we have

$$\mu x' = a - 2b \cos B + 4c \cos B \cos C - 8d \cos B \cos C \cos D.$$

The angles of the two quadrilaterals $EF\dot{G}H$, $E'F'G'H'$ are easily found to be

$$2B - A, \quad 2C - B, \quad 2D - C, \quad 2A - D;$$

$$2A - B, \quad 2B - C, \quad 2C - D, \quad 2D - A.$$

I have not succeeded in getting simple expressions for the intercepts on the sides, or a simple construction for the in-isoscelian quadrilaterals of A, B, C, D .

April, 1888.]

The Free and Forced Vibrations of an Elastic Spherical Shell containing a given Mass of Liquid. By A. E. H. LOVE, B.A.

[Read Feb. 9th, 1888.]

This paper contains an application of the methods of Professor Lamb's papers on the "Vibrations of Elastic Spheres and Spherical Shells" (*Proceedings*, Vols. XIII. and XIV.), to the discussion of the forced vibrations of an elastic spherical shell containing a given mass of liquid and rotating slowly about a diameter, the whole being subject to gravitation and to the action of external disturbing bodies supposed periodic in respect of time. I have prefixed an account of the free vibrations of the shell with and without the liquid nucleus. A summary of aims, methods, and results will be found at the end of the paper.

1. The equations of vibration of an elastic solid mass are

$$\left. \begin{aligned} m \frac{\partial \theta}{\partial x} + n \nabla^2 u &= \rho \frac{\partial^2 u}{\partial t^2} \\ m \frac{\partial \theta}{\partial y} + n \nabla^2 v &= \rho \frac{\partial^2 v}{\partial t^2} \\ m \frac{\partial \theta}{\partial z} + n \nabla^2 w &= \rho \frac{\partial^2 w}{\partial t^2} \end{aligned} \right\} \dots\dots\dots (1),$$

u, v, w being the displacements parallel to the coordinate axes, and θ the cubical dilatation

$$\partial u / \partial x + \partial v / \partial y + \partial w / \partial z.$$

Differentiating these with respect to x, y, z and adding, we obtain in the usual way the equation

$$(m+n) \nabla^2 \theta = \rho \frac{\partial^2 \theta}{\partial t^2} \dots \dots \dots (2).$$

Now, suppose the solid is performing vibrations in period $2\pi/p$, we have to suppose u, v, w proportional to e^{pt} , and the equations become, writing $(m+n)h^2 = p^2\rho = nk^2$,

$$(\nabla^2 + k^2) \theta = 0 \dots \dots \dots (3),$$

$$\left. \begin{aligned} (\nabla^2 + k^2) u &= \left(1 - \frac{k^2}{h^2}\right) \frac{\partial \theta}{\partial x}, & (\nabla^2 + k^2) v &= \left(1 - \frac{k^2}{h^2}\right) \frac{\partial \theta}{\partial y} \\ (\nabla^2 + k^2) w &= \left(1 - \frac{k^2}{h^2}\right) \frac{\partial \theta}{\partial z} \end{aligned} \right\} \dots \dots \dots (4).$$

The last three equations (4) are satisfied by putting

$$u = -\frac{1}{h^2} \frac{\partial \theta}{\partial x}, \quad v = -\frac{1}{h^2} \frac{\partial \theta}{\partial y}, \quad w = -\frac{1}{h^2} \frac{\partial \theta}{\partial z} \dots \dots \dots (5),$$

where θ satisfies $(\nabla^2 + k^2) \theta = 0$.

Hence the complete solution of the equations of vibration (1) will consist of, first, the general solution of (3), secondly, the general solution of the equations

$$\left. \begin{aligned} (\nabla^2 + k^2) u &= 0, & (\nabla^2 + k^2) v &= 0, & (\nabla^2 + k^2) w &= 0 \\ \partial u / \partial x + \partial v / \partial y + \partial w / \partial z &= 0 \end{aligned} \right\} \dots \dots \dots (6),$$

and thirdly, the particular solutions (5).

2. In what follows, we suppose the solid to be bounded by two concentric spheres. In the first place we have to find the solution of

$$(\nabla^2 + k^2) \theta = 0$$

within such a space. Assume $\theta = R_i S_i$, where S_i is a spherical surface harmonic of order i , and R_i a function of r the distance of a point from the centre of the spheres; the equation for R_i is

$$\frac{d^2}{d(hr)^2} (rR_i) + (rR_i) - \frac{i \cdot i + 1}{(hr)^2} (rR_i) = 0 \dots \dots \dots (7).$$

This is the case of Riccati's equation, which is integrable in finite terms, and the solution is

$$rR_i = (hr)^{i+1} \left(\frac{1}{hr} \frac{d}{d(hr)} \right)^{i+1} (A' e^{hr} + B' e^{-hr}),$$

where A', B' are arbitrary complex constants.

