On Recent English Researches in Vortex-motion.

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The object of this paper is to give an account of the more recent researches made by English mathematicians in the theory of the motion of vortices in a perfect liquid. I shall for the most part limit myself to the discussion of investigations which have appeared since the publication of Lamb's "Treatise on the motion of Fluids" *), though for the sake of clearness and continuity it will be necessary to bestow at least a passing notice on some writings of an earlier date.

It may be well to state briefly the laws of vortex-motion as deduced by Helmholtz **) and Sir W. Thomson ***). Defining the circulation in any circuit as the line-integral of the velocity taken round the circuit, it was proved by the latter (1) that the circulation in any circuit which always consists of the same particles of fluid is constant, (2) that the circulation in any circuit is equal to the surface-integral of vortex-strength over any surface having the circuit for an edge. From the second of these, which is a purely analytical theorem, it follows that the circulation is the same in all circuits which can be continuously deformed into one another without passing out of the region occupied by fluid in irrotational motion. Helmholtz's laws of vortex-motion are immediate deductions from these. They are (1) each vortex remains continually composed of the same particles of the fluid, (2) the strength †) of a vortex-filament is constant for all time and for all points of the filament, (3) a vortex-filament must be closed or have its ends on the boundary of the fluid.

The chief physical interest of vortex-motion lies in the speculations of Sir W. Thomson as to the ultimate constitution of matter ††). The adherents of the kinetic theory of gases appear to have been driven to

*) Cambridge University Press, 1879.

**) „Ueber Integrale der hydrodynamischen Gleichungen welche den Wirbelbewegungen entsprechen.“ Crelle, Bd. LV.

***) “On Vortex motion.” Trans. Roy. Soc. Edinburgh vol. 25, 1869.

†) The “spin” is the name given to the vector whose components are ξ , η , ζ , it is also called the elementary or molecular rotation. The “strength” of a vortex-filament is the product of the area of its cross-section and the spin about one of its vortex-lines. The “vorticity” of a vortex-motion is said to be given when the volumes and strengths of all the filaments are given.

††) “Vortex Atoms,” Phil. Mag. (4), XXXIV. 1867.

assume that the atoms of a gas are at once elastic and infinitely strong and rigid, and that they exert on each other a direct action at a distance. The brilliant discoveries of Helmholtz in the theory of this kind of fluid motion, combined it would seem with some striking experiments, suggested the fundamental idea of these speculations. According to the view of their originator, we are to imagine space to be continuously occupied by an incompressible frictionless fluid, i. e. a medium whose motions are determined by the hydrodynamical equations. Some parts of the medium are to be in vortex-motion, and these will necessarily be distributed within certain annular surfaces. The liquid within such a surface is called a vortex, and we are to suppose the atoms of which matter is composed to be such vortices. All the elements of the medium which at any instant form a vortex remain for ever parts of the same vortex. This is the indestructibility and impenetrability of the atom. The vortices may be knotted, or two or more vortices may be linked through each other. In these cases the kind of multiple continuity possessed by the space without or within the vortex is one of its permanent characteristics. This gives us a method of realising different kinds of unalterable atoms which cannot be interchanged. The simple circular ring when undisturbed moves forward with constant velocity in the direction of the circulation through the ring, this realises the external kinetic energy of the atom; it may also vibrate about its circular form, or its section may change shape periodically, and it will thus possess internal energy. The periods of vibration of vortex-atoms for different fundamental modes of deformation should correspond to the lines of the spectrum of the gas composed of them, and the theory of spectrum analysis would require that at least some of the periods should be approximately independent of the energy. It may be observed here that the results obtained later shew that the periods are functions of the energy in all cases except that of a hollow vortex crimped in section.*)

Following up his earlier papers on the general theory of vortex-motion, Sir W. Thomson has considered the theory of the steady motion**) and stability of vortices,***) setting out from the general principle

*) An account of the present state of our knowledge with regard to vortex-motion and the doctrine of vortex-atoms, containing a statement of the results of experimental as well as of theoretical investigations, will be found in a series of memoirs by M. Brillouin, now appearing in the *Annales de la Faculté des Sciences de Toulouse*.

**) A system is said to be in "steady motion" when its motion is similar at any two instants of time. In the case of vortex-motion the term is extended to include all cases in which the forms sizes and velocities and relative configuration of the vortices remain constant.

***) "Vortex Statics." *Proc. Roy. Soc. Edin*: 1876, and *Phil. Mag.* (5) X. 1880.

that in steady motion the energy with given vorticity and impulse must fulfil a stationary condition. The motion will be stable if the energy be a thorough maximum or a thorough minimum, but if it be a maximum-minimum its stability cannot be inferred without further investigation. If for the vortex we substitute a perforated solid of the same shape, and suppose the circulations in the different circuits through its perforations to be those required by the vortex-strength of the original vortex, it is clear that to any steady motion of the solid will correspond a steady motion of the vortex; if further every point of the solid be supposed to carry a mark indicating the direction and amount of the vorticity at the corresponding point of the vortex, the distribution of vorticity in the liquid at any instant will be that expressed by these marks. We know that a solid of any shape will have definite modes of steady motion, and we now see that a vortex of the same form as the solid will, for some determinate distribution of vorticity, be capable of steady motion in the same manner as the solid and with the same impulse. This distribution of vorticity being known, we have a solution of the problem, to make the energy stationary with the given vorticity and impulse. As examples of such steady motions we have the Helmholtz ring, a circular vortex-ring with an inner irrotational core, and many cases of knotted and linked vortices. The author treats more at length the case of the circular ring, for which the impulse reduces to an impulsive force without couple. There is just one determinate figure of revolution for which the energy is a maximum, and the ring is stable as regards changes in the form of its section; there are however figures, not of revolution, differing very little from this, which have the same energy with the same vorticity and impulse, so that from the maximum-minimum problem the stability even of the ordinary thin Helmholtz ring cannot be rigidly inferred.

