certain that if there is any it is extremely small. Hence, the constituent atoms, if aeolotropic as to permeability, must be so, but to an exceedingly small degree. Le Sage's second fundamental assumption given above, under the title "*Constitution of Heavy Bodies*," implies sensibly equal permeability in all directions, even in an aeolotropic structure, unless much greater than Jupiter, provided that the atoms are isotropic as to permeability.

A body having different permeabilities in different directions would, if of manageable dimensions, give us a means for drawing energy from the inexhaustible store laid up in the ultramundane corpuscles, thus:—First, turn the body into a position of minimum weight; Secondly, lift it through any height; Thirdly, turn it into a position of maximum weight; Fourthly, let it down to its primitive level. It is easily seen that the first and third of those operations are performed without the expenditure of work; and, on the whole, work is done by gravity in operations 2 and 4. In the corresponding set of operations performed upon a moveable body in the neighbourhood of a fixed magnet, as much work is required for operations 1 and 3 as is gained in operations 2 and 4; the magnetisation of the moveable body being either intrinsic or inductive, or partly intrinsic and partly inductive, and the part of its aeolotropy (if any), which depends on inductive magnetisation, being due either to magne-crystallic quality of its substance, or to its shape.*

4. Note on Spherical Harmonics. By Professor Tait.

While engaged in some quaternion researches with reference to Spherical Harmonics, which I hope soon to lay before the Society, I was led to imagine that some of my results might produce a simplification of the ordinary modes of treating the subject. The following is the result of the attempt. It seems to make the cal-

* "Theory of magnetic induction in crystalline and non-crystalline substances."—*Phil. Mag.*, March 1851. "Forces experienced by inductively magnetised ferro-magnetic and dia-magnetic non-crystalline substances."—*Phil. Mag.*, Oct. 1850. "Reciprocal action of dia-magnetic particles."—*Phil. Mag.*, Dec. 1855; all to be found in a collection of reprinted and newly written papers on electrostatics and magnetism, nearly ready for publication, (Macmillan, 1872).

culus somewhat more intelligible to the beginner than the methods employed by O'Brien and Murphy, whose works on the subject are usually read in this country. As I am not writing a treatise, but merely sketching a method, I shall run over the principal elementary propositions only.

1. Let

$$\frac{1}{\rho} = \frac{1}{(1 - 2h\mu + h^2)^{\frac{1}{2}}} = \sum_0^{\infty} h^i Q_i \quad .$$

This is possible, if h be always taken less than 1; and, as μ is never beyond the limits ± 1 , Q_i , -1 are in order of magnitude, and the series is always convergent.

Hence we may differentiate, and we thus obtain

$$\frac{d}{d\mu} \frac{1}{\rho} = \frac{h}{\rho^3} = \sum_0^{\infty} h^i \frac{dQ_i}{d\mu},$$

and

$$\begin{aligned} \frac{d}{d\mu} \left((1 - \mu^2) \frac{d}{d\mu} \frac{1}{\rho} \right) &= \frac{d}{d\mu} \frac{(1 - \mu^2)h}{\rho^3} = \rho^{-5} \left\{ -2\mu h \rho^2 + 3(1 - \mu^2)h^2 \right\} \\ &= \sum h^i \frac{d}{d\mu} \left((1 - \mu^2) \frac{dQ_i}{d\mu} \right) \quad . \quad . \quad . \quad (1). \end{aligned}$$

Also

$$h^2 \frac{d}{dh} \frac{1}{\rho} = \frac{\mu h^2 - h^3}{\rho^3} = \sum . i h^{i+1} Q_i,$$

and

$$\begin{aligned} \frac{d}{dh} \left(h^2 \frac{d}{dh} \frac{1}{\rho} \right) &= \rho^{-5} \left\{ (2\mu h - 3h^2) \rho^2 + 3(\mu - h)^2 h^2 \right\} \\ &= \sum . i(i+1) h^i Q_i \quad . \quad . \quad . \quad (2). \end{aligned}$$

