $s$  and  $s_1$ ,

$$\frac{m}{s} = \frac{m_1}{s_1}.$$

The limiting absolute error of ruling is therefore the same in both fine and coarse-ruled gratings. If, for example, the absolute error of spacing of the ruling-machine is  $\frac{1}{100,000}$  inch, equally good definition would be obtained by ruling the grating with 20,000 lines to the inch, and using the first order spectrum only, or by ruling it 4000 lines to the inch, and using the fifth spectrum. But the last grating would

\* The writer has just finished the design of a large ruling-engine, the money for the construction of which has been given by a friend of science in Chicago. Work on it has been begun in the instrument-shop of the Observatory, and every possible precaution will be taken to ensure success.

† The 'Astrophysical Journal,' March 1896, vol. iii. p. 169.

require only  $\frac{1}{5}$  the time for ruling, and hence in general would be only  $\frac{1}{5}$  as difficult to make as the first one. The question of the relative brightness of the spectra in the two gratings would be, as already stated, almost entirely a question of the selection of a ruling-point.

Let us now consider the resolving power of a spectroscope for wide slits (width  $s$ ) and monochromatic radiations. The formula ordinarily given for this is

$$p = \frac{\lambda}{s\psi + \lambda} r. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This is based on the assumption that for distinct resolution of wide lines, the angular distance between the contiguous edges of the two lines must be equal to the resolving power of the aperture through which they are viewed. According to this assumption the angular distance between the centres of the two lines of width  $s$ , which would be just resolved, would have to be

$$\frac{1}{f'} \left( s \frac{f'}{f} + m \frac{\lambda}{b} f' \right) = \left( \frac{s}{f} + m \frac{\lambda}{b} \right) = \frac{1}{b} (s\psi + m\lambda), \quad . \quad (7)$$

where  $\psi$  is the angular magnitude of the aperture  $b$  as viewed from the line  $s$ ,  $f$  is the distance of the line itself from the lens, and  $f'$  the focal length of the observing telescope. But I have recently found that it can be shown, both by theory and by experiment, that this assumption is incorrect, and that the resolving power of an instrument for wide lines is considerably greater than is indicated by the above expression. As this point has apparently escaped notice heretofore it may be considered a little in detail.

The diffraction-pattern due to a line of width  $s$ , or angular width  $\sigma = \frac{s}{f}$ , is found by integrating the effect due to each linear element over the whole width of the line. In the case of a rectangular aperture the diffraction-pattern due to each linear element is represented, as is well-known, by the equation

$$I = C \frac{\sin^2 \frac{\pi}{\alpha} \phi}{\left( \frac{\pi}{\alpha} \phi \right)^2}, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$\phi$  being the angular distance from the centre of the diffraction-image. The intensity at any point  $\gamma$  due to the effect of

all of the elements of a line of uniform brightness will therefore be

$$I' = C \int_{-\frac{\sigma}{2}}^{\frac{\sigma}{2}} \frac{\sin^2 \frac{\pi}{\alpha} (\gamma - \phi)}{\left[ \frac{\pi}{\alpha} (\gamma - \phi) \right]^2} \cdot \cdot \cdot \cdot \quad (9)$$

$$= C' \int_{\frac{\pi}{\alpha} (\gamma - \frac{\sigma}{2})}^{\frac{\pi}{\alpha} (\gamma + \frac{\sigma}{2})} \frac{\sin^2 \chi}{\chi^2} d\chi = f(\gamma, \sigma). \quad (10)$$

The value of the definite integral (10) cannot be found directly in terms of  $\gamma$  and  $\sigma$ , but it can easily be evaluated by mechanical quadrature for different values of these variables. For  $\sigma = \alpha$ , which is about as small a value as is ever used in practice, the values of  $I' = f(\gamma)$  are given in Table I. For the sake of comparison the values of  $I$  (from 8) are also included.

TABLE I.

$\gamma$ .	$I' = f(\gamma)$ .	$I$ .	$\gamma$ .	$I' = f(\gamma)$ .	$I$ .
0	1.00	1.00	.8	.24	.055
.2	.92	.87	1.0	.11	.000
.4	.71	.57	1.2	.044	.024
.6	.45	.25	1.5	.030	.045

The diffraction-curves represented by these values are plotted in fig. 1 (p. 324). Like  $I$ , the curve  $I'$  does not fall off regularly, but passes through a series of maxima and minima whose position is given by the general equation\*

$$\gamma \tan \frac{\pi \gamma}{\alpha} = \frac{\sigma}{2} \tan \frac{\pi \sigma}{2\alpha} \text{ for } 2m < \frac{\sigma}{\alpha} < 2m + 1,$$

$$\frac{1}{\gamma} \tan \frac{\pi \gamma}{\alpha} = \frac{2}{\sigma} \tan \frac{\pi \sigma}{2\alpha} \text{ for } 2m - 1 < \frac{\sigma}{\alpha} < 2m.$$

\* This part of the problem, i.e., that of locating the position of the minima in the diffraction-patterns of a slit and of a circular aperture of finite width, was worked out by the writer (at the suggestion of Professor Michelson) about six years ago, while a student at Clark University. The results were published in Professor Michelson's paper on "Application of Interference Methods to Astronomical Measurements" (Phil. Mag. July 1890, p. 1, see pp. 14-17).



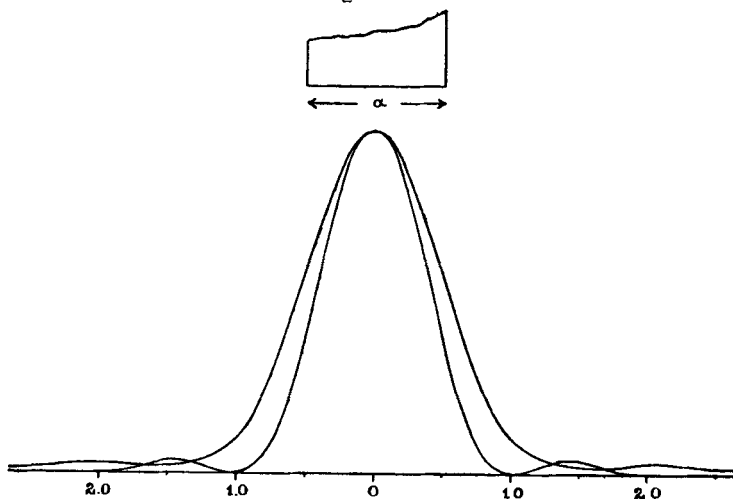
In this case  $\sigma = \alpha$ , and we have therefore

$$\tan \frac{\pi \gamma}{\alpha} = \infty,$$

$$\text{or} \quad \gamma = 1.5\alpha, 2.5\alpha, 3.5\alpha, \&c.,$$

or the minima occur at points  $\frac{1}{2}\alpha$  further from the centre than when the source is a line of negligible width\*.

