



LXII. Table of Zonal Spherical Harmonics, calculated by Messrs. C. E. Holland, P. R. Jones, and C. G. Lamb. With a short explanation and some illustrations of its use

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I wish to express my deep obligations to my assistant, Mr. W. C. Sabine, for his valuable suggestions and for his skill in the mechanical details of this investigation.

CONCLUSIONS.

1. The magnetic permeability of iron wires exercises an important influence upon the decay of electrical oscillations of high frequency. This influence is so great that the oscillations may be reduced to a half oscillation on a circuit of suitable self-induction and capacity for producing them.

2. It is probable that the time of oscillation on iron wires may be changed. Since I have been able to obtain only a half oscillation on iron wires, I have not been able to state this law definitely.

3. Currents of high frequency, such as are produced in Leyden-jar discharges, therefore magnetize the iron.

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LXII. *Table of Zonal Spherical Harmonics, calculated by Messrs. C. E. Holland, P. R. Jones, and C. G. Lamb. With a short Explanation and some Illustrations of its use by Professor JOHN PERRY, D.Sc., F.R.S.**

[Plate I.]

I HAD intended merely to present to the Society for publication a table of Zonal Spherical Harmonics. But some Members whom I have consulted seem to think that I ought also to give one or two examples of the practical use of such a table. The Members of the Society will, I hope, pardon my putting before them one or two elementary examples.

The use of Spherical Harmonics in the numerical solution of practical problems is almost unknown, I believe, except at the Finsbury Technical College, where, every year, I have been accustomed to make some Electrical Engineering students work a few common examples. My students have for some years been in possession of tables of zonal harmonics, but this year I have thought it well to make the tables more complete and to get them published for the general use of students of practical physics.

I have been told that many of the users of such a table would be glad of a few statements of the general principles underlying its use. For the proofs of these statements readers

* Communicated by the Physical Society : read November 14, 1890.

are referred to Mathematical treatises. Many readers will be satisfied with the treatment of the subject in Mr. Ferrer's excellent treatise, which is, however, written only for beginners.

In problems on Heat Conduction (V being temperature), on Hydrodynamics of incompressible fluids (V being velocity-potential*), in Electrostatics (V being electric potential), in Magnetism (V being magnetic potential), and in many other applications of Physics, we require to find V a function of x, y, z which shall satisfy the equation

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = 0, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or, as it is usually written, $\nabla^2 V = 0 \dots (1)$, and which shall also satisfy certain other conditions. Now there are many kinds of function which satisfy equation (1). The definition of a Spherical Harmonic is "a homogeneous function of x, y , and z , which satisfies equation (1)."

If such a function can be found, say of the i th degree, and if we divide it by r^{2i+1} where $r^2 = x^2 + y^2 + z^2$, it can be proved that the resulting expression will also satisfy (1), where i may be a positive or negative integer or fraction.

Now if a Spherical Harmonic of degree i (generally called a Solid Spherical Harmonic) be divided by r^i , we get what is called a *Surface Spherical Harmonic* of degree i .

It was shown by Green that if there is a function V which satisfies equation (1) at every point of any given surface, then it is the only function which satisfies (1) throughout space; and there is always a function V obtainable which satisfies (1). It is the characteristic property of a surface spherical harmonic distribution of density of attracting matter on a spherical surface, that it produces a similar and similarly placed spherical harmonic distribution of potential over any concentric spherical surface throughout space, external and internal.

Instead of using x, y , and z coordinates we may of course use r, θ , and ϕ coordinates.

* When there exists a velocity-potential V in a portion of fluid, we mean that the velocity of the fluid at any place resolved in the direction s is

$$-\frac{dV}{ds}.$$

When the motion is "rotational," as in the wheel of a centrifugal pump or turbine, a velocity-potential does not exist. In any portion of a frictionless fluid, if there is irrotationality, that is, if there is a velocity-potential, the property cannot be destroyed.