The same author has discussed the stability of vortex-motion within spaces bounded externally by closed surfaces, in a communication to the British Association in 1880.*) In this case the condition for stability is that with given form of boundary and given vorticity the energy is a maximum or minimum. By operations on the boundary the energy may in general be indefinitely increased, so that there is no thorough maximum of energy; but in the case of two-dimensional motion, with given cylindric figure and given vorticity, this is no longer true, and there is a maximum of energy. Except in a few cases, there will be an infinite number of stable steady motions. This is illustrated by the case of a cylinder whose section consists of two nearly complete circles connected by a narrow channel; if a small portion of the fluid

*) "On maximum and minimum energy in vortex motion." Brit. Ass. Rep. 1880, or Nature vol. XXII, p. 618.

be in irrotational motion there will be stable steady motions with this portion distributed in different proportions within the circles. The author gave some cases of the annulment of energy by operations on the boundary which withdraw energy. According to the Helmholtz formula for the energy*) it appears that, if infinitely near to any vortex-filament there is an equal and opposite vortex-filament, the volume-integral term consists of elements which destroy each other identically; if the operations on the boundary can be so conducted that thereafter the elements of the fluid in contact with the boundary are at rest, the energy will be annulled. The operations by which this result could be brought about are not described. Taking however the case of liquid enclosed within a circular cylinder two modes of stable steady motion are described. In the first, for which the energy is a maximum, the rotationally-moving fluid is collected in a circular column concentric with the boundary; in the second, for which the energy is a minimum, it is formed into a thick belt at the boundary. The author proceeds to describe a series of operations to be performed on the boundary, whereby the stable motion with maximum energy may be broken up and transformed into the stable motion with minimum energy. The enclosing case may be supposed constructed of some viscous elastic material. If this be deformed and then held from rotating the central vortex will be thrown into vibration. Waves of corrugation will travel round the case and thereby energy will be consumed and the fluid will lose moment of momentum. Now let the case rotate; the fluid will lose energy, but the moment of momentum remains constant. If the case were again held fixed the energy of the fluid and the moment of momentum would be too small for a return to the original stable steady motion to be possible. This state of motion is broken up, and the way in which it is broken up is by the augmentation of the waves of shorter length. If the case be allowed to rotate at intervals this effect will be increased, and the rotationally-moving liquid will be mixed with the irrotationally-moving liquid in what the author calls a „vortex-sponge“. This process may be continued until the sponge occupies the whole enclosure. More energy may now be withdrawn by repeating the operations previously performed on the boundary, and the process may be continued until, when the case is finally held fixed, the condition of minimum energy is arrived at**).

The same author has investigated mathematically the vibrations

*) Lamb. p. 160.

***) Sir W. Thomson is now engaged on this subject, and we may look forward to a revision of this article. See also his papers in the current and forthcoming numbers of the Phil. Mag. „On steady and periodic motions of a perfect incompressible fluid“.

of a columnar vortex*). He transforms the equations of hydrodynamics to columnar coordinates (r, θ, z) and then supposes all the deviations from the condition of steady motion to be expressed by simple harmonic functions of θ, z and the time. If the steady motion be that of a hollow vortex with irrotational circulation about it revolving within a concentric case, the result is that two systems of waves travel round the cylinder with angular velocity $\omega(i \pm \sqrt{N})$, where ω is the angular velocity in the undisturbed motion of points on the internal surface, i is the number of crests in a normal section, and N depends only on the number of crests in a unit length along the generators. In case the vortex contains rotational filaments of uniform vorticity, the periods are given by a transcendental equation which has an infinite number of roots. The solution is simplified if the waves be purely longitudinal or purely sectional. In the first case, if the distance between successive swellings is large compared with the diameter, the velocity of the longitudinal waves is, for the mode of deformation which has the longest period, about $\frac{5}{6}$ of the translational velocity at the surface of the vortex. In the second case the angular velocity of the sectional waves is $i - 1$ times the angular velocity of the vortex, where i is the number of crests on the circumference. The periods are functions of the radius of the vortex column.

In order to complete a vortex theory of matter it is most important to know the way in which the motion of any vortex is altered by the presence of its neighbours. The subject selected for the Adams Prize for 1882 was „A general investigation of the action upon each other of two closed vortices in a perfect incompressible fluid.“ The prize was awarded to *J. J. Thomson*. His essay**) is divided into four parts. In the first some general results are given, viz: the motion being expressed by Helmholtz's forms***), expressions are obtained for the momentum and angular momentum of the liquid containing vortex-rings, and it is verified that these are constant. A new expression is also obtained for the kinetic energy, which is found useful in the application to the theory of gases. In the remainder of the first part the motion of a single ring is treated. The surface of the undisturbed ring is supposed to be a circular tore, whose sectional radius is very small compared with its aperture. Helmholtz's magnetic analogue†) shews that the velocity produced by any distribution of vortices is

*) Phil. Mag. (5) X.