Fig. 1.



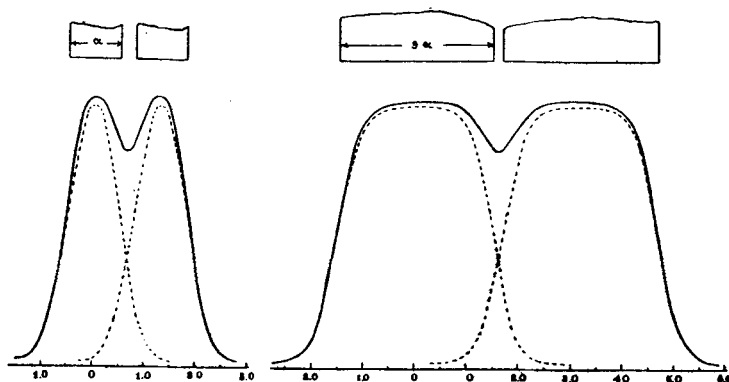
The integral was evaluated in the same manner for different values of  $\sigma$  both smaller and larger up to  $\sigma = 3\alpha$ . The diffraction-patterns of two sources of width  $\sigma = \alpha$  and  $\sigma = 3\alpha$  are shown in fig. 2, in dotted lines.

In order that a double line may be resolved it is necessary

\* Since the position of the minima in this case depends on the angular magnitude of the source  $\sigma$  as well as on the aperture of the telescope, it follows that by covering the objective of the latter by a rectangular opening of known size, and then measuring by means of a micrometer the positions of the minima of the diffraction-pattern, the value of  $\sigma$  may be determined from the above equations. Experiments on a large number of slits of varying width and holes of varying diameters (for which the positions of the minima are slightly different) showed that when the source was sufficiently bright to give well-marked minima, single observations gave results which were at least five times as accurate as could be obtained by direct micrometric measurement of the image with full aperture of the telescope. This method is, however, considerably less accurate than the refractometer method of Professor Michelson which is fully described in the earlier part of the paper referred to, and the observations are therefore not given at length.

that the intensity at the centre of the diffraction-pattern of the double source (shown in full lines in fig. 2) should be

Fig. 2.



about 0.8, the intensity at the maxima corresponding to the centres of the two geometrical images. In order that this may be the case the distance between these centres in the three cases  $\sigma = a$ ,  $\sigma = 2a$ , and  $\sigma = 3a$  must be for

$$\begin{aligned} \sigma &= a, \text{ angular distance between centres} = 1.27a = \sigma + 0.27a, \\ \sigma &= 2a, \text{ " " " " " " } = 2.21a = \sigma + 0.21a, \\ \sigma &= 3a, \text{ " " " " " " } = 3.20a = \sigma + 0.20a, \\ \text{or in general} \quad & \text{ " " " " " " } = \Sigma = \sigma + \delta. \end{aligned}$$

From these and intermediate values the curve in fig. 3 (p. 326), which represents the relation between the angular width of the lines and the angular distance  $\delta$  between the contiguous edges necessary for distinct resolution, was plotted.

In order to test these results experimentally a fine black wire was stretched across the centre of an ordinary double motion slit, thus forming two parallel slits whose widths could be simultaneously varied (by opening the slit), while the distance between the contiguous edges (which was equal to the diameter of the wire) remained constant. The two slits were uniformly illuminated by the light of the sun or an electric arc passing through a screen of white paper, and were viewed by a telescope over whose objective was placed a rectangular opening of width  $b$ .

The slit was set at various measured widths, and the distance of the telescope from it varied until the two halves of the slit were just resolved. If  $D$  represents the distance of

the telescope from the slit,  $d$  the diameter of the wire, and  $S$  the whole length of the slit, we have evidently

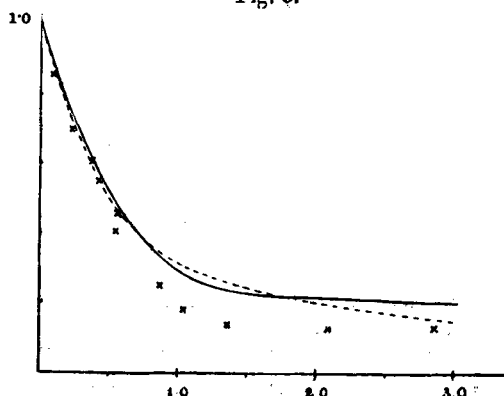
$$\frac{\sigma}{\alpha} = \frac{S-d}{2D} \cdot \frac{b}{\lambda}$$

and

$$\frac{\delta}{\alpha} = \frac{d}{D} \cdot \frac{b}{\lambda}.$$

As a check, a few measurements were also taken with the telescope at a fixed distance from the slit, the aperture  $b$  being varied until the two elements were just resolved. The effect

Fig. 3.



of varying the brightness of the slits by interposing additional screens and by removing them altogether was also tried, as well as the effect of varying the magnifying power of the telescope. As long as the images were bright enough to be clearly seen there was no appreciable effect produced by either of these changes. The results are presented in the following table :—

TABLE II.

S.	D.	$d$ .	$b$ .	$\sigma/\alpha$ .	$\delta/\alpha$ .	
1.97	12760	0.045	20.5	2.8	.13	Sunlight, screened.
1.46	12760	"	"	2.1	.13	" "
0.97	12760	"	"	1.35	.13	" "
0.60	9900	"	"	1.04	.17	" "
0.36	6700	"	"	.84	.25	" screened and unscreened.
0.167	4210	"	"	.55	.40	" vibration very bad.
0.150	3280	"	"	.58	.51	" unscreened, cloudy.
0.100	2770	"	"	.37	.60	" " "
0.075	2370	"	"	.23	.70	" " "
0.250	11000	0.20	26.0	.11	.86	Arc light, screened.
0.500	11000	0.20	17.0	.42	.56	" " "

These results are plotted (crosses) in fig. 3. The agreement with the first part of the curve obtained by theory is very good, but beyond the point  $\sigma = \alpha$  the experimental values are considerably less than the theoretical ones. These last were obtained, it will be remembered, on the assumption that in order to obtain resolution the difference in intensity between the centre and edges of the diffraction-pattern of a double source must be at least 20 per cent. These results indicate that when the lines are broad a falling off in intensity at the centre of considerably less than this is noticeable. Indeed this is what we should expect, since we know that on an extended bright background (such as a planetary surface) faint markings may be distinguished where the variation in intensity from the background itself is not more than from two to five per cent.