In a great number of practical cases V is symmetrical about an axis, and a symmetrical spherical harmonic is said to be a Zonal Spherical Harmonic. Taking the axis of symmetry as the axis of z , V is a function of z and $\sqrt{x^2+y^2}$. Or, in polar coordinates, V is a function of r and θ .

Let O be a point in the axis, the origin of coordinates; let the distance of any point P from O be called r , let the angle between OP and the axis be called θ , then in any distribution which has an axis of symmetry we need only to know r and θ . And over any spherical surface whose centre is O , the distribution will be a function of θ . Any zonal surface spherical harmonic is then merely a function of θ , and I give a table of values of these for values of θ differing by 1° from 0° to 90° , up to the harmonic of the seventh degree. These are indicated by P_0, P_1, P_2 , &c. P_7 .

The surface harmonic of no degree is 1, and is indicated by P_0 .

The student is referred to Mathematical treatises for the proof that, if μ be written to represent $\cos \theta$, then

$$P_0 = 1,$$

$$P_1 = \mu,$$

$$P_2 = \frac{3\mu^2 - 1}{2},$$

$$P_3 = \frac{5\mu^3 - 3\mu}{2},$$

$$P_4 = \frac{35\mu^4 - 30\mu^2 + 3}{8},$$

$$P_5 = \frac{63\mu^5 - 70\mu^3 + 15\mu}{8},$$

$$P_6 = \frac{231\mu^6 - 315\mu^4 + 105\mu^2 - 5}{16},$$

$$P_7 = \frac{429\mu^7 - 693\mu^5 + 315\mu^3 - 35\mu}{16}.$$

Any function of θ may be expanded in terms of P_0, P_1, P_2 , &c., that is any symmetrical function V may be expanded in a series of Zonal Spherical Harmonics. Take, for example, the powers of $\cos \theta$, it may be shown that

$$\mu^0 = 1 = P_0,$$

$$\mu = P_1,$$

$$\mu^2 = \frac{2}{3} P_2 + \frac{1}{3} P_0,$$

$$\mu^3 = \frac{2}{5} P_3 + \frac{3}{5} P_1,$$

$$\mu^4 = \frac{8}{35} P_4 + \frac{4}{7} P_2 + \frac{1}{5} P_0,$$

$$\mu^5 = \frac{8}{63} P_5 + \frac{4}{9} P_3 + \frac{3}{7} P_1.$$

And also

$$\text{Cos } \theta = P_1,$$

$$\text{Cos } 2\theta = \frac{4}{3} P_2 - \frac{1}{3} P_0,$$

$$\text{Cos } 3\theta = \frac{8}{5} P_3 - \frac{3}{5} P_1,$$

$$\text{Cos } 4\theta = \frac{64}{35} P_4 - \frac{16}{21} P_2 - \frac{1}{15} P_0,$$

$$\text{Cos } 5\theta = \frac{128}{63} P_5 - \frac{8}{9} P_3 - \frac{1}{7} P_1.$$

There is an easy rule for expanding any function of θ in terms of P_0, P_1, P_2 &c.

The following table up to P_6 was primarily calculated separately by Messrs. C. E. Holland and P. R. Jones, who checked each other's result by comparison.

Mr. C. G. Lamb applied certain checks to the results of Messrs. Holland and Jones, and then calculated P_6 and P_7 .

It is quite easy to extend the table to P_8 , for it can be proved that there is a law connecting three consecutive harmonics, say the $(n-2)$ th, $(n-1)$ th, and the n th.

