8. It may be noticed here that, since the above note was communicated to the Society, I have found that the circle $UU'V$... passes through an eighth point dependent on U; viz., the intersection of the circles AOU, ABU'.

The coordinates of this point, W, say, are

$$x = \frac{\sin 2\theta}{\sin(2\theta - A)}, \quad y = \frac{\sin\theta}{\sin(\theta + B)}, \quad z = \frac{\sin\theta}{\sin(\theta + C)}.$$

As θ varies, or the circle $UU'V$... moves, the locus of W is a circular cubic having A for a double point.

The equation of this curve is

$$\frac{1}{y\sin B} - \frac{1}{z\sin C} = \cot B - \cot C,$$

which proves that it passes through B, C, and the foot of the perpendicular from A upon BC, and has an asymptote parallel to the median AM.]

On a Theorem of Liouville's. By Mr. G. B. Mathews.

Read December 14th, 1893.

In the first of the series of papers "Sur quelques Formules Générales qui peuvent être Utiles dans la Théorie des Nombres" [*Journ. de Math.*, (2) iii. (1858), p. 143], Liouville has stated without proof the following remarkable proposition:—

Let $2m$, the double of any odd integer, be expressed in all possible ways as the sum of two odd numbers, a and b, where the decompositions $2m = a + b$ and $2m = b + a$ are considered distinct, unless $a = b = m$; let α denote any divisor of a, and β any divisor of b, and let $f(x)$ be any even function of x, that is, such that

$$f(-x) = f(x).$$

Then, if μ denotes any divisor of m,

$$\Sigma\{f(\alpha - \beta) - f(\alpha + \beta)\} = \Sigma\mu\{f(0) - f(2\mu)\},$$

where the summation on the left applies to all pairs of divisors α and

β which can be derived from each of the partitions of $2m$, and the summation on the right applies to all the divisors of m.

A little consideration will show that the theorem must involve the following:—

The number of pairs of *unequal* associated divisors (α, β) for which the sum of α and β has any prescribed value is the same as that of the pairs for which the difference of α and β has the same prescribed value.

Moreover, when this second proposition is proved, it will be easy to infer the truth of the first.

To show the meaning of the theorems, and the way in which they are connected, suppose $m = 7$; then the partitions, and the values of α and β in each case, are

$1+13$,	$\alpha = 1$,	$\beta = 1,\ 13$,
$3+11$,	$1, 3$,	$1, 11$,
$5+9$,	$1, 5$,	$1, 3, 9$,
$7+7$,	$1, 7$,	$1, 7$,
$9+5$,	$1, 3, 9$,	$1, 5$,

and so on, the last partition being $13+1$.

The values of $\alpha+\beta$ are

$$2,\quad 4,\quad 6,\quad 8,\quad 10,\quad 12,\quad 14,$$

occurring respectively

$$7,\quad 4,\quad 2,\quad 4,\quad 2,\quad 2,\quad 7$$

times; and the values of $|\ \alpha-\beta\ |$ are

$$0,\quad 2,\quad 4,\quad 6,\quad 8,\quad 10,\quad 12,$$

occurring respectively

$$8,\quad 6,\quad 4,\quad 2,\quad 4,\quad 2,\quad 2$$

times. Thus, for instance, $|\ \alpha-\beta\ | = 4$, for the combinations (5, 1), (5, 9), (9, 5), (1, 5), and for no others. In this particular case, then,

$$\Sigma\left\{f(\alpha-\beta)-f(\alpha+\beta)\right\} = 8f(0)-f(2)-7f(14),$$

which agrees with $\Sigma\mu\left\{f(0)-f(2\mu)\right\}$, since the values of μ are 1 and 7.

The values of α, β in any associated pair are both odd, and their sum is therefore even. It is easy to see that the number of pairs

(α, β) for which $\alpha+\beta$ has a prescribed value $2t$ is equal to the number of positive integral solutions of the diophantine equations

$$\left.\begin{array}{r} x_1+(2t-1)\,y_1 = 2m \\ 3x_2+(2t-3)\,y_2 = 2m \\ 5x_3+(2t-5)\,y_3 = 2m \\ \vdots \quad\quad \vdots \quad\quad \vdots \\ (2t-1)\,x_t+y_t = 2m \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A).$$

In the same way, the number of pairs (α, β) for which $|\alpha-\beta| = 2t$ is *double* the number of positive integral solutions of

$$\left.\begin{array}{r} \xi_1+(2t+1)\,\eta_1 = 2m \\ 3\xi_2+(2t+3)\,\eta_2 = 2m \\ 5\xi_3+(2t+5)\,\eta_3 = 2m \\ \vdots \quad\quad \vdots \quad\quad \vdots \\ (2h-1)\,\xi_h+(2t+2h-1)\,\eta_h = 2m \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (B),$$

where, for a reason which will sufficiently appear as we proceed, the series of equations (B) is supposed to continue indefinitely, although it is clear that after a certain point, depending on the value of t, the equations cease to have positive solutions.