We are therefore certainly on the safe side in following the curve deduced from theory. The value of  $\Sigma$ , the angular resolution of the telescope for the wide lines, is, moreover, practically the same whichever curve be followed, because, for the values of  $\sigma$  for which the two curves begin to depart to any extent from each other, the value of  $\delta$  is small compared to  $\sigma$  itself\*.

The theoretical curve of fig. 3 may, up to the point  $\sigma = 3\alpha$ , be closely represented by the hyperbola of the form

$$\frac{\sigma}{\alpha} + \frac{1}{2} = \frac{1}{2\left(\frac{\delta}{\alpha}\right)}, \quad \dots \dots \dots (11)$$

whence we get

$$\delta = \frac{\alpha^2}{2\sigma + \alpha}. \quad \dots \dots \dots (12)$$

But  $\sigma = \frac{s}{f}$  and  $\alpha = \frac{\lambda}{b}$  (for rectangular aperture  $m=1$ ). Substituting these values we get

$$\delta = \frac{1}{b} \frac{\lambda^2}{2s\psi + \lambda}. \quad \dots \dots \dots (13)$$

The angular distance between two lines of width  $\sigma$  which can just be resolved is then

$$\Sigma = \sigma + \delta = \frac{1}{b} \left( s\psi + \frac{\lambda}{2s\psi + \lambda} \lambda \right). \quad \dots \dots (14)$$

\* For the value of  $\sigma = 1.5\alpha$ , for which the difference between the two curves is greatest, the two values of  $\Sigma$  differ by only about four per cent. For  $\sigma = 3\alpha$  the difference in  $\Sigma$  is only about two per cent.

An examination of this result develops the interesting fact that the aperture required to separate the components of a double line is *less when the lines have a small finite width than when they are infinitely narrow*. For, as may be easily proved, the expression for  $\Sigma$  becomes a maximum when

$$s\psi = \frac{\lambda}{2(1 + \sqrt{2})} \approx \frac{1}{5}\lambda.$$

Thus for a line of angular width  $\sigma = \frac{1}{5}\lambda$  we have

$$\Sigma = \cdot 91 \frac{\lambda}{b} = \cdot 91\alpha,$$

or, what amounts to the same thing, a telescope of given aperture has 10 per cent. greater resolving power for lines of width  $\frac{1}{5}\alpha$  than for lines infinitely narrow.

To find the width of line for which the resolving power of the instrument is the same as the theoretical resolving power we put

$$s\psi + \frac{\lambda^2}{2s\psi + \lambda} = \lambda,$$

which gives at once

$$s\psi = 0, \text{ or } \frac{1}{2}\lambda,$$

or it is just as easy to resolve the components of a double line when these have a width equal to one-half the angular resolution of the telescope as when their width is zero. This increased resolving power resulting from increasing the width of the lines from 0 up to  $\frac{1}{2}\alpha$  is due to the same effect as is produced by stopping out the central portion of the telescope objective, *i. e.*, by a strengthening of the centre of the resulting diffraction-pattern relative to the edges.

For the spectroscopic resolution we have, as in the first case,

$$\frac{d\theta}{d\lambda} (d\lambda)_2 = \Sigma = \frac{1}{b} \left( s\psi + \frac{\lambda}{2s\psi + \lambda} \lambda \right), \quad \dots (15)$$

or

$$\frac{\lambda}{(d\lambda)_2} = p = \frac{\lambda}{s\psi + \frac{\lambda}{2s\psi + \lambda} \lambda} r, \quad \dots (16)$$

which differs from the expression ordinarily given for the purity of a spectrum by the presence of the factor  $\frac{\lambda}{2s\psi + \lambda}$  as a coefficient of the second term of the denominator. The

existence of this factor necessitates a considerable modification of certain statements based on the old formula for purity. Instead of diminishing continuously with increased slit-width, the purity of the spectrum first actually increases up to the point

$$s\psi = \frac{1}{5}\lambda,$$

and is still equal to the theoretical resolving power of the instrument when  $s\psi = \frac{1}{2}\lambda$  \*. As the slit is widened still further, the purity begins to diminish, although much less rapidly than is indicated by the old formula for purity. In his remarks on the practical purity of a bright line-spectrum in the article "Spectroscopy" (*Enc. Brit.* vol. xxii. p. 374), Schuster says :—"The maximum illumination for any line is obtained when the angular width of the slit is equal to the angle subtended by one wave-length at a distance equal to the collimator aperture. In that case  $s\psi = \lambda$  and the purity is half the resolving power. Hence when light is a consideration we shall not as a rule realize more than half the resolving power of the spectroscope." Equation (16) shows, however, that under this condition for maximum illumination† the purity is really 75 per cent. of the theoretical resolving power instead of 50 per cent. as indicated by Schuster. A similar erroneous conclusion (based upon the commonly accepted formula for purity) was drawn by the writer in one of his earlier papers‡, in which it was stated that the purity in case of stellar spectra could never exceed one-third the theoretical resolving power (unless the slit-width is made less than the diameter of the diffraction-image of the star). Equation (16) shows us that this limit should be nearly one-half instead of one-third.

*Third Case.*—If the radiation is not monochromatic, but is made up of wave-lengths ranging over a interval from  $\lambda$  to  $\lambda + \Delta\lambda$ , the dispersion of the spectroscope train will spread out the image of an infinitely narrow slit into a band in which the distribution of intensity (supposing the dispersion over the small range  $\Delta\lambda$  to be strictly proportional to  $\lambda$ ) will be the same as in the source of radiation. This image will be further broadened by diffraction, and the distribution of intensity in the image formed by the spectroscope objective

\* Unfortunately it is not generally possible to profit by this fact, because for such narrow slits the spectrum is in most cases too faint to be well seen.

† As is readily seen, this condition holds only for absolutely monochromatic sources of radiation (see 'Astrophysical Journal,' January 1895, pp. 62, 63).

‡ 'Astrophysical Journal,' January 1895, pp. 68, 69.

will be given by an expression similar to (9), but containing a term  $f(\phi)$  which represents the distribution of intensity in the source of radiation.