$$nP_n = (2n-1)\mu P_{n-1} - (n-1)P_{n-2}.$$

θ .	P_1 .	P_2 .	P_3 .	P_4 .	P_5 .	P_6 .	P_7 .
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	.9998	.9995	.9991	.9985	.9977	.9965	.9955
2	.9994	.9982	.9963	.9939	.9909	.9872	.9829
3	.9986	.9959	.9918	.9863	.9795	.9713	.9617
4	.9976	.9927	.9854	.9758	.9638	.9495	.9329
5	.9962	.9886	.9773	.9623	.9437	.9216	.8961
6	.9945	.9836	.9674	.9459	.9194	.8881	.8522
7	.9925	.9777	.9557	.9267	.8911	.8476	.7986
8	.9903	.9709	.9423	.9048	.8589	.8053	.7448
9	.9877	.9633	.9273	.8803	.8232	.7571	.6831
10	.9848	.9548	.9106	.8532	.7840	.7045	.6164
11	.9816	.9454	.8923	.8238	.7417	.6483	.5461
12	.9781	.9352	.8724	.7920	.6966	.5892	.4732
13	.9744	.9241	.8511	.7582	.6489	.5273	.3940
14	.9703	.9122	.8283	.7224	.5990	.4635	.3219
15	.9659	.8995	.8042	.6847	.5471	.3982	.2454
16	.9613	.8860	.7787	.6454	.4937	.3322	.1699
17	.9563	.8718	.7519	.6046	.4391	.2660	.0961
18	.9511	.8568	.7240	.5624	.3836	.2002	.0289
19	.9455	.8410	.6950	.5192	.3276	.1347	-.0443
20	.9397	.8245	.6649	.4750	.2715	.0719	-.1072
21	.9336	.8074	.6338	.4300	.2156	.0107	-.1662
22	.9272	.7895	.6019	.3845	.1602	-.0481	-.2201
23	.9205	.7710	.5692	.3386	.1057	-.1038	-.2681
24	.9135	.7518	.5357	.2926	.0525	-.1559	-.3095
25	.9063	.7321	.5016	.2465	.0009	-.2053	-.3463
26	.8988	.7117	.4670	.2007	-.0489	-.2478	-.3717
27	.8910	.6908	.4319	.1553	-.0964	-.2869	-.3921
28	.8829	.6694	.3964	.1105	-.1415	-.3211	-.4052
29	.8746	.6474	.3607	.0665	-.1839	-.3503	-.4114
30	.8660	.6250	.3248	.0234	-.2233	-.3740	-.4101
31	.8572	.6021	.2887	-.0185	-.2595	-.3924	-.4022
32	.8480	.5788	.2527	-.0591	-.2923	-.4052	-.3876
33	.8387	.5551	.2167	-.0982	-.3216	-.4126	-.3670
34	.8290	.5310	.1809	-.1357	-.3473	-.4148	-.3409
35	.8192	.5065	.1454	-.1714	-.3691	-.4115	-.3096
36	.8090	.4818	.1102	-.2052	-.3871	-.4031	-.2738
37	.7986	.4567	.0755	-.2370	-.4011	-.3898	-.2343
38	.7880	.4314	.0413	-.2666	-.4112	-.3719	-.1918
39	.7771	.4059	.0077	-.2940	-.4174	-.3497	-.1469
40	.7660	.3802	-.0252	-.3190	-.4197	-.3234	-.1003
41	.7547	.3544	-.0574	-.3416	-.4181	-.2938	-.0534
42	.7431	.3284	-.0887	-.3616	-.4128	-.2611	-.0065
43	.7314	.3023	-.1191	-.3791	-.4038	-.2255	.0398
44	.7193	.2762	-.1485	-.3940	-.3914	-.1878	.0846
45	.7071	.2500	-.1768	-.4062	-.3757	-.1485	.1270
46	.6947	.2238	-.2040	-.4158	-.3568	-.1079	.1666
47	.6820	.1977	-.2300	-.4252	-.3350	-.0645	.2054
48	.6691	.1716	-.2547	-.4270	-.3105	-.0251	.2349
49	.6561	.1456	-.2781	-.4286	-.2836	+.0161	.2627
50	.6428	.1198	-.3002	-.4275	-.2545	+.0563	.2854
51	.6293	.0941	-.3209	-.4239	-.2235	+.0954	.3031
52	.6157	.0686	-.3401	-.4178	-.1910	+.1326	.3153
53	.6018	.0433	-.3578	-.4093	-.1571	+.1677	.3221
54	.5878	.0182	-.3740	-.3984	-.1223	+.2002	.3234

Table (continued).

