8. The investigation of the determinant which differs from L_n in having *cosines* everywhere instead of *sines* leads to results more or less closely analogous with the foregoing. Thus we have

$$\left\{ \begin{array}{l} \left| \cos \frac{1 \cdot 1 \pi}{n+1}, \cos \frac{2 \cdot 2 \pi}{n+1}, \dots, \cos \frac{n \cdot n \pi}{n+1} \right| \\ = (-1)^{\frac{n}{2}} \left\{ \frac{1}{2} (n+1) \right\}^{\frac{n-1}{2}} \quad (n \text{ even}) \\ = 0 \quad (n \text{ odd}) \end{array} \right\} \dots (F),$$

$$\begin{aligned} & \left| \cos \frac{1 \cdot 1 \pi}{2m+1}, \cos \frac{2 \cdot 2 \pi}{2m+1}, \dots, \cos \frac{2m \cdot 2m \pi}{2m+1} \right| \\ = & (-1)^m 2^m \left| \cos \frac{1 \cdot 1 \pi}{2m+1}, \cos \frac{2 \cdot 3 \pi}{2m+1}, \dots, \cos \frac{m(2m-1) \pi}{2m+1} \right|^2 \dots (G), \end{aligned}$$

$$\begin{aligned} & \left| \cos \frac{1 \cdot 1 \pi}{2m+1}, \cos \frac{2 \cdot 3 \pi}{2m+1}, \dots, \cos \frac{m(2m-1) \pi}{2m+1} \right| \\ & = \frac{1}{2} \left\{ \frac{1}{2} (2m+1) \right\}^{\frac{m-1}{2}} \dots (H), \end{aligned}$$

$$\begin{aligned} \text{and} \quad & \left| \cos \frac{1 \cdot 1 \pi}{2m+1}, \cos \frac{2 \cdot 3 \pi}{2m+1}, \dots, \cos \frac{m(2m-1) \pi}{2m+1} \right| \\ & = (-1)^m \left| \cos \frac{1 \cdot 1 \cdot 2 \pi}{2m+1}, \cos \frac{2 \cdot 2 \cdot 2 \pi}{2m+1}, \dots, \cos \frac{m \cdot m \cdot 2 \pi}{2m+1} \right| \dots (I). \end{aligned}$$

Note on the Condensation of Skew Determinants, which are partially zero-axial. By THOMAS MUIR, M.A., F.R.S.E.

[Read April 6th, 1882.]

1. Taking, in the first place, a skew determinant having a zero only in the last place of the principal diagonal, e.g.,

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ -a_2 & a_2 & b_2 & b_4 & b_5 & b_6 \\ -a_3 & -b_3 & a_3 & c_4 & c_5 & c_6 \\ -a_4 & -b_4 & -c_4 & a_4 & d_5 & d_6 \\ -a_5 & -b_5 & -c_5 & -d_5 & a_5 & e_6 \\ -a_6 & -b_6 & -c_6 & -d_6 & -e_6 & 0 \end{vmatrix}, \text{ or } D_1,$$

we see that it is clearly equal to

$$\frac{1}{a_1^4} \begin{vmatrix} x_1 & a_2 a_3 & a_3 a_3 & a_4 a_3 & a_5 a_3 & a_6 \\ -a_2 & x_2 a_3 & b_3 a_3 & b_4 a_3 & b_5 a_3 & b_6 \\ -a_3 & -b_3 a_3 & x_3 a_3 & c_4 a_3 & c_5 a_3 & c_6 \\ -a_4 & -b_4 a_3 & -c_4 a_3 & x_4 a_3 & d_5 a_3 & d_6 \\ -a_5 & -b_5 a_3 & -c_5 a_3 & -d_5 a_3 & x_5 a_3 & e_6 \\ -a_6 & -b_6 a_3 & -c_6 a_3 & -d_6 a_3 & -e_6 a_3 & 0 \end{vmatrix}.$$

Multiplying the latter by x_1 in the form

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_1 \\ -b_6 & 1 & 0 & 0 & 0 & -a_3 \\ -c_6 & 0 & 1 & 0 & 0 & -a_4 \\ -d_6 & 0 & 0 & 1 & 0 & -a_5 \\ -e_6 & 0 & 0 & 0 & 1 & -a_6 \end{vmatrix},$$

we obtain

$$D_1 = \frac{1}{x_1 a_1^4} \begin{vmatrix} x_1 & a_2 x_1 & -b_6 x_1 & -c_6 x_1 & -d_6 x_1 & -e_6 x_1 \\ -a_2 & b_6 x_1 & a_3 x_2 & || a_3 b_3 c_6 || & || a_3 b_4 d_6 || & || a_3 b_5 e_6 || \\ -a_3 & c_6 x_1 & -|| a_3 b_3 c_6 || & a_6 x_3 & || a_3 c_4 d_6 || & || a_3 c_5 e_6 || \\ -a_4 & d_6 x_1 & -|| a_3 b_4 d_6 || & -|| a_3 c_4 d_6 || & a_6 x_4 & || a_4 d_5 e_6 || \\ -a_5 & e_6 x_1 & -|| a_3 b_5 e_6 || & -|| a_3 c_5 e_6 || & -|| a_4 d_5 e_6 || & a_6 x_5 \\ -a_6 & 0 & 0 & 0 & 0 & 0 \end{vmatrix},$$