**) „A Treatise on the motion of Vortex-Rings.“ London, 1883. (Macmillan & Co.)

***) Lamb, p. 150.

†) Lamb, p. 153. Hicks, Report „On Recent Progress in Hydrodynamics.“ Brit. Ass. Rep. 1881, p. 64.

proportional to the magnetic force produced by electric currents coinciding in position with the vortex-lines, and such that the strength of the current is proportional to the strength of the vortex; and Maxwell*) has proved that if currents of electricity flow round a tore, whose section is small compared with its aperture, the magnetic effects of the currents are the same as if all the currents were collected into one flowing along the central-line of the tore. Thus the action of a vortex-ring of this shape will be the same as that of one of equal strength condensed along the circular axis. The central-line of the ring when disturbed is supposed to be very nearly circular, viz: referring it to columnar coordinates (ϱ, z, ψ) , ϱ differs from the radius of the ring, and z from that at the centre of the ring, by series containing sines and cosines of multiples of ψ with small coefficients. The velocity of the fluid at any point is calculated, as far as the second order in these coefficients, by means of Helmholtz's forms, the vorticity being supposed condensed along the axis. The shape of the ring is taken to be such that all sections through the straight axis remain circular, and have their radius unaltered. The equations of the surface of the ring must satisfy the differential equation of surfaces which always contain the same particles. Substituting for the velocities at the surface the expressions obtained by the method just described, this differential equation gives rise to sufficient equations to determine all the unknown quantities in terms of the time.

Taking m for the strength of the vortex, a for the radius of the central-line, and e for the sectional radius of the undisturbed ring, the results are (1) The angular velocity of the fluid elements round the tangent to the central-line is approximately equal to the spin. (2) The velocity of translation is $V = \frac{m}{2\pi a} \left(\log \frac{8a}{e} - 1 \right)$. In the steady motion this is the velocity of points on the central-line; points on any other circle within the vortex (radius ϱ) whose centre is on the axis of the vortex move with velocity $V \frac{a}{\varrho}$. (3) When the ring is disturbed the period of the n^{th} harmonic component vibration is $2\pi / (L.n. \sqrt{n^2 - 1})$ where

$$4\pi a^2 L = m [\log 64a^2/e^2 - 4f(n) - 1]$$

and

$$f(n) = 1 + 1/3 + 1/5 + \dots + 1/(2n-1).$$

The first of these was stated by Sir W. Thomson in his note to Tait's translation of Helmholtz's memoir**, and the velocity of translation

*) Electricity and Magnetism vol. II, § 633.

**) Phil. Mag. (4) XXXIII, 1867. p. 511.

was there given as $\frac{m}{2\pi a} \left(\log \frac{8a}{e} - \frac{1}{4} \right)$, which nearly agrees with the result (2). J. J. Thomson's result was also obtained by Lewis*), but, as will be seen below, Hicks's investigation confirms Sir W. Thomson's statement. As the section of the ring is not in the steady motion an exact circle, the effect is not strictly the same as if the vorticity were condensed along the axis, so that we can hardly expect the method to yield results in which any but the largest terms are correct. The vibrations of the ring cannot be very different from those of the columnar vortex investigated by Sir W. Thomson, and it is accordingly found that the periods for corresponding modes of deformation of the two systems are approximately the same. It may be noted that the periods found are functions of the energy.

In part II is discussed the problem „To find the action of two vortices upon each other which move so as never to approach closer than a large multiple of the diameter of either.“

Since the vortices never approach each other very closely the central-line of each vortex-core may be taken to be approximately circular throughout the collision. The velocity of the fluid due to the vortices is then known from the investigation in part I, and at any point of either vortex this must be equal to the velocity of the vortex at that point. We have thus two expressions for the velocity along the radius, and two for that perpendicular to the plane of either vortex, each of them being a series proceeding according to sines and cosines of multiples of the vectorial angle of a point on the corresponding central-line. Equating the coefficients of these series, we obtain sufficient equations to determine the coefficients in the series forming the equation to the central-line of either vortex, *i. e.* the effect of one vortex on the other. The effects of the collision may be divided into (1) the effect upon the radii of the vortex-rings, (2) the deflection of their paths perpendicular to the plane parallel to the original paths of both rings, this we will call the „plane of motion“, (3) the deflection of their paths in this plane, (4) the effect on the form of the central-lines. The collisions may be divided into two classes, (a) those in which the shortest distance between the vortices is greater than twice the shortest distance between their original paths, (b) those in which it is less. The effects are given as follows: (p. 63). —

Class (a). If the vortex II be the first to intersect the shortest distance between the directions of motion of the vortices its radius is increased. If its velocity is greater than the velocity of I resolved along the direction of motion of II, it is bent away from the plane of motion,

*) „On the Images of Vortices in a spherical vessel.” Quarterly Journal of Pure and Applied Mathematics. vol. XVI.

and towards the direction of motion of I in this plane. If its velocity is less than that of I resolved along the direction of motion of II, it is bent away from the plane of motion, and away from the direction of motion of I in this plane. In the same case, if the velocity of I is greater than the velocity of II resolved along the direction of motion of I, then I is bent away from the plane of motion and towards the direction of motion of II in this plane. If the velocity of I is less than that of II resolved along the direction of motion of I, then I is bent away from the plane of motion and away from the direction of motion of II in this plane.