The law of distribution (in a normal source) is not yet definitely known. The one ordinarily assumed is that which follows from Maxwell's kinetic theory, which is \*

$$f(\phi) = e^{-\kappa\phi^2} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where  $\kappa$  is a constant whose value varies with the substance emitting radiation, and with the temperature and pressure in the source. A law of distribution more recently proposed by Michelson is †

$$f(\phi) = e^{-\frac{\sin^2 r\phi}{\phi^2}} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

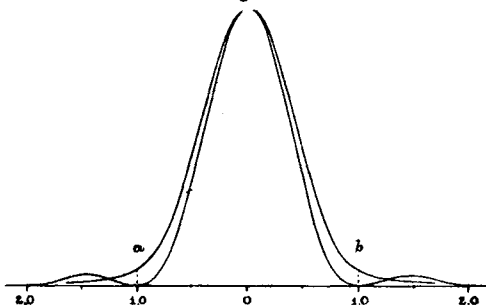
If the first law is assumed, we have for the intensity in the diffraction-pattern

$$I_1 = \int_{-\infty}^{+\infty} \frac{e^{-\kappa\phi^2} \sin^2 \frac{\pi}{\alpha} (\gamma - \phi)}{\left\{ \frac{\pi}{\alpha} (\gamma - \phi) \right\}^2} d\phi = \psi_1(\kappa, \gamma, \alpha); \quad . \quad (19)$$

and if the second,

$$I_2 = \int_{-\infty}^{+\infty} \frac{\sin^2 r\phi \sin^2 \frac{\pi}{\alpha} (\gamma - \phi)}{\phi^2 \left\{ \frac{\pi}{\alpha} (\gamma - \phi) \right\}^2} d\phi = \psi_2(r, \gamma, \alpha). \quad . \quad (20)$$

Fig. 4.



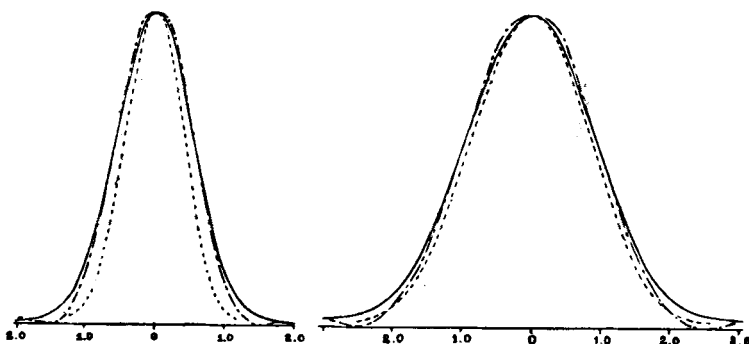
I have not succeeded in integrating either of these integrals

\* See Rayleigh, *Phil. Mag.* April, 1880, p. 298; also Michelson, *Phil. Mag.* September 1892.

† 'Astrophysical Journal,' Nov. 1895, p. 251.

in finite form. They may be integrated by developing into a series, but I have found it easier and quicker to integrate by mechanical quadrature. Owing to the very close correspondence between the curves represented by (17) and (18) (see fig. 4), the result will be practically the same whichever law be adopted. The expression for  $I_1$  is the one which has actually been integrated, and the resulting curves  $\psi_1(\kappa, \gamma, \alpha,)$  for two values of  $\kappa$  are given in fig. 5. The dotted lines represent the curves  $f(\phi)$  and the full lines the resulting diffraction-pattern  $\psi_1(\gamma)$ .

Fig. 5.



For convenience the values of  $\kappa$  are expressed in terms of the "half-width" of the line (Michelson) and  $\alpha$  the limiting resolving power of the spectroscope objective. The "half-width"  $\delta$  is defined to be the value of  $\phi$  for which  $f\phi = \frac{1}{2}$ . Hence

$$\kappa = \frac{\text{Nap. log } 2}{\delta^2} \dots \dots \dots (21)$$

What we may call the effective width of the line  $w$  is the width  $ab$  (fig. 4), which is equal to  $4\delta$ . At the points  $a$  and  $b$  the intensity  $f(\phi)$  is only about one-twentieth the intensity at the centre, and the part of the curve beyond this point may therefore be considered as having but little effect either on the eye or on the photographic plate.

The values of  $w$  in the curves of fig. 5 are  $w = 2\alpha, w = 4\alpha$ .

In fig. 6 *a* the diffraction-curve for a double source, of which each component is of width  $w = 2\alpha$ , is shown. Adopting the same rule as before, *i. e.* that for resolution the intensity at the middle of the diffraction-pattern must not be

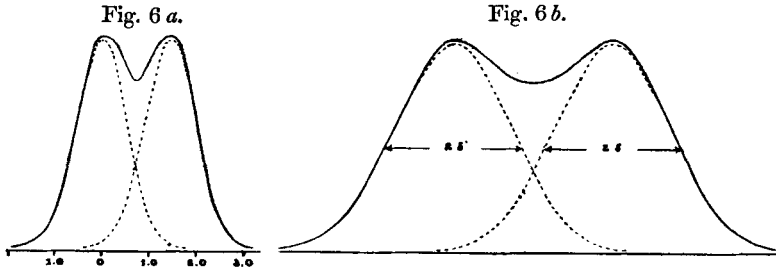


more than 0.8 the intensity at the two maxima on each side, we find that for resolution the distance between the components in different cases must be

$$\begin{aligned} w = \alpha, & \quad \text{dist.} = 1.12\alpha = \Omega_1, \\ w = 2\alpha, & \quad \text{,,} = 1.45\alpha = \Omega_2, \\ w = 3\alpha, & \quad \text{,,} = 1.90\alpha = \Omega_3, \\ w = 4\alpha, & \quad \text{,,} = 2.45\alpha = \Omega_4. \end{aligned}$$

For lines so wide that the broadening by diffraction can be entirely neglected we find (fig. 6*b*) that the distance between the components necessary for resolution is

$$2.3\delta = 0.575w \simeq \frac{4}{7}w.$$



Expressing the preceding results in the form

$$\Omega = \frac{4}{7}w + f(w)\alpha,$$

we have

for $w=0$ ,	$f(w)=1.00$ ,	$\Omega = \alpha = \frac{\lambda}{b},$
for $w=\alpha$ ,	$f(w)=0.55$ ,	$\Omega = \frac{4}{7}w + 0.53 \frac{\lambda}{b},$
for $w=2\alpha$ ,	$f(w)=0.31$ ,	$\Omega = \frac{4}{7}w + 0.31 \frac{\lambda}{b},$
for $w=3\alpha$ ,	$f(w)=0.18$ ,	$\Omega = \frac{4}{7}w + 0.18 \frac{\lambda}{b},$
for $w=4\alpha$ ,	$f(w)=0.15$ ,	$\Omega = \frac{4}{7}w + 0.15 \frac{\lambda}{b},$
for $w=\infty$ ,	$f(w)=0.00$ ,	$\Omega = \frac{4}{7}w + 0.00.$

The coefficients  $f(w)$  of the last term are plotted in fig. 7 as a function of  $w$ . The first portion of this curve may, as in