For convenience the r^{th} equation of the set (A) or (B) will be referred to as (A_r) or (B_r) respectively.

The equation (A_1) may be written

$$(x_1-2y_1)+(2t+1)\,y_1 = 2m\,;$$

hence, if there is a positive solution of the equation, such that $x_1 > 2y_1$, the values

$$\xi_1 = x_1-2y_1, \quad \eta_1 = y_1$$

give a positive solution of (B_1). Conversely, writing (B_1) in the form

$$(\xi_1+2\eta_1)+(2t-1)\,\eta_1 = 2m,$$

we see that from every positive solution of (B_1) may be derived a solution of (A_1) by putting

$$x_1 = \xi_1+2\eta_1, \quad y_1 = \eta_1,$$

and, moreover, this gives $\quad x_1-2y_1 = \xi_1,$

so that $x_1 > 2y_1$.

In the same way, every positive integral solution of (A_r) for which

$x_r > 2y_r$ is associated with a solution of (B_r); and, conversely, from every solution of (B_r) may be derived a solution of (A_r) for which $x_r > 2y_r$.

Again, the equation (A_1) may be written

$$(2t-1)(y_1-2x_1)+(4t-1)\,x_1 = 2m;$$

consequently, if there is a solution for which $y_1 > 2x_1$, we obtain a solution of (B_t) by putting

$$\xi_t = y_1 - 2x_1, \quad \eta_t = x_1,$$

and, conversely, from every positive solution of (B_t), we derive a solution of (A_1) by putting

$$x_1 = \eta_t,$$

$$y_1 = \xi_t + 2\eta_t,$$

and this is a solution for which $y_1 > 2x_1$.

In the same way, every positive solution of (A_r) for which $y_r > 2x_r$ is associated with a solution of (B_{t+1-r}) by means of the relations

$$\xi_{t+1-r} = y_r - 2x_r,$$

$$\eta_{t+1-r} = x_r;$$

and, conversely, from every positive solution of (B_{t+1-r}) may be derived a solution of (A_r) by putting

$$x_r = \eta_{t+1-r},$$

$$y_r = \xi_{t+1-r} + 2\eta_{t+1-r},$$

and this is a solution for which $y_r > 2x_r$.

It will be observed that the conditions $x_r > 2y_r$ and $y_r > 2x_r$ are mutually exclusive; so that, on the whole, each positive solution of the equations (B_1), (B_2), ... (B_t) is associated with two distinct solutions of the equations (A).

Now the equation (B_{t+1}) is

$$(2t+1)\,\xi_{t+1} + (4t+1)\,\eta_{t+1} = 2m,$$

which may be written

$$(2\xi_{t+1}+3\eta_{t+1}) + (2t-1)(\xi_{t+1}+2\eta_{t+1}) = 2m;$$

hence we obtain a solution of (A_1) by putting

$$x_1' = 2\xi_{t+1} + 3\eta_{t+1},$$
$$y_1' = \xi_{t+1} + 2\eta_{t+1}.$$

This gives

$$x_1' - 2y_1' = -\eta_{t+1},$$
$$2x_1' - 3y_1' = \xi_{t+1},$$

so that the solution is one for which

$$2y_1' > x_1' > \tfrac{3}{2}y_1';$$

and, therefore, distinct from those already considered.

In the same way, every positive solution of (B_{t+r}), where $r < t+1$, is associated with a solution of (A_r), say (x_r', y_r'), for which

$$2y_r' > x_r' > \tfrac{3}{2}y_r';$$

and, conversely, every solution of (A_r) for which these conditions of inequality are satisfied leads to a positive solution of (B_{t+r}) in the form

$$\xi_{t+r} = 2x_r' - 3y_r',$$
$$\eta_{t+r} = 2y_r' - x_r'.$$

Again, the equation (A_t) may be written

$$(4t-1)(2y_1 - 3x_1) + (6t-1)(2x_1 - y_1) = 2m,$$

so that, if there is a solution (x_r', y_r') for which

$$2x_1' > y_1' > \tfrac{3}{2}x_1',$$

we obtain a positive solution of (B_{2t}) by putting

$$\xi_{2t} = 2y_1' - 3x_1',$$
$$\eta_{2t} = 2x_1' - y_1';$$

and, conversely, every positive solution of (B_{2t}) leads to a corresponding solution of (A_1) for which