θ .	P_1 .	P_2 .	P_3 .	P_4 .	P_5 .	P_6 .	P_7 .
55°	·5736	—·0065	—·3886	—·3852	—·0868	+·2297	·3191
56	·5592	—·0310	—·4016	—·3698	—·0510	+·2559	·3095
57	·5446	—·0551	—·4131	—·3524	—·0150	+·2787	·2949
58	·5299	—·0788	—·4229	—·3331	·0206	+·2976	·2752
59	·5150	—·1021	—·4310	—·3119	·0557	+·3125	·2511
60	·5000	—·1250	—·4375	—·2891	·0898	+·3232	·2231
61	·4848	—·1474	—·4423	—·2647	·1229	+·3298	·1916
62	·4695	—·1694	—·4455	—·2390	·1545	+·3321	·1571
63	·4540	—·1908	—·4471	—·2121	·1844	+·3302	·1203
64	·4384	—·2117	—·4470	—·1841	·2123	+·3240	·0818
65	·4226	—·2321	—·4452	—·1552	·2381	+·3138	·0422
66	·4067	—·2518	—·4419	—·1256	·2615	+·2996	·0021
67	·3907	—·2710	—·4370	—·0955	·2824	+·2819	—·0375
68	·3746	—·2896	—·4305	—·0650	·3005	+·2605	—·0763
69	·3584	—·3074	—·4225	—·0344	·3158	+·2361	—·1135
70	·3420	—·3245	—·4130	·0038	·3281	+·2089	—·1485
71	·3256	—·3410	—·4021	·0267	·3373	+·1786	—·1811
72	·3090	—·3568	—·3898	·0568	·3434	+·1472	—·2099
73	·2924	—·3718	—·3761	·0864	·3463	+·1144	—·2347
74	·2756	—·3860	—·3611	·1153	·3461	+·0795	—·2559
75	·2588	—·3995	—·3449	·1434	·3427	+·0431	—·2730
76	·2419	—·4112	—·3275	·1705	·3362	+·0076	—·2848
77	·2250	—·4241	—·3090	·1964	·3267	—·0284	—·2919
78	·2079	—·4352	—·2894	·2211	·3143	—·0644	—·2943
79	·1908	—·4454	—·2688	·2443	·2990	—·0989	—·2913
80	·1736	—·4548	—·2474	·2659	·2810	—·1321	—·2835
81	·1564	—·4633	—·2251	·2859	·2606	—·1635	—·2709
82	·1392	—·4709	—·2020	·3040	·2378	—·1926	—·2536
83	·1219	—·4777	—·1783	·3203	·2129	—·2193	—·2321
84	·1045	—·4836	—·1539	·3345	·1861	—·2431	—·2067
85	·0872	—·4886	—·1291	·3468	·1577	—·2638	—·1779
86	·0698	—·4927	—·1038	·3569	·1278	—·2811	—·1460
87	·0523	—·4959	—·0781	·3648	·0969	—·2947	—·1117
88	·0349	—·4982	—·0522	·3704	·0651	—·3045	—·0735
89	·0175	—·4995	—·0262	·3739	·0327	—·3105	—·0381
90	·0000	—·5000	—·0000	·3750	·0000	—·3125	·0000

N.B.—To find P_i , if θ lies between 90° and 180° , look up the value for the angle $180^\circ - \theta$, and change the sign if i be odd.

When P_1 , P_2 , &c. are plotted as radial heights and depths above and below a quadrant, the resulting curves, especially if coloured, are exceedingly interesting. A student can draw them to a useful scale in about one hour.

Example I.