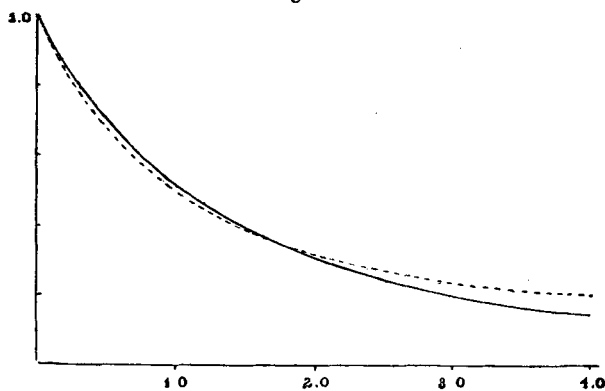
the case of fig. 3, be closely represented by an empirical hyperbola (dotted curve) whose equation is

$$\frac{w}{\alpha} + 1 = \frac{1}{f(w)} \text{ or } f(w) = \frac{\alpha}{w + \alpha}, \quad . \quad . \quad . \quad (22)$$

whence

$$\Omega = \frac{4}{7}w + \frac{\alpha^2}{w + \alpha}. \quad . \quad . \quad . \quad . \quad (23)$$

Fig. 7.



The angular width,  $w$ , of the line, since this is produced by the dispersion of the spectroscopic train, is

$$w = D\Delta\lambda = \frac{r}{b}\Delta\lambda, \quad . \quad . \quad . \quad . \quad (24)$$

$$\therefore \quad \Omega = \frac{1}{b} \left( \frac{4}{7} r\Delta\lambda + \frac{\lambda}{r\Delta\lambda + \lambda} \lambda \right); \quad . \quad . \quad . \quad (25)$$

and therefore for the spectroscope resolution

$$\frac{d\theta}{d\lambda}(d\lambda)_3 = \Omega,$$

and

$$R = \frac{\lambda}{(d\lambda)_3} = \frac{\lambda}{\frac{4}{7} r\Delta\lambda + \frac{\lambda}{r\Delta\lambda + \lambda}} r, \quad . \quad . \quad . \quad (26)$$

a formula very similar in form to that derived for the purity  $P$  in the case of a wide slit and monochromatic radiations.

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Since the spectrum lines must always have a certain "width," the expression for  $R$  last deduced, which for convenience we will call the limiting resolving power, is more generally useful in determining the greatest resolving power of a spectroscope under practical conditions than the usual expression for  $r$  (the theoretical resolution of the instrument). For very small values of  $r\Delta\lambda$ , *i. e.* for very small resolving powers or very narrow lines, the value of  $R$  will, as in the case of  $p$ , slightly exceed  $r$ . But for large values of either  $r$  or  $\Delta\lambda$  the limiting resolving power will be very much less than the theoretical power of the instrument, particularly for large values of  $r$ . No matter how narrow the line may be there is a limit beyond which an increase in the theoretical resolving power is without effect in increasing  $R$ . This maximum value of  $R$  will evidently be

$$R_{\max.} = \frac{\lambda}{\frac{4}{7}\Delta\lambda} = 1.75 \frac{\lambda}{\Delta\lambda},$$

or the maximum resolving power that can be attained with any instrument with infinitely narrow slit is not more than one and three-quarter times the ratio between the mean wave-length and "width" of the spectral lines under examination\*.

Our knowledge of the width of spectral lines under different conditions is at present very limited. Various hypotheses, of which the most noted are those of Lommel, Jauman, Galitzin, and Michelson, have been advanced to account for the broadening of the lines under varying conditions of temperature and pressure, and to give us a numerical measure of the amount, but they are all more or less unsatisfactory. Michelson's recent experimental work with the interferometer has given us our most definite knowledge of the widths of some few bright lines in the spark-spectra of some of the metals under different pressures. In each case the exponential law of distribution is assumed, and the quantity given is  $\delta$ , the "half-width" which has already been defined. It has been assumed as before that the effective range of wave-length  $\Delta\lambda$  is about  $4\delta$ .

Table III. contains a brief summary of some of the results obtained.

\* As will be presently seen, however, we may attain a somewhat greater practical purity  $P$  than this.

TABLE III.

Substance.	Line $\lambda$ .	Character of Source.	Pressure in mm.	$\delta$ (tenth-metres).	$\Delta\lambda$ - $\delta\delta$ .
Hydrogen .....	H $\alpha$ * 6565	Vacuum tube.	Very low.	·047	$\Delta\lambda' = 0.328^*$
	" "	" "	50	·098	$\Delta\lambda' = 0.532^*$
	" "	" "	100	·134	$\Delta\lambda' = 0.696^*$
	" "	" "	200	·230	$\Delta\lambda' = 1.06^*$
Sodium .....	D $\alpha$ * 5890	Vacuum tube.	Very low.	·005	0.020
	Not stated,	Not stated.	100	·09	0.36†
	D $\beta$ * 5890	Bunsen flame.	200	·16	0.64†
			Atmospheric.	About ·05 †	0.27*
Cadmium .....	Red 6439	Vacuum tube, temp. about 280°.	Very low.	·0065	0.026
	Green 5086	" "	100	·0050	0.020
	Not stated.	Not stated, probably spark.	200	·05	0.200†
	" "	" "	400	·08	0.32†
				·14	0.56
Mercury .....	Green 5461	Vacuum tube, temp. about 100°.	Very low.	·003 †	0.12

\* The red hydrogen line is a double, the distance between the components being about 0.14 tenth-metre. The value given for  $\delta$  is for each component, and the total effective width of the double line is therefore  $\Delta\lambda' = \delta\delta + 0.14$ . The same is true of each of the D lines (according to Michelson each is made up of at least four components), the distance between the centres of the principal components being 0.07. When the density is low, these components are therefore separated by much more than their own width; but when it is high (as in the Bunsen flame) each component broadens and overlaps the other, so that the total effective width is, as in the case of the H $\alpha$  line,  $\Delta\lambda' = \delta\delta + 0.07$ .

† There would seem to be some discrepancy between these results, which are given in the *Astrophysical Journal* for November 1895, p. 251, and the results previously obtained with the vacuum tube (*Phil. Mag.* September 1892, p. 280).

‡ Calculated from data given in *Phil. Mag.* September 1892, p. 280.

For convenience the values of  $R$ ,  $R_{\max}$ , and  $r/R$ , have been computed for various values of  $\Delta\lambda$ , ranging from 0.01 to 1.0 tenth-metre, and for values of  $r$  from 25000 to 1000000. They are given in Table IV.

The vertical columns show the decrease in the value of  $R$  with an increase in  $\Delta\lambda$  for a given value of  $r$ ; the horizontal lines show the increase in  $R$  with  $r$  for a given width of line. The last column gives the maximum resolving power  $R_{\max}$  that can be attained when the lines have the width  $\Delta\lambda$  given in the first column.