The sum of the multipliers of ρ^{-5} in (1) and (2) is obviously **zero**. Thus we have the equation for Q_i

$$i(i+1) Q_i + \frac{d}{d\mu} \left((1 - \mu^2) \frac{dQ_i}{d\mu} \right) = 0 \quad . \quad (3).$$

2. From this equation, by differentiation $s-1$ times with respect to μ , we have

$$i(i+1) \frac{d^{s-1} Q_i}{d\mu^{s-1}} + (1 - \mu^2) \frac{d^{s+1} Q_i}{d\mu^{s+1}} - 2s\mu \frac{d^s Q_i}{d\mu^s} - s(s-1) \frac{d^{s-1} Q_i}{d\mu^{s-1}} = 0,$$

or,

$$\left(i(i+1) - s(s-1) \right) (1-\mu^2)^{s-1} \frac{d^{s-1}Q_i}{d\mu^{s-1}} + \frac{d}{d\mu} \left((1-\mu^2)^s \frac{d^s Q_i}{d\mu^s} \right) = 0 \quad (4).$$

3. Let Q_j be any one of the values of Q above defined, then

$$\begin{aligned} \int (1-\mu^2)^s \frac{d^s Q_i}{d\mu^s} \frac{d^s Q_j}{d\mu^s} d\mu &= (1-\mu^2)^s \frac{d^s Q_i}{d\mu^s} \frac{d^{s-1} Q_j}{d\mu^{s-1}} - \int d\mu \frac{d^{s-1} Q_j}{d\mu^{s-1}} \frac{d}{d\mu} \left((1-\mu^2)^s \frac{d^s Q_i}{d\mu^s} \right) \\ &= (1-\mu^2)^s \frac{d^s Q_i}{d\mu^s} \frac{d^{s-1} Q_j}{d\mu^{s-1}} + (i+s)(i-s+1) \int (1-\mu^2)^{s-1} \frac{d^{s-1} Q_i}{d\mu^{s-1}} \frac{d^{s-1} Q_j}{d\mu^{s-1}} d\mu. \end{aligned}$$

Hence, integrating between the limits ∓ 1 of μ , we have

$$\int_{+1}^{-1} (1-\mu^2)^s \frac{d^s Q_i}{d\mu^s} \frac{d^s Q_j}{d\mu^s} d\mu = (i+s)(i-s+1) \int_{+1}^{-1} (1-\mu^2)^{s-1} \frac{d^{s-1} Q_i}{d\mu^{s-1}} \frac{d^{s-1} Q_j}{d\mu^{s-1}} d\mu \quad (5).$$

Applying the reduction s times, we evidently obtain

$$\int_{+1}^{-1} (1-\mu^2)^s \frac{d^s Q_i}{d\mu^s} \frac{d^s Q_j}{d\mu^s} d\mu = \frac{|i+s|}{|i-s|} \int_{+1}^{-1} Q_i Q_j d\mu \quad (6).$$

4. To find the value of the integral on the right, note that $Q_i Q_j$ is the co-efficient of $h^i h^j$ in the expansion of

$$\frac{1}{(1-2\mu h + h^2)^{\frac{1}{2}} (1-2\mu h' + h'^2)^{\frac{1}{2}}}$$

Now

$$\begin{aligned} &\int_{+1}^{-1} \frac{d\mu}{\sqrt{(1+h^2-2h\mu)} \sqrt{(1+h'^2-2h'\mu)}} \\ &= \frac{1}{\sqrt{hh'}} \log. \frac{\sqrt{\frac{1+h^2}{2h}} - 1 + \sqrt{\frac{1+h'^2}{2h'}} - 1}{\sqrt{\frac{1+h^2}{2h}} + 1 + \sqrt{\frac{1+h'^2}{2h'}} + 1} \\ &= \frac{1}{\sqrt{hh'}} \log. \frac{\sqrt{h'}(1-h) + \sqrt{h}(1-h')}{\sqrt{h'}(1+h) + \sqrt{h}(1+h')} \\ &= \frac{1}{\sqrt{hh'}} \log. \frac{1 - \sqrt{hh'}}{1 + \sqrt{hh'}} \end{aligned}$$

$$= -2 \sum_0^\infty \frac{(hh')^i}{2i+1}.$$

In this there is no term in which the powers of h and h' are different, hence we have

$$\int_{+1}^{-1} Q_i Q_j d\mu = 0 \quad . \quad . \quad . \quad (7).$$

in all cases unless $j=i$. In this special case we have

$$\int_{+1}^{-1} Q_i^2 d\mu = \frac{2}{2i+1} \quad . \quad . \quad . \quad (8.)$$