This may be written

$$R_i = r^i \left(\frac{1}{r} \frac{d}{dr} \right)^i \left(A \frac{\sin hr}{hr} + B \frac{\cos hr}{hr} \right) \dots\dots\dots (8).$$

And hence, ω_i , Ω_i denoting spherical solid harmonics of positive degree i , we may take

$$\theta = \sum_{i=0}^{i=\infty} \{ \omega_i \psi_i(hr) + \Omega_i \Psi_i(hr) \} \dots\dots\dots (9),$$

where $r^i \Psi_i(hr)$ and $r^i \psi_i(hr)$ are the two particular integrals of the equation (7) for R_i , of which $\psi_i(hr)$ is the one which does not become infinite for $r = 0$.

The first term of (9) is the solution used by Prof. Lamb in his discussion of the vibrations of an elastic sphere.

3. *Properties of the functions ψ and Ψ .*

The differential equation satisfied by ψ_i and Ψ_i is found from (7) to be

$$\left(\frac{d^2}{dr^2} + \frac{2(i+1)}{r} \frac{d}{dr} + h^2 \right) \frac{\psi_i}{\Psi_i}(hr) = 0,$$

and, observing that $\left(\frac{1}{r} \frac{d}{dr} \right)^2 = \frac{1}{r^2} \frac{d^2}{dr^2} - \frac{1}{r^3} \frac{d}{dr}$, this is

$$\left[r^2 \left(\frac{1}{r} \frac{d}{dr} \right)^2 + \frac{2i+3}{r} \frac{d}{dr} + h^2 \right] \frac{\psi_i}{\Psi_i}(hr) = 0 \dots\dots\dots (10).$$

But, since $\psi_i(hr) = \left(\frac{1}{r} \frac{d}{dr} \right)^i \left(\frac{\sin hr}{hr} \right)$,

$$\Psi_i(hr) = \left(\frac{1}{r} \frac{d}{dr} \right)^i \left(\frac{\cos hr}{hr} \right),$$

it follows that $\frac{d}{dr} \left\{ \frac{\psi_i}{\Psi_i}(hr) \right\} = r \left\{ \frac{\psi_{i+1}}{\Psi_{i+1}}(hr) \right\} \dots\dots\dots (11),$

so that

$$r^2 \left\{ \frac{\psi_{i+2}}{\Psi_{i+2}}(hr) \right\} + (2i+3) \left\{ \frac{\psi_{i+1}}{\Psi_{i+1}}(hr) \right\} + h^2 \left\{ \frac{\psi_i}{\Psi_i}(hr) \right\} = 0;$$

hence the ψ and Ψ satisfy the same difference equation

$$r^2 \psi_{i+1}(hr) + (2i+1) \psi_i(hr) + h^2 \psi_{i-1}(hr) = 0.$$

Combining this with (11), we find, for ψ and Ψ ,

$$\frac{d\psi_i}{dr} = -\frac{1}{r} \{ h^2 \psi_{i-1} + (2i+1) \psi_i \}.$$

It is convenient now to change the notation, so that

$$\left. \begin{aligned} \psi_i(hr) &= (-)^i (1.3 \dots 2i+1) \left\{ \frac{1}{hr} \frac{d}{d(hr)} \right\}^i \left(\frac{\sin hr}{hr} \right) \\ \text{and } \Psi_i(hr) &= (-)^i (1.3 \dots 2i+1) \left\{ \frac{1}{hr} \frac{d}{d(hr)} \right\}^i \left(\frac{\cos hr}{hr} \right) \end{aligned} \right\} \dots\dots(12).$$

Then the above relations are replaced by

$$\left. \begin{aligned} r \frac{d}{dr} \psi_{i-1}(hr) &= -\frac{h^2 r^3}{2i+1} \psi_i(hr) = (2i-1) \{ \psi_{i-2}(hr) - \psi_{i-1}(hr) \} \\ \text{and } \psi_i(hr) - \psi_{i-1}(hr) &= \frac{h^2 r^3}{(2i+1)(2i+3)} \psi_{i+1}(hr) \end{aligned} \right\} \dots\dots\dots(13),$$

with exactly similar relations between the functions Ψ .

This change of notation does not affect the form of θ , which is still given by (9).

4. We have now to consider the solution of equations (6). Professor Lamb has shown that within a sphere the general solution is expressed by

$$\begin{aligned} u = \Sigma \left[\psi_i(kr) \left(\frac{\partial \phi_{i+1}}{\partial x} + y \frac{\partial \chi_i}{\partial z} - z \frac{\partial \chi_i}{\partial y} \right) \right. \\ \left. - \frac{i+1}{i+2} \frac{h^2 r^{2i+5}}{(2i+3)(2i+5)} \psi_{i+2}(kr) \frac{\partial}{\partial x} \left(\frac{\phi_{i+1}}{r^{2i+3}} \right) \right] \dots(14), \end{aligned}$$

with similar expressions for v and w , where ϕ_{i+1} , χ_i are spherical solid harmonics whose order is indicated by the suffixes.