We see that in general we shall very nearly reach this limit when the theoretical resolving power  $r$  is about twice  $R_{\max}$ . The additional gain in  $R$ , obtained by a further increase in  $r$ , would not be worth the expense of the larger instruments required and the sacrifice in brightness necessary. Indeed, in most cases it would hardly be advisable to use a value of  $r$  greater than one to one and one-half times  $R_{\max}$ , as with this we shall have already attained from  $\frac{2}{3}$  to  $\frac{7}{8}$  of the limiting resolving power. The finest lines so far found (see Table IV.) have a width  $\Delta\lambda$  of not less than 0.01 tenth-metre. For this width the value of  $R_{\max}$  is 950000, and the maximum theoretical power which it would be advisable to use would therefore be about 1400000, corresponding in the case of a grating to an aperture of from 18 to 20 inches. On the other hand, for some of the wider lines, such as those of hydrogen in the vacuum tube, and of many bright metallic lines in arc spectra, there would be no advantage whatever for visual work in using a resolving power greater than 20000 to 25000, for which a grating of  $\frac{1}{2}$ -inch aperture, or 5 60-prisms of  $1\frac{1}{4}$  inches aperture would suffice. For solar spectrum work, in which the lines are not likely to be narrower than  $\frac{1}{20}$  tenth-metre\*, our present 5 and 6 inch gratings will do nearly all that we could hope to attain with larger apertures†, unless indeed there should be some marked advantage in particular cases in the use of the first and second orders of spectra, rather than the higher orders.

The preceding conclusions are all based on the assump-

\* In the case of faint lines the apparent width may sometimes be much less than this, because of the rapid falling off in intensity towards the edge of the line. Indeed, for faint lines, it is not likely that the apparent width of the line is greater than  $2\delta$ , and in some cases even less. Hence estimates of pressure based upon direct visual observations of the widening of lines may be considerably in error.

† The latter would, however, be advantageous in photographic work in giving increased accuracy and increased photographic resolution by reason of the greater linear dispersion. See 'Astrophysical Journal,' vol. i. p. 233, and vol. ii. p. 264.

TABLE IV.  
 $\lambda = 5500$  tenth-metres.

$\Delta\lambda$ , tenth- metres.	$r = 25000$ .		$r = 50000$ .		$r = 100000$ .		$r = 200000$ .		$r = 500000$ .		$r = 1000000$ .		$R_{\max}$ .
	$r/R$ .	R.	$r/R$ .	R.	$r/R$ .	R.	$r/R$ .	R.	$r/R$ .	R.	$r/R$ .	R.	
0.01	0.98	25400	0.97	51600	0.95	105600	0.94	212800	1.04	480000	1.39	722000	962000
0.02	0.97	25800	0.95	52800	0.94	106400	1.00	200000	1.39	361000	2.29	437000	481000
0.04	0.95	26400	0.94	53200	1.00	100000	1.24	161700	2.29	219000	4.27	234000	240000
0.06	0.94	26600	0.96	52400	1.10	90900	1.56	128500	3.27	153000	6.30	159000	160000
0.08	0.94	26600	1.00	50000	1.24	80800	1.91	104600	4.27	117000	8.35	120000	120000
0.10	0.95	26400	1.04	48000	1.39	71900	2.29	87300	5.28	95000	10.41	96000	96000
0.12	0.96	26200	1.10	45500	1.56	64300	2.67	75000	6.30	79400	12.50	80000	80000
0.14	0.97	25800	1.16	42900	1.73	57700	3.06	63000	7.33	69000	14.50	69000	69000
0.16	1.00	25000	1.24	40400	1.91	52300	3.46	58000	8.35	60000	16.60	60000	60000
0.18	1.02	24600	1.31	38100	2.10	47700	3.86	52900	9.38	53000	18.70	53000	53000
0.20	1.04	24000	1.39	36000	2.29	43700	4.27	46800	10.41	48000	20.75	48000	48000
0.25	1.12	22400	1.60	31200	2.77	36100	5.28	37900	13.00	38000	25.9	38000	38000
0.30	1.20	20800	1.85	27000	3.27	30600	6.30	31800	15.60	32000	31.1	32000	32000
0.35	1.29	19300	2.05	24400	3.76	26600	7.33	27000	18.17	27000	36.3	27000	27000
0.40	1.39	18000	2.29	21800	4.27	23400	8.35	24000	20.75	24000	41.4	24000	24000
0.50	1.60	15600	2.77	18000	5.28	18900	10.41	19000	25.90	19000	51.8	19000	19000
0.60	1.82	13700	3.27	15300	6.30	15900	12.47	16000	31.1	16000	62.2	16000	16000
0.80	2.29	10900	4.27	11700	8.35	12000	16.61	12000	41.4	12000	82.9	12000	12000
1.00	2.77	9000	5.28	9500	10.41	9600	20.75	9600	51.8	9600	103.6	9600	9600

tion that the maximum practical resolving power  $r_0$ , which has been assumed to be equal to  $1.5\lambda/b$ , and which corresponds to an angle of deviation of about  $90^\circ$  ( $\theta = i = 45^\circ$  to  $50^\circ$ ), can be utilized. When for any reason this is not the case, whether because of the inaccuracies of ruling, the faintness of the higher orders of spectra, or the character of the mounting, a correspondingly larger aperture must be made use of. If, for example, we consider the maximum angle of deflexion to be  $60^\circ$  (which from purely mechanical considerations is about the largest possible angle that can be used in the ordinary Rowland mounting), we have for  $r_0$

$$r_0 \approx \frac{7}{8} \frac{b}{\lambda}.$$

In order to attain the same resolving powers,  $R$ , as before, the apertures must be increased about 75 per cent. If we assume a maximum angle of  $45^\circ$ , which in practice is not often exceeded in our present gratings, the apertures would have to be increased by over 100 per cent., and we should therefore need to attain the full limiting resolving power  $R_{\max}$ .

For lines  $\Delta\lambda = .01$  tenth-metre, an aperture  $b$  of at least 1 metre.

„	$\Delta\lambda = .02$	„	„	$b$	„	50 cm.
„	$\Delta\lambda = .05$	„	(solar work)	„	$b$	„ 25 cm.