The density σ of attracting matter on a spherical shell 1 centim. in radius is proportional to the square of the

distance of any point from a diametral plane, being 6 per square centimetre where greatest : find the potential A inside and B outside. Taking the diametral plane as the equator and θ as the co-latitude, it is obvious that

$$\sigma = 6\mu^2.$$

The expansion of μ^2 in spherical harmonics is already given as

$$\mu^2 = \frac{1}{3} P_0 + \frac{2}{3} P_2.$$

So that we have σ in spherical harmonics,

$$\sigma = 2P_0 + 4P_2.$$

Hence, as A and B are derivable from the same surface harmonics,

$$A = A_0 P_0 + A_2 r^2 P_2,$$

$$B = \frac{B_0}{r} P_0 + \frac{B_2}{r^3} P_2;$$

where A_0, A_2, B_0, B_2 are constants to be found.

Now at the surface, that is where $r=1$, $A=B$, and we can apply this to every harmonic separately. Hence

$$A_0 = B_0, \quad A_2 = B_2.$$

Again, we know from the theory of attraction that the resultant force just outside and just inside the shell differs by the amount $4\pi\sigma$, or

$$\frac{dA}{dr} - \frac{dB}{dr} = 4\pi\sigma;$$

and this is to be applied to every harmonic separately. Thus, taking terms involving P_0 , we have

$$0 + B_0 r^{-2} = 4\pi \times 2;$$

or putting $r=1$,

$$B_0 = 8\pi.$$

Again, taking the second terms,

$$2A_2 r + 3B_2 r^{-4} = 4\pi \times 4;$$

or, as $A_2 = B_2$, and putting $r=1$,

$$5B_2 = 16\pi, \quad \therefore B_2 = \frac{16}{5}\pi.$$

Hence we have

$$\text{Inside potential } A = 8\pi P_0 + \frac{16}{5} \pi r^2 P_2,$$

$$\text{Outside potential } B = \frac{8\pi}{r} P_0 + \frac{16}{5} \pi \frac{1}{r^3} P_2;$$

$$\text{or } \frac{A}{8\pi} = P_0 + \frac{2}{5} r^2 P_2 = \alpha \text{ say, } \quad . \quad . \quad . \quad (1)$$

$$\frac{B}{8\pi} = \frac{1}{r} P_0 + \frac{2}{5} \frac{1}{r^3} P_2 = \beta \text{ say. } \quad . \quad . \quad . \quad (2)$$

As, for our purpose, the actual unit of potential is unimportant, we will use α for $\frac{A}{8\pi}$, and β for $\frac{B}{8\pi}$.

I had no notion of how the equipotential surfaces would shape, and I tried to avoid forming any such notion, as it was my object to test the usefulness of the tables in working out any new problem. The first thing that it strikes one to do in this case is to find the potential on the sphere itself. This can be done from either (1) or (2) by putting $r=1$, and then

$$\alpha = P_0 + \frac{2}{5} P_2, \quad \text{or } 1 + \frac{2}{5} P_2.$$

We can now find α for various values of θ , using the table. Thus when $\theta=0$, $P_2=1$, and therefore $\alpha=1.4$. Thus we have the following values:—

θ	0	15	30	45	60	75	90
P_2	1	.9	.625	.25	-.125	-.3995	-.5000
α	1.4	1.36	1.25	1.1	0.95	0.84	0.80

Next, take any value of θ , say $\theta=45^\circ$. Then from the table, $P_2=0.2500$, so that

$$\text{inside, } \alpha = 1 + \frac{1}{10} r^2;$$

$$\text{outside, } \beta = \frac{1}{r} + \frac{1}{10} \frac{1}{r^3}, \quad \text{or } \frac{10r^2 + 1}{10r^3}.$$

For any value of r less than 1 we calculate α ; for any value of r greater than 1 we calculate β .

For $\theta = 45^\circ$.