$$x_1' = \xi_{2t} - 2\eta_{2t},$$
$$y_1' = 2\xi_{2t} + 3\eta_{2t},$$

and

$$2x_1' > y_1' > \tfrac{3}{2}x_1';$$

and, in the same way, every positive solution of (B_{2t+1-r}) is associated with a solution of (A_r) for which

$$2x_r' > y_r' > \tfrac{3}{2}x_r',$$

by means of the relations

$$\xi_{2t+1-r} = 2y'_r - 3x'_r, \quad \eta_{2t+1-r} = 2x'_r - y'_r,$$

and conversely.

It is now easy to see that if the equations (B) are considered in successive groups, each containing t equations, then, if (ξ, η) is any positive solution of the r^{th} equation of the i^{th} group, we obtain a corresponding solution of (A_r) by putting

$$x_r^{(i)} = i\xi + (i+1)\,\eta,$$
$$y_r^{(i)} = (i-1)\,\xi + i\eta\,;$$

in fact, this gives

$$\begin{aligned}(2r-1)\,x_r^{(i)} + (2t-2r+1)\,y_r^{(i)} \\ &= \left\{(2r-1)\,i + (2t-2r+1)(i-1)\right\}\xi \\ &\quad + \left\{(2r-1)(i-1) + (2t-2r+1)\,i\right\}\eta \\ &= \left\{2\,(i-1)\,t + 2r - 1\right\}\xi + \left\{2it + 2r - 1\right\}\eta \\ &= 2m\,;\end{aligned}$$

because (ξ, η) is a solution of (B_k), where

$$k = (i-1)\,t + r.$$

It will be found that

$$\xi = ix_r^{(i)} - (i+1)\,y_r^{(i)},$$
$$\eta = iy_r^{(i)} - (i-1)\,x_r^{(i)},$$

so that the solution of (A_r) is one for which

$$\frac{i}{i-1}\,y_r^{(i)} > x_r^{(i)} > \frac{i+1}{i}\,y_r^{(i)}\,;$$

and, conversely, from every such solution may be derived a corresponding solution of (B_k).

Similarly, to every solution $(x_r^{(i)}, y_r^{(i)})$ for which

$$\frac{i}{i-1}\,x^{(i)} > y_r^{(i)} > \frac{i+1}{i}\,x_r^{(i)}$$

corresponds a solution of (B_l), where

$$l = it + 1 - r,$$

which is given by
$$\xi = iy_r^{(i)} - (i+1)\,x_r^{(i)},$$
$$\eta = ix_r^{(i)} - (i-1)\,y_r^{(i)}.$$

Each solution (ξ, η) belonging to the i^{th} group of equations (B) is therefore associated with two solutions of the equations (A), while every solution of an equation (A) which satisfies the inequalities last written is associated with one, and only one, equation (B) of the group. Moreover, if we consider the inequalities

$$x > 2y,\quad 2y > x > \tfrac{3}{2}y,\quad \tfrac{3}{2}y > x > \tfrac{4}{3}y,\ \dots\ \frac{i}{i-1}y > x > \frac{i+1}{i}y,\ \dots,$$

$$y > 2x,\quad 2x > y > \tfrac{3}{2}x,\quad \tfrac{3}{2}x > y > \tfrac{4}{3}x,\ \dots\ \frac{i}{i-1}x > y > \frac{i+1}{i}x,\ \dots,$$

where i assumes all positive integral values, it is clear that any two positive integers x, y must satisfy one, and only one, of these conditions, except when
$$x = y,$$
or
$$x = i+1,\quad y = i,$$
or
$$x = i,\qquad y = i+1.$$

In the case we are considering, it is easy to see that, since m is odd, x and y must be both odd or both even if (x, y) is a solution of an equation (A_r); consequently, the only exceptional case to be considered is when $x = y$.

This gives, for each of the equations (A),
$$2tx = 2m,\quad \text{or}\quad tx = m;$$
therefore t is a divisor of m, say μ, and
$$x = y = \mu',$$
where
$$\mu\mu' = m.$$

If, then, μ is any divisor of m, there will be exactly μ combinations (α, β) for which
$$\alpha+\beta = 2\mu,$$
namely, $(1, 2\mu-1),\ (3, 2\mu-3),\ \dots\ (2\mu-1, 1),$

not associated with corresponding combinations (α', β') for which
$$|\,\alpha'-\beta'\,| = 2\mu.$$

These pairs are connected with the partitions

$$\mu' + (2\mu - 1)\,\mu', \quad 3\mu' + (2\mu - 3)\,\mu', \; \ldots \; (2\mu - 1)\,\mu' + \mu'$$

(where $\mu' = m/\mu$).