Hence the left hand member of (6) vanishes unless $j=i$, and in that case we have

$$\int_{+1}^{-1} (1-\mu^2)^s \left(\frac{d^s Q_i}{d\mu^s} \right)^2 d\mu = \frac{2}{2i+1} \frac{|i+s|}{|i-s|} \quad . \quad . \quad (9).$$

We might have proved (7) from (6) by exchanging i and j , and showing that unless $i=j$, we cannot have

$$\frac{|i+s|}{|i-s|} = \frac{|j+s|}{|j-s|}.$$

5. The equation (3), which is satisfied by Q_i , is a mere particular case of the general equation of surface harmonics—

$$i(i+1) S_i + \frac{1}{1-\mu^2} \frac{d^2 S_i}{d\phi^2} + \frac{d}{d\mu} \left((1-\mu^2) \frac{dS_i}{d\mu} \right) = 0 \quad (10).$$

which may be obtained by putting $V_i = S_i$ in the ordinary equation of Laplace—

$$r \frac{d^2(rV_i)}{dr^2} + \frac{1}{1-\mu^2} \frac{d^2 V_i}{d\phi^2} + \frac{d}{d\mu} \left((1-\mu^2) \frac{dV_i}{d\mu} \right) = 0,$$

after differentiating the first term. That differentiation gives, in fact,

$$r \frac{d}{dr} \left(V_i + r \frac{dV_i}{dr} \right) = (i+1) r \frac{dV_i}{dr} = i(i+1) V_i.$$

From equation (10) we may prove, as usual, by multiplying by S_j and integrating over the unit sphere, that

$$i(i+1) \int d\sigma S_i S_j = j(j+1) \int d\sigma S_i S_j,$$

the expression for either being symmetrical in i and j , so that the integral vanishes unless $i=j$: or, if negative values be admitted, unless $i+j+1=0$.

6. We must now express S_i in terms of φ and Q_i . Let, then,

$$S_i = \sum_0 A_s \cos. (s\varphi + \alpha_s) \Theta_i^{(s)} \quad . \quad . \quad . \quad (11).$$

where A_s, α_s are virtually $2i+1$ arbitrary constants. Substituting this value in (10), and supposing all the coefficients A to vanish except A_s , we have

$$\left(i(i+1) - \frac{s^2}{1-\mu^2} \right) \Theta_i^{(s)} + \frac{d}{d\mu} \left((1-\mu^2) \frac{d\Theta_i^{(s)}}{d\mu} \right) = 0 \quad . \quad (12).$$

This equation is materially simplified by assuming (as is suggested by (6) and (9))

$$\Theta_i^{(s)} = (1-\mu^2)^{\frac{s}{2}} \theta_s \quad . \quad . \quad . \quad (13),$$

for with this substitution it becomes, by a process the same as that of section 2 above,

$$\left(i(i+1) - s(s+1) \right) (1-\mu^2)^s \theta_s + \frac{d}{d\mu} \left((1-\mu^2)^{s+1} \frac{d\theta_s}{d\mu} \right) = 0.$$

But, by (4), putting $s+1$ for s ,

$$\left(i(i+1) - s(s+1) \right) (1-\mu^2)^s \frac{d^s Q_i}{d\mu^s} + \frac{d}{d\mu} \left((1-\mu^2)^{s+1} \frac{d^{s+1} Q_i}{d\mu^{s+1}} \right) = 0.$$