where the Pfaffian notation employed is explained by the example

$$|| a_3 \ b_3 \ c_6 || \equiv \begin{vmatrix} a_3 & a_3 & a_6 \\ & b_3 & b_6 \\ & & c_6 \end{vmatrix} = a_3 c_6 - a_3 b_6 + a_6 b_3.$$

We thus have the result

$$x_1 a_1^4 D_1 = \begin{vmatrix} a_6 x_1 & b_6 x_1 & c_6 x_1 & d_6 x_1 & e_6 x_1 \\ -b_6 x_1 & a_6 x_2 & || a_3 b_3 c_6 || & || a_3 b_4 d_6 || & || a_3 b_5 e_6 || \\ -c_6 x_1 & -|| a_3 b_3 c_6 || & a_6 x_3 & || a_3 c_4 d_6 || & || a_3 c_5 e_6 || \\ -d_6 x_1 & -|| a_3 b_4 d_6 || & -|| a_3 c_4 d_6 || & a_6 x_4 & || a_4 d_5 e_6 || \\ -e_6 x_1 & -|| a_3 b_5 e_6 || & -|| a_3 c_5 e_6 || & -|| a_4 d_5 e_6 || & a_6 x_5 \end{vmatrix} \dots (I.),$$

where it has to be noted that the determinant on the right is still skew with respect to the principal diagonal.

2. Secondly, let us take a skew determinant having the last two elements of the principal diagonal each equal to zero, *e.g.*, the deter-

minant, D , say, got from D_1 by making $x_5 = 0$. From (I) we have at the outset

$$x_1 a_6^2 D_3 = \begin{vmatrix} a_6 x_1 & b_6 x_1 & c_6 x_1 & d_6 x_1 & e_6 x_1 \\ -b_6 x_1 & a_6 x_3 & || a_3 b_5 c_6 || & || a_3 b_4 d_6 || & || a_3 b_5 e_6 || \\ -c_6 x_1 & -|| a_3 b_5 c_6 || & a_6 x_3 & || a_3 c_4 d_6 || & || a_3 c_5 e_6 || \\ -d_6 x_1 & -|| a_3 b_4 d_6 || & -|| a_3 c_4 d_6 || & a_6 x_4 & || a_4 d_5 e_6 || \\ -e_6 x_1 & -|| a_3 b_5 e_6 || & -|| a_3 c_5 e_6 || & -|| a_4 d_5 e_6 || & 0 \end{vmatrix}$$

But the determinant on the right is exactly of the form dealt with in § 1, and thus we know that it is transformable into

$$\frac{1}{a_6 x_1 (e_6 x_1)^3} \begin{vmatrix} a_6 e_6 x_1^2 & a_6 x_1 || a_3 b_5 e_6 || & a_6 x_1 || a_3 c_5 e_6 || & a_6 x_1 || a_4 d_5 e_6 || \\ -a_6 x_1 || a_3 b_5 e_6 || & a_6 e_6 x_1 x_3 & P & Q \\ -a_6 x_1 || a_3 c_5 e_6 || & -P & a_6 e_6 x_1 x_4 & R \\ -a_6 x_1 || a_4 d_5 e_6 || & -Q & -R & a_6 e_6 x_1 x_4 \end{vmatrix} \dots\dots\dots(\omega),$$

where

$$P = \begin{vmatrix} b_6 x_1 & c_6 x_1 & e_6 x_1 \\ || a_3 b_5 e_6 || & || a_3 b_5 e_6 || & || a_3 b_5 e_6 || \\ || a_3 c_5 e_6 || & & \end{vmatrix}, \quad Q = \begin{vmatrix} b_6 x_1 & d_6 x_1 & e_6 x_1 \\ || a_3 b_4 d_6 || & || a_3 b_5 e_6 || & || a_4 d_5 e_6 || \end{vmatrix},$$

and

$$R = \begin{vmatrix} c_6 x_1 & d_6 x_1 & e_6 x_1 \\ || a_3 c_4 d_6 || & || a_3 c_5 e_6 || & || a_4 d_5 e_6 || \end{vmatrix}.$$