Consequently, the sum of the uncompensated terms $f(\alpha+\beta)$ is

$$\Sigma \mu f(2\mu).$$

The number of terms for which $\alpha - \beta = 0$ is very easily found. If $\alpha = \beta$, each must be a divisor of m; suppose

$$\alpha = \beta = \mu.$$

Then the combination (μ, μ) arises from each of the partitions

$$\mu + (2m - \mu), \quad 3\mu + (2m - 3\mu), \; \ldots \; (2m - \mu) + \mu,$$

and no others. The number of these partitions is

$$\frac{m}{\mu} = \mu', \text{ say};$$

therefore the number of times $\alpha - \beta = 0$ is $\Sigma\mu'$, or, which is the same thing, $\Sigma\mu$.

In the triple sum $\quad \Sigma\,\{f(\alpha-\beta) - f(\alpha+\beta)\}$,

every term $f(\alpha-\beta)$ in which $\alpha-\beta$ is not zero is cancelled by a corresponding term $f(\alpha'+\beta')$, and, therefore, finally,

$$\begin{aligned} \Sigma\,\{f(\alpha-\beta) - f(\alpha+\beta)\} &= f(0)\,\Sigma\mu - \Sigma\mu f(2\mu) \\ &= \Sigma\mu\,\{f(0) - f(2\mu)\}, \end{aligned}$$

which is Liouville's theorem.

Thursday, January 11th, 1894.

Mr. A. B. KEMPE, F.R.S., President, in the Chair.

Mr. A. E. Daniels, B.A., late Scholar of Peterhouse, Cambridge, Mathematical Master of Nottingham High School, was elected a member. Messrs. H. M. Macdonald and C. Morgan were admitted into the Society.

The President communicated to the meeting the intelligence which had just reached him of the death, on the 10th January, of Dr. Heinrich Rudolf Hertz, who was elected an honorary foreign member of the Society on April 14th, 1892.*

The following communications were made:—

The Types of Wave-Motion in Canals: Mr. H. M. Macdonald.

On Green's Function for a System of non-Intersecting Spheres: Prof. W. Burnside.

The following presents were made to the Library:—

"University of Japan—Calendar for 1892-3"; Tokyo, 1893.

"Beiblätter zu den Annalen der Physik und Chemie," 1893, Bd. xvii., St. 11; Leipzig, 1893.

"Nieuw Archief voor Wiskunde," 2e Reeks, Deel i., 1; Amsterdam, 1894.

"Proceedings of the Royal Society," Vol. liv., No. 328.

"Nyt Tidsskrift for Mathematik," A. 4e Aargang, Nr. 4-6.

"Nyt Tidsskrift for Mathematik," B. 4e Aargang, Nr. 3; Copenhagen, 1893.

"Bulletin of the New York Mathematical Society," Vol. iii., No. 3; December, 1893.

"Proceedings of the Cambridge Philosophical Society," Vol. viii., Pt. 2; 1894.

"Revue Semestrielle des Publications Mathématiques," Tome ii., 1ère Partie; Amsterdam, 1894.

"Wiskundige Opgaven met de Oplossingen," Zesde Deel, 2de Stuk; Amsterdam, 1893.

"Bulletin de la Société Mathématique de France," Tome xxi., Nos. 7, 8; Paris.

"Bulletin des Sciences Mathématiques," Tome xvii., October, 1893; Paris.

Byerly, W. E.—"Elementary Treatise on Fourier's Series, and Spherical, Cylindrical, and Ellipsoidal Harmonics," 8vo; Boston, 1893.

"L'Intermédiaire des Mathématiciens," Tome i., No. 1; Paris, 1894.

"Annali di Matematica," Serie 2, Tomo xxi., Fasc. 4; Milano.

"Educational Times," January, 1894.

"Atti della reale Accademia dei Lincei—Rendiconti," Vol. ii., Fasc. 10, 11, 2 Sem.; Roma, 1893.

"Annals of Mathematics," Vol. viii., No. 1.

"Indian Engineering," Vol. xiv., Nos. 22-5.

"Transactions of the Royal Irish Academy," Vol. xxx., Pts. 5-10.

"Proceedings of the Royal Irish Academy," Vol. iii., No. 1; Dublin, 1893.

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* *Cf. Nature*, January 18th, 1894 (p. 265), for a Notice by D. E. J.