Comparing these equations, and remembering that all the permissible arbitrary constants have already been introduced into the solution of (10), we have

$$\theta_s = \frac{d^s Q_i}{d\mu^s}.$$

Hence, finally,

$$S_i = \sum_0^i A_s \cos. (s\varphi + \alpha_s) (1-\mu^2)^{\frac{s}{2}} \frac{d^s Q_i}{d\mu^s} \quad . \quad . \quad (14.)$$

7. We may now easily find the value of

$$\int S_i S_j d\sigma$$

taken over the whole spherical surface. For

$$\int (\quad) d\sigma = \int_0^{2\pi} \int_{-1}^{+1} (\quad) d\phi d\mu :$$

and

$$\int_0^{2\pi} d\phi \cos. (s\phi + \alpha_s) \cos. (s'\phi + \alpha_{s'})$$

vanishes unless s and s' be equal, in which case its value is π .

Hence, attending to § 4, and to (14),

$$\int S S d\sigma = 0 ,$$

and

$$\int S_i^2 d\sigma = \frac{2\pi}{2i+1} \sum_0^i A_s^2 \left\{ \frac{i+s}{i-s} \right\} . \quad . \quad . \quad (15).$$

8. Another curious expression for $\Theta_i^{(s)}$ is given by (4). For that equation gives

$$\begin{aligned} (1-\mu^2)^s \frac{d^s Q_i}{d\mu^s} &= - \left(i(i+1) - s(s-1) \right) \int (1-\mu^2)^{s-1} \frac{d^{s-1} Q_i}{d\mu^{s-1}} d\mu \\ &= + \{ i(i+1) - s(s-1) \} \{ i(i+1) - (s-1)(s-2) \} \iint (1-\mu^2)^{s-2} \frac{d^{s-2} Q_i}{d\mu^{s-2}} d\mu^2 \\ &= (-)^s \frac{i+s}{i-s} \int^{(s)} Q_i d\mu^s . \quad . \quad . \quad . \quad (16). \end{aligned}$$

Hence

$$\Theta_i^{(s)} = (-)^s \frac{i+s}{i-s} (1-\mu^2)^{-\frac{s}{2}} \left(\int d\mu \right)^s Q_i . \quad (17).$$

10. Let

$$\sqrt{1+2\mu h+h^2} = 1+hy . \quad . \quad . \quad (19),$$

where y is a function of h and μ , never beyond the limits $+1$ and -1 .

Then

$$\frac{h}{\sqrt{1+2\mu h+h^2}} = h \frac{dy}{d\mu} .$$

But the first equation gives, at sight,

$$y = \mu + h \frac{1-y^2}{2} . \quad . \quad . \quad (20),$$

whence,

$$y = \mu + h \frac{1 - \mu^2}{2} + \frac{h^2}{1 \cdot 2} \frac{d}{d\mu} \left(\frac{1 - \mu^2}{2} \right)^2 + \&c.,$$

and therefore,

$$\frac{1}{\sqrt{1 + 2\mu h + h^2}} = \frac{dy}{d\mu} = 1 + h \frac{d}{d\mu} \left(\frac{1 - \mu^2}{2} \right) + \frac{h^2}{1 \cdot 2} \frac{d^2}{d\mu^2} \left(\frac{1 - \mu^2}{2} \right)^2 + \&c.,$$

which shows that

$$Q_i = (-)^i \left(\frac{d}{d\mu} \right)^i \left(\frac{1 - \mu^2}{2} \right)^i \quad . \quad . \quad (21),$$

and suggests obvious simplifications of preceding results, *e.g.*,

$$\theta_s = (-)^i \left(\frac{d}{d\mu} \right)^{i+s} \left(\frac{1 - \mu^2}{2} \right)^i = (\text{by } \S 8) (-)^{i+s} (1 - \mu^2)^{-s} \frac{[i+s]}{[i-s]} \left(\frac{d}{d\mu} \right)^{i-s} \left(\frac{1 - \mu^2}{2} \right)^i, \\ \&c., \&c.,$$