*Fourth Case.*—In order to determine the limit of resolution or the practical purity  $P$  in this, the most important case, we must first determine the diffraction-curve resulting from a superposition of all the elements of the slit, each one of which has a dispersion-pattern similar to those represented in full lines in fig. 5. If, as before, these elements are equal in intensity, *i.e.*, if the illumination over the whole width of the slit is uniform, the intensity-curve of the diffraction-image will be

$$I_{//} = \int_{-\sigma/2}^{+\sigma/2} \psi_1(\xi - \gamma, w, \alpha) d\xi = \psi_{//}(\sigma, \gamma, w, \alpha), \quad (28)$$

where

$$\psi_1(\gamma, w, \alpha) = \int_{-\infty}^{+\infty} 2 - \left(\frac{2\phi}{w}\right)^2 \frac{\sin^2 \frac{\pi}{\alpha} (\gamma - \phi)}{\left[\frac{\pi}{\alpha} (\gamma - \phi)\right]^2} d\phi, \quad (29)$$

as derived from (19) and (21).

Since the function  $\phi_1$  is not known in finite terms,  $\psi_{//}$  cannot be directly found. We may, however, approximate very

closely indeed to it by replacing the function  $\psi_{\alpha}$  by the function

$$\frac{\sin^2 \frac{\pi}{\Omega} \gamma}{\left[ \frac{\pi}{\Omega} \gamma \right]^2}, \quad \dots \quad (30)$$

which between the points  $\gamma = \frac{1}{3}\Omega$  and  $\gamma = \frac{2}{3}\Omega$ , or over all that part of the curve which is important in determining the resolution of a double line, coincides, as seen in fig. 5 (dashed curve), almost exactly with the curve  $\psi_1(\kappa, \gamma, \alpha)$ .

The expression for  $I_{//}$  then becomes

$$I_{//} = \int_{-\sigma/2}^{+\sigma/2} \frac{\sin^2 \frac{\pi}{\Omega} (\xi - \gamma)}{\frac{\pi}{\Omega} (\xi - \gamma)} d\xi, \quad \dots \quad (31)$$

which is exactly similar in form to (9), the only difference being that  $\alpha$  has been replaced by  $\Omega$ .

We may therefore obtain at once the limit of resolution for this case from (12) and (14) by replacing  $\alpha$  by  $\Omega$ , giving us

$$\Sigma_1 = \text{limiting angular resolution} = \sigma + \frac{\Omega^2}{2\sigma + \Omega}. \quad (32)$$

Replacing  $\sigma$  and  $\Omega$  by their values in terms of  $s, \psi, R, r$ , and  $\lambda$  and reducing, we finally obtain for  $\Sigma_1$ ,

$$\Sigma_1 = \frac{1}{b} \left[ s\psi + \frac{\left( \lambda \frac{r}{R} \right)^2}{2s\psi + \lambda \frac{r}{R}} \right], \quad \dots \quad (33)$$

and for purity

$$P = \frac{\lambda}{(d\lambda)_4} = \frac{\lambda}{s\psi + \frac{\lambda \left( \frac{r}{R} \right)}{2s\psi + \lambda \frac{r}{R}}} r. \quad \dots \quad (34)$$

This expression differs from (16) only in the presence of the factor  $\frac{r}{R}$  as a coefficient of  $\lambda$  in the denominator. When this ratio is unity  $P=p$ , or the practical purity is equal to the theoretical purity for monochromatic radiations.

By an inspection of Table IV. it will be seen that while



for narrow lines and small resolving powers the ratio  $\frac{r}{R}$  is very nearly unity, and that formula (16) therefore represents very closely the purity of the spectrum, the same is by no means true for wide lines and large resolving powers. In the extreme case figured in the table the value of this ratio rises as high as 100. In order to show more clearly the influence of this factor on the purity of the spectrum under different conditions, Table V. has been prepared, showing the values of  $P$  for different slit apertures, from 0.005 mm. to 0.3 mm., different widths of lines from 0.01 to 1.00 tenths-metres, and resolving powers varying from 25000 to 1000000. For comparison the values of  $p$  are given for each slit-width and resolving power, and also the value of  $p'$  calculated from the old formula for purity (6). An inspection of the table shows at once how greatly in error estimates of purity based upon this old formula may be in some very common cases.

Take for example the case of a spectroscope having a resolving power of 100000 (5-inch grating, 20000 lines, 2nd order); working with angular slit-width such that  $s\psi = 0.005$  ( $s = \frac{1}{50}$  mm.,  $\psi = \frac{1}{40}$ , as in the concave grating). The value of  $p$  (16) is about 158000, while the value of  $P$  varies from 163000 to 10000. The value of  $p'$  (the old formula for purity) for the same case is only 105000. It is therefore in this case from 50 per cent. to 1000 per cent. in error. In case of larger resolving-powers ( $r = 1000000$ ) it may be as much as 60 times too great. In general, of course, the large values of  $r\Delta\lambda$  that give rise to the smaller values of  $P$  will not be used for visual work, as there is, as already indicated, but little gain in practical resolving power or purity when the value of  $r$  is greater than the value of  $R_{\max}$  given in Table V. But in photographic work it is, as has already been shown in a previous paper, a great advantage to use (for extended sources) a short camera and very high resolving power, in order to attain a given degree of photographic purity. Another point which is of considerable practical importance in this connexion is that for these large values

of  $\frac{r}{R}$  the purity of the spectrum may be maintained constant or even actually improved over a wide range of those slit-widths actually used in practice. For the maximum value of  $P$  (as of  $p$ ) will be attained when

$$s\psi \simeq \frac{1}{2}\lambda \left( \frac{r}{R} \right).$$

For  $r=200000$ ,  $\Delta\lambda=1\cdot00$ ,  $\frac{r}{R}=20\cdot75$ , and the maximum value of  $P$  is therefore attained when the value of  $s\psi$  is about  $4\cdot15\lambda$  or about  $\cdot0023$ , corresponding for the usual spectro-scope ( $\psi=\frac{1}{15}$ ) to a slit-width of about  $\frac{1}{30}$  mm. Under the same circumstances the practical purity is still as great when the slit-width is  $\frac{1}{15}$  mm. as when it is zero. For still higher resolving powers the maximum allowable widths of slit are still greater. Even with such low values of  $\frac{r}{R}$  as 2 or 3 (corresponding to lines as fine as those sometimes found in the solar chromosphere, *i. e.*,  $0\cdot2$  to  $0\cdot25$  tenth-metre), and resolving powers of only 100000, the purity remains undiminished up to values of  $s\psi=\lambda$  to  $1\frac{1}{2}\lambda$  ( $\cdot0005$  to  $\cdot0008$ ), or to slit-widths (with the concave grating) of from  $\frac{1}{50}$  mm. to  $\frac{1}{30}$  mm.