Now it is readily shown that these compound Pfaffians are equal to

$$x_1 a_6 || b_5 c_6 e_6 ||, \quad x_1 a_6 || b_4 d_5 e_6 ||, \quad x_1 a_6 || c_4 d_5 e_6 ||,$$

respectively; hence (ω) becomes

$$\frac{a_6^3}{e_6^2 x_1} \begin{vmatrix} e_6 & || a_3 b_5 e_6 || & || a_3 c_5 e_6 || & || a_4 d_5 e_6 || \\ -|| a_3 b_5 e_6 || & e_6 x_1 x_3 & x_1 || b_5 c_6 e_6 || & x_1 || b_4 d_5 e_6 || \\ -|| a_3 c_5 e_6 || & -x_1 || b_5 c_6 e_6 || & e_6 x_1 x_4 & x_1 || c_4 d_5 e_6 || \\ -|| a_4 d_5 e_6 || & -x_1 || b_4 d_5 e_6 || & -x_1 || c_4 d_5 e_6 || & e_6 x_1 x_4 \end{vmatrix};$$

and thus we almost immediately obtain the result

$$e_6^2 D_3 = \begin{vmatrix} e_6 x_1 & || a_3 b_5 e_6 || & || a_3 c_5 e_6 || & || a_4 d_5 e_6 || \\ -|| a_3 b_5 e_6 || & e_6 x_3 & || b_5 c_6 e_6 || & || b_4 d_5 e_6 || \\ -|| a_3 c_5 e_6 || & -|| b_5 c_6 e_6 || & e_6 x_4 & || c_4 d_5 e_6 || \\ -|| a_4 d_5 e_6 || & -|| b_4 d_5 e_6 || & -|| c_4 d_5 e_6 || & e_6 x_4 \end{vmatrix} \dots\dots(II).$$

3. Putting $x_1 = x_3 = x_4 = 0$ in the foregoing, and extracting the

square root, we have

$$e_6 \begin{vmatrix} a_2 & a_3 & a_4 & a_5 & a_6 \\ & b_3 & b_4 & b_5 & b_6 \\ & & c_4 & c_5 & c_6 \\ & & & d_5 & d_6 \\ & & & & e_6 \end{vmatrix} = \begin{vmatrix} \| a_2 b_3 e_6 \| & \| a_3 c_5 e_6 \| & \| a_4 d_5 e_6 \| \\ & \| b_3 c_5 e_6 \| & \| b_4 d_5 e_6 \| \\ & & \| c_4 d_5 e_6 \| \end{vmatrix}$$

an instance of a theorem in Pfaffians corresponding to the well-known theorem in determinants which gives the relation between a determinant and a minor of the adjugate determinant.

4. Proceeding now to the third case, let us denote by D_3 the determinant obtained from D_1 by making $x_1 = x_5 = 0$. To begin with, we have from (II.), if we write $F_1, F_2, \dots, F_6$ for the Pfaffians there occurring,

$$e_6^2 D_3 = \begin{vmatrix} e_6 x_1 & F_1 & F_2 & F_3 \\ -F_1 & e_6 x_2 & F_4 & F_5 \\ -F_2 & -F_4 & e_6 x_3 & F_6 \\ -F_3 & -F_5 & -F_6 & 0 \end{vmatrix}.$$

But this determinant, again, is of the form considered in § 1; therefore, from (I.), we have

$$e_6^2 D_3 = \begin{vmatrix} e_6 x_1 F_3 & e_6 x_1 F_5 & e_6 x_1 F_6 \\ -e_6 x_1 F_3 & e_6 x_2 F_5 & \Phi \\ -e_6 x_1 F_5 & -\Phi & e_6 x_3 F_6 \end{vmatrix} \div e_6 x_1 F_3 \dots \dots \dots (\xi),$$

where

$$\Phi = \begin{vmatrix} F_1 & F_2 & F_3 \\ & F_4 & F_5 \\ & & F_6 \end{vmatrix}.$$

Now the compound Pfaffian just written has been shown (§ 3) to be equal to

$$e_6 \| a_2 b_3 c_4 d_5 e_6 \|.$$

Making, therefore, this substitution in (ξ), and simplifying, we obtain

$$x_1 \| a_4 d_5 e_6 \| D_3 = \begin{vmatrix} x_1 \| a_4 d_5 e_6 \| & x_1 \| b_4 d_5 e_6 \| & x_1 \| c_4 d_5 e_6 \| \\ -x_1 \| b_4 d_5 e_6 \| & x_3 \| a_4 d_5 e_6 \| & \| a_2 b_3 c_4 d_5 e_6 \| \\ -x_1 \| c_4 d_5 e_6 \| & -\| a_2 b_3 c_4 d_5 e_6 \| & x_5 \| a_4 d_5 e_6 \| \end{vmatrix} \dots \dots \dots (\text{III.})$$

5. In a similar manner and with a similar notation, it is found that

$$D_4 = \begin{vmatrix} x_1 \| c_4 d_5 e_6 \| & \| a_2 b_3 c_4 d_5 e_6 \| \\ -\| a_2 b_3 c_4 d_5 e_6 \| & x_3 \| c_4 d_5 e_6 \| \end{vmatrix} \dots \dots \dots (\text{IV.}),$$

a result readily verifiable otherwise. When we come to D_5 and D_6 , we find ourselves upon ground already explored.