$r \dots$	0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1·0
$\alpha \dots$	1	1·001	1·004	1·009	1·016	1·025	1·036	1·049	1·064	1·081	1·100
$r \dots$	1·2	1·4	1·6	1·8	2·0	2·2	2·6	3·0	3·5	4	5
$\beta \dots$	·8012	·7507	·65	·572	·512	·2

Now on a sheet of squared paper I plotted the values of r and α or β , and so found the values of r for such particular values of α or β as seemed suitable for curve-drawing. In fact I found the values of r for $\theta = 45^\circ$ for various equipotential surfaces.

Repeating this for other values of θ and drawing radial lines on a sheet of paper, it was easy to draw the equipotential surfaces.

The figure (Pl. I.) was obtained in this way by Mr. Joselin. It shows the equipotential surfaces from 0·5 to 1·4. These surfaces are surfaces of revolution. Of course the resultant force anywhere is inversely as the normal distance apart of the equipotential surfaces, and the direction of the force is everywhere normal to the equipotential surfaces.

Example II.

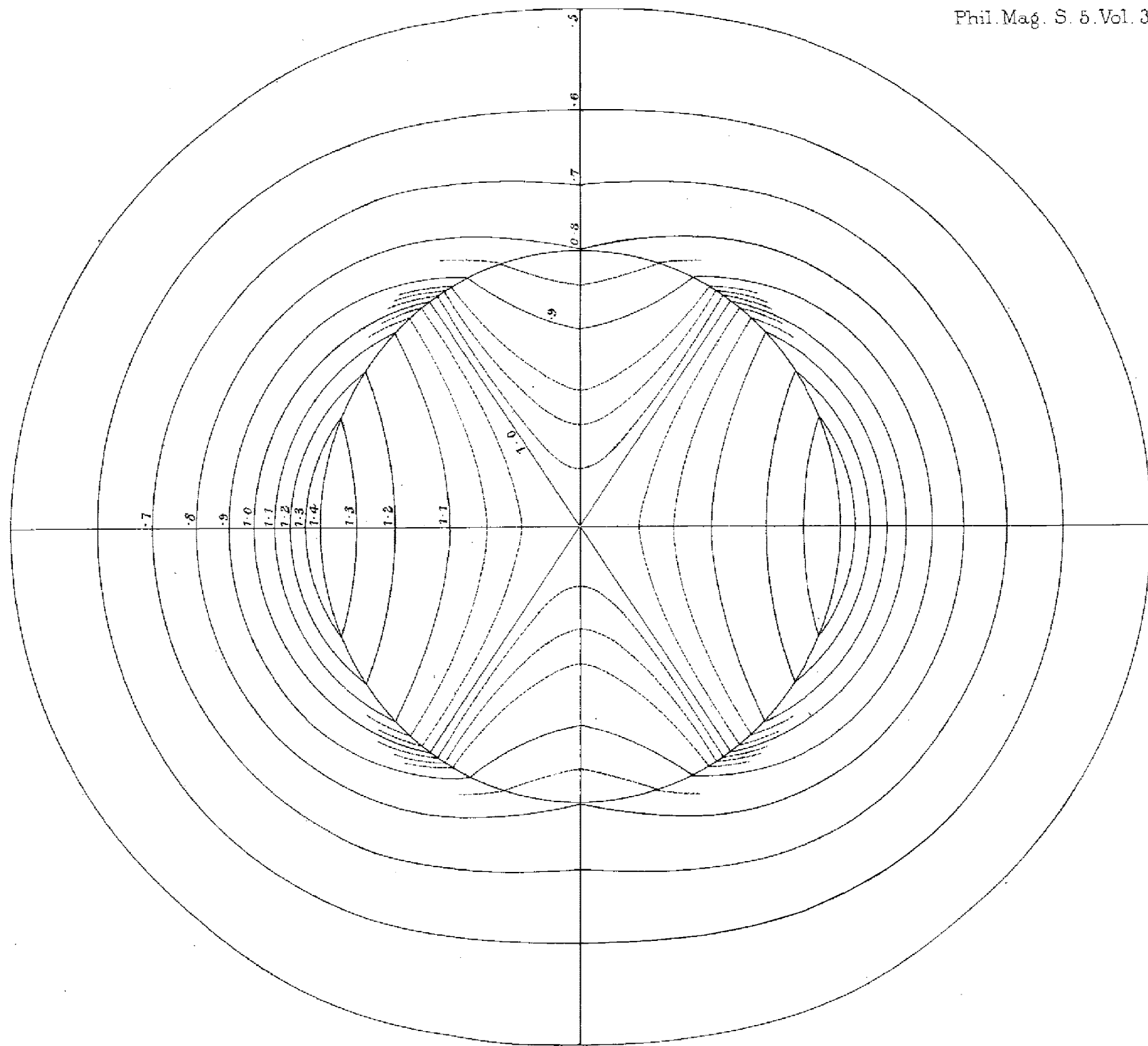
One circular spire of wire of radius a has an electric current C flowing in it. Find the electromagnetic potential everywhere.