Class (b). If the vortex II be the first to intersect the shortest distance between the directions of motion of the vortices its radius is diminished. It is bent from or towards the plane of motion according as the shortest distance between the vortices is greater or less than $2/\sqrt{3}$ times the shortest distance between their original paths, and it is bent from or towards the direction of motion of I according as its velocity is greater or less than the velocity of I resolved along the direction of motion of II. If I be the first to intersect the shortest distance, II is bent in the same manner as before with reference to the plane of motion, but in the opposite way in this plane.

It is also shewn that the central-lines do not remain circular after the collision is over, but become twisted ellipses, vibrating about the circular form with constantly changing ellipticities. The time of vibration is the corresponding free period. It may be noted that the ring which has its radius increased gains energy, but loses translational velocity, and the external energy of the system is diminished by the collision.

Very similar to the above problem is that of finding the motion of a vortex-ring in a fluid in which there is a given irrotational motion. The case where the motion is that of a vortex past a fixed obstacle is discussed, and worked out in the particular case where the obstacle is spherical. The radius of the ring is increased by a quantity whose greatest value is proportional to the volume of the sphere, and varies inversely as the sixth power of the least distance between the vortex and the centre of the sphere. The vortex is attracted towards the sphere, and its path is turned through an angle which varies inversely as the sixth power of the same distance.

In part III the author proceeds to discuss the case of linked vortices. Two rings are supposed to be linked in such a way that the distance between them is small compared with the aperture, but large compared with the sectional diameter, of either. Before the method of part I can be applied, it has to be shewn that the sections of the rings remain approximately circular. The effect of those parts

of ring II which are nearest to any part of ring I in altering the shape of that part, will, under the circumstances contemplated, be great compared with the effect of the remainder; and if these parts of rings I and II can be treated as approximately straight and parallel, the alteration of shape will be comparable with that which takes place in the case of two parallel columnar vortices. It is shewn that two such vortices of strengths m , m' , and whose centres are at a distance d , will remain approximately circular, vibrating about the circular form, if their distance be great compared with the diameter of either. In this case they spin round their mean centre with angular velocity $(m + m') / \pi d^2$. We have now to find a method of linking by means of which the rings shall have their nearest parts approximately parallel, and nearly at a constant distance. This is done by the following construction: — describe a tore whose apertural radius is a and sectional radius $\frac{m'}{m + m'} d$, the central-line of the vortex of strength m lies on this tore; describe a second tore, having the same circular axis, but whose sectional radius is $\frac{m}{m + m'} d$, then the central-line of the vortex of strength m' lies on this tore. The central-lines are now to be so arranged that in any plane through the straight axis of the tore the points on the central-lines of the two vortices are at opposite points on the same sectional diameter. The method of part I is applied to find the condition that the motion may be steady. This is a relation between the constants determining the size of the system and the components of the impulse. The motion if steady is stable. The periods of vibration of the system are found, and it is shewn that corresponding to each mode of deformation there are two periods, one long and the other short; these vary with the energy, and are longer the more complex is the linking. The author proceeds to discuss the problem of the motion of n linked vortices of equal strength. It is pointed out that, if n be large, the surface of the tore on which the vortices are linked will approximate to a vortex-sheet, and the motion will become unstable. To throw light on this problem the analogous one of n columnar vortices placed at the corners of a regular polygon inscribed in a circle is solved, and it is shewn that the motion is stable if 3, 4, 5, or 6 vortices are arranged in this way, but unstable if there are 7. It is inferred that probably not more than six vortices can be linked symmetrically round a tore so as to form a stable system.

In part IV the preceding theory is applied to sketch out a vortex-atom theory of gases and a theory of chemical combination. According to the formula for the kinetic energy of a system of vortex-rings investigated in part I, it appears that the temperature of the gas com-

posed of vortex-atoms is very nearly proportional to the sum of the products of the momentum and velocity of its vortex-rings. It is shewn that the pressure of the gas is nearly proportional to the same quantity, and thus we re-find Boyle's Law with the kind of defect indicated by Regnault's experiments. After discussing a crucial but probably very difficult experiment to decide between the vortex-atom and the solid-particle theories, the author proceeds to apply his results to the theory of chemical combination. When two vortex-rings of the same sort approach each other in such a way that when nearest together they have nearly equal velocities and nearly coincident planes, they do not separate, but pass alternately through each other. Supposing the vortex-rings to be atoms of a gas, two atoms paired will form a molecule. Any disturbance will tend to separate the molecule into its component atoms. Thus the atoms will be constantly changing partners. If the rings are of different sorts the combination is supposed to form a molecule of a chemical compound. In all cases if the compound is stable the ratio of the time during which the atoms are paired to that during which they are free must be large. The ring-atoms may be simple or complex-ly linked. If two rings are linked in an atom, this can only combine stably with two or an even number of simple rings. Hence from the degree of complexity of an atom of any element we may determine what stable compounds can be formed containing this element, and conversely we may hence infer the kind of complexity of each atom. The valency of several elements is discussed on these principles. The principle that not more than six vortices can be linked together so as to form a stable system is applied to find a limit to the complexity of arrangement of atoms, and shewn to agree with the results of experiment.