11. The complete integral of

$$i(i+1)Q_i + \frac{d}{d\mu} \left((1 - \mu^2) \frac{dQ_i}{d\mu} \right) = 0 \quad . \quad . \quad (3)$$

may easily be found, since a particular integral is known. Let it be MQ_i , where M is a function of μ . Then (3) gives at once

$$\left(-2\mu Q_i + 2(1 - \mu^2) \frac{dQ_i}{d\mu} \right) \frac{dM}{d\mu} + (1 - \mu^2) Q_i \frac{d^2 M}{d\mu^2} = 0,$$

or,

$$\frac{-2\mu}{1 - \mu^2} + \frac{2}{Q_i} \frac{dQ_i}{d\mu} + \frac{1}{\frac{dM}{d\mu}} \frac{d^2 M}{d\mu^2} = 0,$$

whence

$$\frac{dM}{d\mu} = \frac{C}{(1 - \mu^2)Q_i^2}.$$

Thus the complete integral is

$$CQ_i \int \frac{d\mu}{(1 - \mu^2)Q_i^2} \quad . \quad . \quad . \quad (22).$$

12. Let us now suppose

$$S_i = P_i Q_i \quad . \quad . \quad . \quad (23),$$

where Q_i is as in § 1, and P_i is a function of μ and φ . The equation (10) becomes successively

$$\begin{aligned} \frac{d}{d\mu} \left((1-\mu^2) \frac{d(P_i Q_i)}{d\mu} \right) + \frac{1}{1-\mu^2} \frac{d^2(P_i Q_i)}{d\varphi^2} + i(i+1) P_i Q_i &= 0, \\ -2\mu Q_i \frac{dP_i}{d\mu} + (1-\mu^2) \left(2 \frac{dP_i}{d\mu} \frac{dQ_i}{d\mu} + Q_i \frac{d^2 P_i}{d\mu^2} \right) + \frac{Q_i}{1-\mu^2} \frac{d^2 P_i}{d\varphi^2} &= 0, \\ (1-\mu^2) Q_i \frac{dP_i}{d\mu} \left(-\frac{2\mu}{1-\mu^2} + \frac{2}{Q_i} \frac{dQ_i}{d\mu} + \frac{1}{\frac{dP_i}{d\mu}} \frac{d^2 P_i}{d\mu^2} \right) + \frac{Q_i}{1-\mu^2} \frac{d^2 P_i}{d\varphi^2} &= 0, \\ (1-\mu^2) Q_i \frac{dP_i}{d\mu} \frac{\frac{d}{d\mu} \left((1-\mu^2) Q_i^2 \frac{dP_i}{d\mu} \right)}{(1-\mu^2) Q_i^2 \frac{dP_i}{d\mu}} + \frac{Q_i}{1-\mu^2} \frac{d^2 P_i}{d\varphi^2} &= 0, \end{aligned}$$

and, finally,

$$(1-\mu^2) Q_i^2 \frac{d}{d\mu} \left((1-\mu^2) Q_i^2 \frac{dP_i}{d\mu} \right) + Q_i^4 \frac{d^2 P_i}{d\varphi^2} = 0.$$

If we put, for a moment,

$$\frac{d\mu}{(1-\mu^2) Q_i^2} = d\nu \quad (\text{which has a real meaning, see § 11}),$$

and suppose Q_i to be expressed in terms of ν instead of μ , calling it q_i , the equation may be written

$$\frac{d^2 P_i}{d\nu^2} + q_i^4 \frac{d^2 P_i}{d\varphi^2} = 0 \quad . \quad . \quad . \quad (24).$$

Hence it appears at once that P_i cannot contain φ except in the form of factors, such as $\cos. s\varphi$, $\sin. s\varphi$, in the several terms of which (as an integral of a linear equation) it must be composed. Hence, as before,

$$P_i = \sum_0^i A_s \Theta_i^{(s)} \cos.(s\varphi + \alpha),$$

and, keeping to one value of s ,

$$\frac{d^2 \Theta_i^{(s)}}{d\nu^2} - s^2 q_i^4 = 0.$$