At any point on the axis, z centimetres from the centre, the potential is

$$V = 2\pi C \left(1 - \frac{z}{\sqrt{z^2 + a^2}} \right). \quad \dots \quad (1)$$

It is always worth while at first to get an idea of the range of values of V .

$$\begin{aligned} \text{Putting} \quad z=0, \quad V &= 2\pi C. \\ z=a, \quad V &= \cdot 293 \times 2\pi C. \\ z=\infty, \quad V &= 0. \end{aligned}$$



For the sake of ease of calculation let $2\pi C=1$, and let $\alpha=1$ centim.

Then
$$V=1-\frac{z}{\sqrt{z^2+1}}.$$

We can expand this in powers both of z and of $\frac{1}{z}$, and we have either

$$V=1-z+\frac{1}{2}z^3-\frac{3}{8}z^5+\frac{5}{16}z^7-\frac{35}{128}z^9+\&c. \quad (2)$$

or

$$V=\frac{1}{2}\frac{1}{z^2}-\frac{3}{8}\frac{1}{z^4}+\frac{5}{16}\frac{1}{z^6}-\frac{35}{128}\frac{1}{z^8}+\&c. \quad (3)$$

Now, if we can find V as a function of r and θ which is correct along the axial line, then it must be correct everywhere. [This is Green's theorem, assuming that the axial line to infinity is a cylindric surface of no lateral dimensions.] But it is obvious that

$$V=1-rP_1+\frac{1}{2}r^3P_3-\frac{3}{8}r^5P_5+\frac{5}{16}r^7P_7-\frac{35}{128}r^9P_9+\&c. \quad (4)$$

becomes (2) when $\theta=0$; and

$$V=\frac{1}{2}\frac{P_1}{r^2}-\frac{3}{8}\frac{P_3}{r^4}+\frac{5}{16}\frac{P_5}{r^6}-\frac{35}{128}\frac{P_7}{r^8}+\&c. \quad (5)$$

becomes (3) when $\theta=0$.

Hence these express the potential everywhere. The first of these is useful for calculation only when r is less than 1. The second is useful only when r is greater than 1.

And this shows a defect of the Spherical Harmonic method. For if r is nearly 1 we cannot easily calculate V from either of the series, having to use too many terms. It will, however, be found, even in this case, that when we use the harmonics up to P_{11} for any value of θ we can plot V on squared paper from $r=0$ to $r=0.9$, and from $r=1.1$ to values of r as large as we please. If the intermediate part of the curve from $r=0.9$ to $r=1.1$ be drawn with a little judgment, it is astonishing how quickly and accurately the equipotential surfaces may be drawn. The result may be compared with the lines of force as worked out by the elliptic integral method of Sir William Thomson.

Any such Electromagnetic solution is also the solution of a Hydrodynamic problem.