On the same subject we may notice a recent paper by the same author*). The principles laid down in his essay are applied to obtain a theory of the electric discharge in gases. It is shewn that, on the vortex-atom theory, the kind of stress which exists in the dielectric medium would be found between the vortically-moving elements of the perfect fluid. Now the electric field is taken to consist of a distribution of velocity in the irrotationally-moving part of the fluid. This disturbance makes the molecules break up more quickly than they otherwise would. If the disturbance in some places becomes so violent as to bring down the ratio of the paired time to the free time to somewhere near the value which it has when the gas is about to be dissociated, any further diminution of the ratio will cause the absorption of a large amount of energy. This energy is drawn from

*) Phil. Mag. (5) XV. 1883.

the electric field, and we have the phenomenon of the electric discharge. Considerations were also adduced to shew that at pressures comparable with the atmospheric pressure the electric strength of a gas would increase as the density decreases, and that at very low pressures the reverse would be the case. Both these results are in accordance with experiment.

I may mention here that a theory of the constitution of the luminiferous ether on the vortex-atom theory has been given by Hicks*). The perfect fluid required by the theory is incapable of transmitting light-vibrations. The author supposes the medium to be filled with very small and closely packed vortex-atoms, which are portions of the fluid in vortex-motion. A medium thus constructed could transmit light-vibrations and would probably act as a fluid for large motions. The atoms of matter are to be other vortices in the same perfect fluid.

One of the problems presented to a speculator in kinetic theories of matter is the explanation of gravitation between distant masses by the transmission of stress through the intervening medium. Whenever a body in a fluid changes its volume there will be a term proportional to the inverse distance in the velocity-potential**), and two such bodies will appear to act on each other with a force whose principal term varies inversely as the square of the distance between them. A sketch of a vortex-theory of gravitation was first given by Hicks***). He observes that, unless an infinite pressure can be provided, the velocity of elements of the fluid in vortex-motion may be so increased that the pressure sinks to zero; in this case a hollow will be formed. A hollow once existing with circulation round it has similar properties to a vortex, inasmuch as it cannot be destroyed by the action of conservative forces. A hollow vortex with or without rotational filaments may pulsate†), and thus two such vortices will appear to act on each other with a force whose greatest term varies inversely as the square of the distance, when that is great. It was shewn that in the case of spheres the force is an attraction if the periods of pulsation are the same and the phases do not differ by more than a quarter of a period. If this result be true of vortex-atoms, then, in order that gravity may be always an attraction, it is necessary that the period of pulsation should be approximately inde-

*) Brit. Ass. Rep. 1885. p. 930.

**) Stokes. "On some cases of Fluid motion." Camb. Phil. Trans. vol. VIII, p. 119. "Reprint of mathematical and physical papers." vol. I, p. 40.

***) "On the problem of two pulsating spheres in a fluid." Camb. Phil. Proc. vol. III, pp. 283 sq.

†) A body is said to pulsate when it changes its volume periodically.

pendent of the energy. If it should be thought that results subsequently obtained are unfavourable to this speculation, it would not follow that we are reduced to explaining the gravitation of matter by attributing the same property arbitrarily to the vortex-atom; on the other hand, when it is remembered how small is the force of gravitation compared with electric and magnetic forces and the molecular forces of cohesion &c., it will be felt as a possibility that the gravitation of masses compacted of vortex-atoms may be a small residual effect, of which our approximate mathematical work has so far failed to take account. At the same time it may be mentioned that, as remarked by Sir W. Thomson*), it is as improbable, not more so, that vortex-atoms should exert on each other a direct action at a distance as that any other atoms should.

Partly to test his theory of gravitation, and partly for its intrinsic interest, Hicks has made an elaborate investigation of the motion of a single vortex-ring. The first step is the determination of functions satisfying Laplace's equation, and suitable for expressing conditions given over the surface of a tore. This was done in the paper on Toroidal Functions**). Taking the two systems of orthogonal circles having common radical axes and limiting points, the figure is rotated round the radical axis perpendicular to the line joining the real limiting points. We thus obtain a system of orthogonal surfaces. Calling the circle traced out by the limiting points the critical circle, these are

- (1) tores $u = \text{const.}$ having the same critical circle,
- (2) spheres $v = \text{const.}$ intersecting in the critical circle,
- (3) planes $w = \text{const.}$ through the straight axis of the tores.

It is shewn that the functions depend upon zonal spherical harmonics of fractional index, and the zonal functions are expressible in terms of elliptic integrals of the first and second kind whose modulus (k) is $\frac{r}{R + \sqrt{R^2 - r^2}}$; where r , $R - r$ are the radii of the section and aperture of the tore. When the section is small compared with the aperture k is small and is nearly equal to $\frac{r}{2R}$. A function satisfying Laplace's equation, and symmetrical about the straight axis, can be expressed in the form $\sqrt{\cosh u - \cos v} \Sigma (A_n P_n + B_n Q_n) \cos(nv + a)$, where

*) "Vortex Atoms." Phil. Mag. (4) XXXIV. p. 16.

**) Phil. Trans. 1881. p. 609. In the discovery of these functions, and in an important theorem regarding the solution of Laplace's equation with conditions given over surfaces of revolution, Hicks was anticipated by C. Neumann. The results of his investigation with some applications were given in a pamphlet, "Theorie der Elektricitäts- und Wärme-Vertheilung in einem Ringe", published at Halle in 1864.

P_n , Q_n are zonal harmonics of the first and second kinds having $\sqrt{-1} \cdot u$ for argument and $n - \frac{1}{2}$ for index.