One further case remains to be considered, *viz.* that of a wide slit and non-monochromatic radiations in which the slit-image is not uniformly brought across the whole width. The expression for the intensity in the diffraction-pattern then becomes

$$I_{III} = \int_{-\sigma/2}^{+\sigma/2} f(\xi) \psi_1(\xi - \gamma, w, \alpha) d\xi, \quad \dots \quad (35)$$

where  $f(\xi)$  expresses the intensity at any part of the slit at a distance  $\xi$  from its centre. The only case of importance of this kind is the case of stars. If the star-image is perfect, *i. e.* unaffected by atmospheric or instrumental aberration, the distribution in intensity for any one wave-length is represented by the law

$$\frac{\sin^2 \frac{\pi}{\alpha_0} \xi}{\left(\frac{\pi}{\alpha_0} \xi\right)^2},$$

$\alpha_0$  being the resolving power of the telescope-lens which forms an image of the star.

As before, the integration could only be effected by mechanical quadrature or by development into a series ( $\psi_1$  not being known in finite terms). It has not been thought worth while to go through the necessary labour of integration for the reason that, practically, such conditions are never realized, at least in stellar spectrographic work. There might be moments at which, if the star were kept perfectly centred on the slit, the full resolving power resulting from

TABLE V.

S.	$\psi$ radians.	S $\psi$ .	$w$ 48	$r$ 25000	$r$ 50000	$r$ 100000	$r$ 200000	$r$ 500000	$r$ 1000000
·005	$\frac{1}{10}$	·0005	P {	·01 20000	40200	81200	163200	389000	662000
				·05 20300	40600	77800	132400	194000	207000
·010	$\frac{1}{5}$			·10 20300	38900	66200	91400	103400	102200
				·50 15100	19400	20700	20400	19900	19900
·020	$\frac{1}{5}$		$p$ (from 16)	1·00 9700	10300	10200	10000	9800	9700
			$p'$ (from 6)	19800	39600	79100	158200	396000	792000
				13100	26200	52400	104800	262000	524000
·010	$\frac{1}{10}$	0·001	P {	·01 12400	24800	49700	99400	243000	454000
				·05 12400	24800	48700	90900	166000	202000
				·10 12400	24400	45500	74300	101500	108000
·015	$\frac{1}{5}$			·50 10900	16600	20200	21500	20400	19900
			$p$ (from 16)	1·00 8300	10100	10800	10300	10000	9800
			$p'$ (from 6)	12300	24600	49100	98200	245000	491000
				8900	17800	35500	71000	177000	355000
·020	$\frac{1}{10}$	0·002	P {	·01 6700	13400	26700	53500	133000	259000
				·05 6700	13400	26600	51900	113600	171000
				·10 6700	13300	25900	47800	85700	103000
·030	$\frac{1}{5}$			·50 6400	11400	17100	20600	21000	20600
			$p$ (from 16)	1·00 5700	8500	10300	10600	10300	10000
			$p'$ (from 6)	6650	13350	26700	53400	133500	267000
				5400	10800	21600	43200	108000	216000
·030	$\frac{1}{10}$	0·003	P {	·01 4500	91000	18100	36200	90200	178000
				·05 4500	91000	18000	35600	83100	141000
				·10 4500	9000	17800	34200	75700	95300
·045	$\frac{1}{5}$			·50 4400	8300	14100	19100	21100	20800
			$p$ (from 16)	1·00 4200	7600	9500	10500	10400	10100
			$p'$ (from 6)	4500	9100	18100	36200	90500	181000
				3900	7800	15300	31000	77500	155000
·050	$\frac{1}{10}$	0·005	P {	·01 2700	5400	10900	21800	54500	108000
				·05 2700	5400	10900	21800	52900	97300
				·10 2700	5400	10800	21400	48800	77100
·075	$\frac{1}{5}$			·50 2700	5300	9700	15400	20600	21100
			$p$ (from 16)	1·00 2600	4900	7700	9900	10500	10300
			$p'$ (from 6)	2700	5400	10900	21800	54500	109000
				2500	5000	9900	19800	49500	99000
·10	$\frac{1}{10}$	0·010	P {	·01 1400	2800	5500	11000	27500	54900
				·05 1400	2800	5500	11000	27200	53000
				·10 1400	2800	5500	10900	26500	48800
·15	$\frac{1}{5}$			·50 1400	2700	5300	9800	17300	26600
			$p$ (from 16)	1·00 1400	2600	4900	7800	10300	10500
			$p'$ (from 6)	1400	2800	5500	11000	27500	55000
				1300	2600	5200	10400	26000	52000
·20	$\frac{1}{10}$	0·020	P {	·01 700	1400	2800	5600	14000	27500
				·05 700	1400	2800	5500	13700	27200
				·10 700	1400	2800	5500	13600	26500
·30	$\frac{1}{5}$			·50 700	1400	2700	5300	11600	17300
			$p$ (from 16)	1·00 700	1400	2600	4900	8600	10300
			$p'$ (from 6)	700	1400	2800	5600	14000	28000
				650	1300	2600	5200	13000	26000

the superposition of two such diffraction-patterns as are represented by (35) might be realized, but in general the star-image will be so broadened and disturbed in position by continual atmospheric disturbance (to say nothing of chromatic aberration in the case of the image being formed with a lens), that the effect on the photographic plate will in the long run be practically the same as would be produced by a uniformly illuminated slit.

Yerkes Observatory,  
University of Chicago,  
February 1897.

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XLVI. *On the Measurement of Alternate Currents by means of an obliquely situated Galvanometer Needle, with a Method of determining the Angle of Lag.* By LORD RAYLEIGH, F.R.S.\*

IT is many years† since, as the result of some experiments upon induction, I proposed a soft iron needle for use with alternate currents in place of the permanently magnetized steel needle ordinarily employed in the galvanometer for the measurement of steady currents. An instrument of this kind designed for telephonic currents has since been constructed by Giltay; but, so far as I am aware, no application has been made of it to measurements upon a large scale, although the principle of alternately reversed magnetism is the foundation of several successful commercial instruments.

The theory of the behaviour of an elongated needle is sufficiently simple, so long as it can be assumed that the magnetism is made up of two parts, one of which is constant and the other proportional to the magnetizing force. If internal induced currents can be neglected, this assumption may be regarded as legitimate so long as the forces are small‡. In the ordinary case of alternate currents, where upon the whole there is no transfer of electricity in either direction, the constant part of the magnetism has no effect; while the variable part gives rise to a deflecting couple proportional on the one hand to the mean value of the square of the magnetizing force or current, and upon the other to the sine of twice the angle between the direction of the force and the

\* Communicated by the Author.

† Brit. Assoc. Report, 1868; Phil. Mag. vol. iii. p. 43 (1877).

‡ Phil. Mag. vol. xxiii. p. 225 (1887).