The principle adopted in this example is very useful. It is

this :—If the potential at any point along an axis is expressible as

$A_0 + A_1z + A_2z^2 + \&c.$, where $A_0, A_1, A_2, \&c.$ are constants, then the potential *anywhere* is

$$A_0P_0 + A_1P_1r + A_2P_2r^2 + \&c.$$

If the potential along an axis is

$$\frac{B_0}{z} + \frac{B_1}{z^2} + \frac{B_2}{z^3} + \&c., \text{ where } B_0, B_1, B_2, \&c. \text{ are constants,}$$

the potential anywhere is

$$\frac{B_0P_0}{r} + \frac{B_1P_1}{r^2} + \frac{B_2P_2}{r^3} + \&c.$$

I gave to Mr. Holland, as an example, the case of a hollow cylindric coil of wire, 2 centim. long, 1 centim. in radius inside, and 2 centim. radius outside: to find the magnetic potential everywhere when there are n turns of wire in the coil per unit length of the coil and there is unit current in the wire. Mascart gives for such a case the force at any point of the axis, and it is possible to expand Mascart's expression in powers of z and again in powers of $\frac{1}{z}$; so that it

is easy to get the potential in powers of z and of $\frac{1}{z}$, and therefore, as in the last case, the potential everywhere. Now inside the coil and well outside it only a few terms of the series need calculation; but just at the ends of the coil the calculation is troublesome because many terms are required.

Mr. Holland tried to shorten the work in the following way. He found that the potential at points along the axis could be expressed *approximately* from $z=0$ to $z=2.5$, with a maximum error anywhere of only 2 per cent., by

$$V = 2\pi n^2 (2 - .4884z + .05513z^3 - .00518z^5 + .00022z^7),$$

z being measured from the middle point of the coil.

Hence, he said, the magnetic potential at a point r, θ is

$$V = 2\pi n^2 \{ 2 - .4884 P_1 r + .05513 P_3 r^3 - .00518 P_5 r^5 + .00022 P_7 r^7 \}.$$

From this he plotted equipotential surfaces and lines of force. He found that, inside the coil, and indeed everywhere near the coil except certain critical positions, these were approximately correct. But at the flat ends close to the wire they were absurdly wrong. It is easy now to see the physical meaning of Mr. Holland's approximation, and why we cannot use this

ingenious quick way of working ; but we had to be taught by experience.

Example III.

A solid bounded by a surface of revolution moves axially in an infinite mass of incompressible fluid which has no other motion than this gives to it. Find the motion.

In this case

$$\nabla^2 V = 0 :$$

and $-\frac{dV}{dn}$ at any point at the surface of the solid (dn being an element of the normal) in the fluid is really the normal velocity of the solid itself. Again, V must be constant at any infinite distance.

This may easily be worked out for any surface of revolution. Applying it to a sphere of 1 centim. radius. Let v be the velocity of the centre of the sphere in the direction parallel to the axis of z . Then $\frac{dV}{dr} = v \cos \theta$ at any point at the surface of the sphere.

Expressing V in zonal harmonics, and taking it as 0 at an infinite distance,

$$V = \frac{A_0 P_0}{r} + \frac{A_1 P_1}{r^2} + \frac{A_2 P_2}{r^3} + \&c.$$

$\frac{dV}{dr}$ when $r=1$ is

$$v \cos \theta = -A_0 P_0 - 2A_1 P_1 - 3A_2 P_2 - \&c.$$

But $P_1 = \mu = \cos \theta$, so that the other coefficients vanish, and

$$A = -\frac{v}{2},$$

so that

$$V = -\frac{1}{2} \frac{v}{r^2} P_1. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The equipotential surfaces which a student will draw from (1) are perfectly well known. It is a good exercise to draw them.

If now a velocity $-v$ is impressed upon the whole system, sphere and fluid, we have the case of the sphere at rest and the fluid moving past it. We now merely add the term $-vr \cos \theta$ to (1) and obtain

$$V = -v \left(r + \frac{1}{2r^2} \right) P_1, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which again is easily represented upon paper in equipotential surfaces.