In his next paper Hicks discusses the steady motion and small vibrations of a hollow vortex*). It is shewn that Stokes's stream-function (ψ) can be expressed in the form

$$\psi = \frac{1}{V \cosh u - \cos v} \sum (A_n R_n + B_n T_n) \cos(nv + \alpha),$$

where

$$R_n = \sinh u \frac{dP_n}{du}, \quad T_n = \sinh u \frac{dQ_n}{du}.$$

The motion of a rigid ring with circulation through its aperture is investigated; and the author proceeds to discuss the motion of a hollow vortex. The form of such a vortex and its motion are conditioned by the fact that the velocity of the fluid relatively to the hollow, when the motion is steady, must be constant over the whole free surface. When the section is small compared with the aperture it is clear that the section is approximately circular, and a first approximation to the motion can be found from the work relating to the rigid tore. The stream-function for the tore held fixed and with fluid streaming past it consists of a term $-\frac{1}{2} V \varrho^2$, where V is the velocity of the stream and ϱ the distance from the axis, together with a series of terms of the form given above, the coefficients in these terms being each one higher in k (the modulus) than the preceding. From this we find the velocity, and the condition is that, to the first order in k , the term in $\cos v$ in the velocity at the surface of the tore is to disappear. This will ensure that, to the lowest order in k , the velocity is constant at the surface of the hollow. The velocity (V) of translation hence found is $V = \frac{\mu}{4\pi a} (L - \frac{1}{2})$ where μ is the circulation, $L = \log \frac{4}{k}$, and a is the radius of the critical circle.

The value of k determines the size of the tore. If the section is not exactly circular the defect from circularity can be expressed by making the value of k differ from that for the mean section by a series of sines and cosines of multiples of v with small coefficients; the stream-function will then differ from that in the case of the rigid tore by a new series satisfying the differential equation of Stokes's stream-function, and consequently of the form given above. The condition to be satisfied is that the new stream-function is constant over the surface of the hollow. Since in the old stream-function the terms diminish rapidly, the coefficients in the new stream-function are easily found

*) Phil. Trans. 1834.

in terms of those in the new k . The first coefficient in the latter series is found from the fact that it must destroy the term in $\cos 2\psi$ in the surface-velocity. The form of the hollow is, to this order of approximation, slightly elliptic, with major axis perpendicular to the plane of the ring and the inner side slightly flatter than the outer*).

It is shewn that the fluid carried forward with the ring is ring-shaped only when the section is very small**), e. g. for $R/r = 100$ the mass is without aperture. The energy of the motion is also found and to the lowest order its expression is

$$\frac{\mu^3}{4\pi V} (L - 2) \left(L - \frac{1}{2}\right),$$

where V is the velocity of translation.

The small vibrations of the ring are next investigated. It is shewn that if the section is crimped the alteration in volume is small enough to allow us to use the stream-function. The conditions that the surface of the hollow always contain the same particles and that the pressure there is constant solve the problem. The time of vibration is

$\frac{\mu \varrho}{2\pi \sqrt{n(n+1)}} ***)$, where ϱ is the density, Π the pressure at infinity, and n the number of crests running round the hollow. It is important to notice that this period is independent of the energy of the motion. The pulsations are investigated by means of the velocity-potential, since the stream-function is many-valued. The time of pulsation is $\frac{\mu \varrho}{2\pi} \sqrt{\log \frac{4}{k}}$, which varies slowly with the energy.

The same author has discussed in a more general manner the motion of the single ring†). He assumes a rotational core of density different from that of the surrounding fluid, an inner hollow with circulation round it, and an additional circulation round the outer surface of the core. For the sake of simplicity it is supposed that the vorticity is constant throughout the rotational core. In the steady motion, the conditions to be satisfied are that the inner and outer surfaces of the core always contain the same particles, and that the pressure is constant over the surface of the hollow and is the same on both sides of the outer boundary of the core. It is shewn that these conditions enable us to solve the problem to any desired order of approximation. The steady motion is known when we know the

* c. f. "Vortex Statics" Phil. Mag. (5) X. p. 105 and M. J. M. Hill's paper Phil. Trans. 1884, p. 403.

***) Stated by Sir W. Thomson, "Vortex Atoms" Phil. Mag. (4) XXXIV. p. 21.

Correctly given in the next paper "Researches on the Theory of Vortex-Rings" Part II. Phil. Trans. 1885, p. 767.

†) "Researches on the Theory of Vortex-Rings" Part. II. Phil. Trans. 1885.

position and size of the hollow, the form of the internal and external boundaries of the core, and the velocity of translation. These are all found for the general case, and the particular cases of no core, continuous core with added circulation, ordinary ring, and hollow vortex with no added circulations, receive special discussion. As mentioned before, the velocity of translation for the ordinary ring is that given by Sir W. Thomson. It is remarked that the effect of increased aperture on the velocity of translation is much diminished after a hollow has been formed. The internal and external energy are also found, and it is noticed that with an internal additional circulation the internal energy is comparable with the external, but decreases as the whole energy increases. The remainder of the memoir is devoted to the discussion of the small sectional vibrations. In the general case, when the section is crimped, the periods depend on a certain biquadratic equation; several special cases are considered. When there is no rotational core the motion is always stable, and the period for each mode of deformation is independent of the energy. For a simple continuous core there are two sets of fluted vibrations, and the ring is unstable unless the ratio of the outer circulation to that due to the core is $> \sqrt{\frac{1}{2} \sigma(\sigma + 2)/(\sigma + 1)}$, where σ is the ratio of the density of the core to that of the fluid. When there is no added circulation this requires $\sigma < \sqrt{2}$. When there is a hollow and no internal added circulation the simple ring is still stable; but if the core is denser than the surrounding fluid the ring is only stable when the outer added circulation exceeds a certain critical value depending on the densities and the circulations. When there is an inner additional circulation the periods depend on a biquadratic equation. This case is not discussed, but it is stated that the outer added circulation will still be necessary to maintain the stability of a ring with dense core. The pulsations of the hollow ring with rotational filaments are investigated, and it is shewn that the motion is always stable for pulsations.

The memoirs of Hicks and J. J. Thomson are occupied with an attempt to work out some of the more elementary problems of a vortex-atom theory of matter. A quite different train of thought has been followed out by *M. J. M. Hill*. His earlier researches*) are principally connected with the general equations of fluid motion, and need not detain us long here. The main results are similar to those

*) "On some properties of the equations of Hydrodynamics." Quarterly Journal vol. XVII and "On some general equations which include the equations of Hydrodynamics." Camb. Phil. Trans. 1883.

obtained by Clebsch*) a year after the publication of Helmholtz's memoir. It was shewn that the velocity-components can be put in the forms

$$u = \frac{\partial \chi}{\partial x} + \lambda \frac{\partial \Omega}{\partial x} \text{**}) \text{ etc.,}$$

where λ and Ω are surfaces which move so as always to contain the same vortex-lines, and the pressure is given by

$$\begin{aligned} \frac{p}{\rho} + V = & -\frac{\partial \chi}{\partial t} - \frac{1}{2} \left\{ \left[\frac{\partial \chi}{\partial x} \right]^2 + \left[\frac{\partial \chi}{\partial y} \right]^2 + \left[\frac{\partial \chi}{\partial z} \right]^2 \right\} \\ & + \frac{1}{2} \lambda^2 \left\{ \left[\frac{\partial \Omega}{\partial x} \right]^2 + \left[\frac{\partial \Omega}{\partial y} \right]^2 + \left[\frac{\partial \Omega}{\partial z} \right]^2 \right\}. \end{aligned}$$

The functions χ , λ , Ω satisfy certain partial differential equations, viz: the equation of continuity, and the equations expressing that λ and Ω always contain the same particles. This gives us a general method of treating vortex-motion quite different from Helmholtz's, and more allied to the stream-function method employed with such effect by Hicks. Hill***) has developed this method in the cases where there is a stream-function. He shews that the surface separating the fluid moving rotationally from that moving irrotationally may be taken to be any one of the family $\lambda = \text{const.}$ That λ , the spin (ξ), and the stream-function (ψ) are connected by certain equations, and that λ and ξ satisfy certain equations. The equation for ψ was known before. Hence λ and ξ equally with ψ serve for the complete determination of the motion. The author proposes the following methods for solving problems in vortex-motion, when there is a stream-function.

(a) Suppose that any particular integral of the equation in λ is taken, then we know ξ and ψ at every point within $\lambda = 0$, and thus u, v, w, p are known for all points within $\lambda = 0$. Now find a stream-function ψ outside $\lambda = 0$ satisfying the equation of the stream-function, and make ψ outside continuous with ψ inside $\lambda = 0$, the velocity outside is continuous with that inside. If now the value of p calculated from ψ outside is continuous at the surface with that inside $\lambda = 0$, the integral taken of the equation in λ does correspond to a possible case of fluid motion. If on the other hand it be found theoretically impossible to find a stream-function which shall make the velocity and pressure continuous at the surface, then the integral taken of the equation in λ does not correspond to a possible case of fluid motion. It by no means follows that no fluid motion can take place for which

*) Crelle, Bd. LVI.

**) I have changed the notation, to avoid confusion.

***) "On the motion of fluid part of which is moving rotationally and part irrotationally." Phil. Trans. 1884, and — "On the differential equations of cylindrical and annular vortices." Proc. Lond. Math. Soc. vol XVI

the assumed surface separates the rotational from the irrotational parts of the fluid. *)

(b) When the rotational motion is given a method may be derived from Clebsch's forms: — Calculate the quantity $-\frac{1}{4\pi}\nabla^2\chi = \varrho$ (say). Let the potential of a distribution of matter of density ϱ inside $\lambda = 0$ be χ' inside, and φ' outside. χ' will in general differ from χ , because χ may contain many-valued terms P satisfying $\nabla^2 P = 0$, and also because $\chi - P$ may be the potential of a distribution of matter part of which is outside $\lambda = 0$. Then $\varphi' + P$ is the velocity-potential of a motion outside $\lambda = 0$ continuous with the given rotational motion inside.

These methods are applied to the solution of the following problems, (1) the motion of an elliptic cylindrical vortex parallel to its major axis, (2) the rotation of such a vortex about its axis in an infinite fluid, and (3) when enclosed in a confocal cylinder, (4) the motion of a fluid in which is a circular cylindrical vortex with vorticity a function of the distance from the axis, (5) the rotational motion within a certain annular surface. The result of (2) is the same as that of Kirchhoff's investigation. **) In (1) and (3) the motion as given by Helmholtz's forms is calculated, and shewn to lead to expressions for the velocity which make the velocity and pressure discontinuous in crossing the surface of the vortex, but in both cases the fluid is shewn to be bounded, and we know that in general for motion in a bounded space the Helmholtz forms require correction unless every vortex-element is within a finite distance of the origin. ***) The results are new. The methods proposed would seem to be capable of further extension and application, but no fresh advance has yet been made.

A large class of problems usually considered under vortex-motion is that concerning the motion due to *isolated vortex-filaments*. In two dimensions the velocity- and stream-functions for a vortex are θ and $\log r$, where r is the distance from any point to the vortex, and θ the angle this line makes with a fixed line. When the fluid is bounded internally or externally by a cylindrical surface, we have to find a distribution of vortices which, with the given vortex, will make the surface one across which there is no flow. This distribution is called the "image" of the vortex in the boundary. When the image is known we have only to add the solutions corresponding to the elements of the "object" and the image to obtain the solution of our problem. I shall state the cases solved by English writers, classifying them according to the form of the boundary.

*) See the account of Sir W. Thomson's "Vortex Statics." ante p. 328.

**) "Vorlesungen über Mathematische Physik." Vorlesung XX, § 4, pp. 261 sq.

***) Lamb, p. 154.

(1) Straight lines. Greenhill*) has given the solutions for a single vortex in fluid bounded by one line, by two lines meeting at an angle $\frac{\pi}{n}$, by the sides of an equilateral triangle, by a rectangle. The limiting case of the rectangle when one side moves off to infinity has been discussed by Hicks.**)

(2) Circles. The solutions for a vortex within or without the circle, two vortices equidistant from the centre and equal, or equal and opposite, and within or without the circle, are given by Greenhill in his paper on Plane Vortex-motion.

(3) Straight lines and circles. The solutions for a single vortex in a sector of angle $\frac{\pi}{n}$, or within the curvilinear rectangle bounded by two concentric circles and two radii, are due to Greenhill.*)

(4) Ellipse. The stream-function for a vortex outside or inside an ellipse, and inside a semi-ellipse, has been investigated by Coates,**) the results were deduced from Greenhill's solutions for circles. Some cases are also discussed by Hicks, †) proceeding directly by the method of images.

(5) Other curves. Greenhill ††) has given the solutions for a vortex between two confocal Cartesians, proceeding by a method similar to that of Hicks for ellipses, and Basset †††) has found the solution for a vortex inside or outside a cylinder whose transverse section is the inverse of an ellipse with respect to its centre, by inversion from the results obtained by Hicks and Coates.

In three dimensions the problem has not the same simplicity, as the vortex-filament forms a closed curve. It is however still true that an element of a vortex-filament in an unbounded fluid implies at any point a certain velocity determined by Helmholtz's forms.*†) If the fluid be bounded, the expressions must be modified so as to satisfy the

*) "Plane Vortex-motion", Quarterly Journal vol XV, and "Solution by means of Elliptic Functions of some Problems" etc., Quarterly Journal vol XVII also "Some Applications of Weierstrass's Elliptic Functions", Proc. Lond. Math. Soc. vol XVII.

**) "Velocity and Electric Potentials between Parallel Planes," Quarterly Journal vol XV.

***) "Vortex-motion in and about elliptic cylinders." Quarterly Journal vols XV & XVI.

†) "Functional images in ellipses." Quarterly Journal vol. XVII.

††) "On Functional images in Cartesians." Quarterly Journal vol XVIII, and "Some Applications of Weierstrass's Elliptic Functions." Proc. Lond. Math. Soc. vol XVII.

†††) "On the motion of liquid in and about cylinders whose transverse sections are the inverses of confocal ellipses with respect to their centre." Quarterly Journal XIX.

*†) Lamb. p. 152.

boundary conditions, and it may happen that the image of a vortex-filament in the boundary consists of a vortex or a group of vortices. The image of an element of a vortex-filament will be one or more elements of vortex-filaments, but the strength of the image may be a function of position on the object in which case it will vary from point to point on the image. The only case which has been fully worked out is that of a circular vortex-filament in space bounded by a sphere.*) The image of an element of a filament is an element of a filament at the inverse point, and the strengths of the image and the object are inversely proportional to the square roots of their distances from the centre. It is only when the object lies on a sphere concentric with the boundary that the image is a vortex-filament of constant strength. By means of these results Lewis has investigated the motion of a circular vortex-filament within a sphere when it moves symmetrically with respect to a diameter.

Cambridge, May 1887.

Note added September 1887.

Since the above was written, Sir W. Thomson has published two papers on steady and stable vortex-motion in continuation of those considered above.**) The first of these is devoted (1) to shewing how the energy of the motion of a perfect fluid may be indefinitely increased by doing work in a systematic manner on the containing vessel, and then reducing the vessel to rest; (2) to illustrating the subsidence of the motion within a viscously-elastic containing vessel down to the appropriate type of steady stable motion. The second is a revision of the article published in 1880. Some errors are corrected, and some of the statements receive mathematical proof. The conception of vortex-sponge is illustrated by considering a vortex to whose elements is attributed a mutual attraction insensible at sensible distances. In this case, instead of the indefinitely fine drawing out of the sheets of vortex-filaments, we find the vortex breaking up into innumerable detached columns, which, in the steady motion corresponding to any given moment of momentum, revolve uniformly in a cloud. The density of distribution of the vortex-sponge is always similar to that of the cloud of "spindrift" in the new problem. Finally a very simple method of conducting the operations to be performed in order to transform the stable motion with maximum energy to that with minimum energy is described. This gives a clearer view of the whole problem.

*) Lewis. "On the images of vortices in a spherical vessel." *Quarterly Journal* vol. XVI.

**) *Phil. Mag.* (5) XXIII, pp. 459 and 529.