The solutions of the equations

$$(\nabla^2 + k^2) u = 0, \quad (\nabla^2 + k^2) v = 0, \quad (\nabla^2 + k^2) w = 0$$

within a sphere are functions of the form

$$\Sigma [\psi_i(kr) V_i],$$

where V_i denotes a spherical solid harmonic, and the equation

$$\partial u / \partial x + \partial v / \partial y + \partial w / \partial z = 0$$

reduces the number of independent harmonics from three to two. The reasoning employed by Professor Lamb proves that, if u, v, w are three functions of the above form, where ψ_i satisfies the differential equation

$$\frac{d^2}{d(kr)^2} (r^{i+1} \psi_i) + \left[1 - \frac{i(i+1)}{(kr)^2} \right] (r^{i+1} \psi_i) = 0,$$

and if $\partial u / \partial x + \partial v / \partial y + \partial w / \partial z = 0$,

then the most general form of solution for u, v, w is that given by (14). We conclude that the most general values of u, v, w which are of the forms

$$\Sigma [\psi_i(kr) \omega_i + \Psi_i(kr) \Omega_i],$$

where ψ_i, Ψ_i are the functions defined by equations (12) with k in place of h , and which also satisfy the relation

$$\partial u / \partial x + \partial v / \partial y + \partial w / \partial z = 0,$$

are three expressions of which the type is

$$\begin{aligned} u = & \Sigma \left[\psi_i(kr) \left(\frac{\partial \phi_{i+1}}{\partial x} + y \frac{\partial \chi_i}{\partial z} - z \frac{\partial \chi_i}{\partial y} \right) \right. \\ & \left. - \frac{i+1}{i+2} \frac{k^3 r^{2i+5}}{(2i+3)(2i+5)} \psi_{i+2}(kr) \frac{\partial}{\partial x} \left(\frac{\phi_{i+1}}{r^{2i+3}} \right) \right] \\ & + \Sigma \left[\Psi_i(kr) \left(\frac{\partial \Phi_{i+1}}{\partial x} + y \frac{\partial \chi_i}{\partial z} - z \frac{\partial \chi_i}{\partial y} \right) \right. \\ & \left. - \frac{i+1}{i+2} \frac{k^3 r^{2i+5}}{(2i+3)(2i+5)} \Psi_{i+2}(kr) \frac{\partial}{\partial x} \left(\frac{\Phi_{i+1}}{r^{2i+3}} \right) \right] \dots \dots (15), \end{aligned}$$

where the ϕ, Φ, χ, X are spherical solid harmonics, and the values of v and w are to be obtained from that of u by cyclical interchange of the letters x, y, z .

Adding to (15) the particular solutions (5) and rewriting, we have

$$\begin{aligned} u = & - \frac{1}{h^3} \frac{\partial}{\partial x} [\Sigma \{ \omega_i \psi_i(hr) \}] \\ & + \Sigma \left[\psi_i(kr) \left(y \frac{\partial \chi_i}{\partial z} - z \frac{\partial \chi_i}{\partial y} \right) + \psi_{i-1}(kr) \frac{\partial \phi_i}{\partial x} \right. \\ & \left. - \frac{i}{i+1} k^3 r^3 \psi_{i+1}(kr) \frac{r^{2i+1}}{(2i+1)(2i+3)} \frac{\partial}{\partial x} \left(\frac{\phi_i}{r^{2i+1}} \right) \right] \\ & - \frac{1}{h^3} \frac{\partial}{\partial x} [\Sigma \{ \Omega_i \psi_i(hr) \}] \\ & + \Sigma \left[\Psi_i(kr) \left(y \frac{\partial \chi_i}{\partial z} - z \frac{\partial \chi_i}{\partial y} \right) + \Psi_{i-1}(kr) \frac{\partial \Phi_i}{\partial x} \right. \\ & \left. - \frac{i}{i+1} k^3 r^3 \Psi_{i+1}(kr) \frac{r^{2i+1}}{(2i+1)(2i+3)} \frac{\partial}{\partial x} \left(\frac{\Phi_i}{r^{2i+1}} \right) \right] \dots (16), \end{aligned}$$

with expressions for v and w to be obtained from this by cyclical interchange of the letters x, y, z .

Equation (16) is in a form suitable for the calculation of the surface-tractions.

5. If F, G, H be the components of the surface-traction at any sphere $r = \text{const.}$, we know that for a solid sphere Fr takes the form

$$\Sigma \left[p_i \left(y \frac{\partial \chi_i}{\partial z} - z \frac{\partial \chi_i}{\partial y} \right) + a_i \frac{\partial \omega_i}{\partial x} + b_i \frac{\partial}{\partial x} \left(\frac{\omega_i}{r^{2i+1}} \right) + c_i \frac{\partial \phi_i}{\partial x} + d_i \frac{\partial}{\partial x} \left(\frac{\phi_i}{r^{2i+1}} \right) \right],$$

where p_i, a_i, b_i, c_i, d_i are functions of r , and Gr, Hr are to be obtained by cyclical interchange of the letters x, y, z . No reductions are made in obtaining this except such as arise from the χ, ϕ, ω being solid harmonics, and the ψ satisfying equations (10), (13), with, if necessary, k in the place of h . It follows that in the case of the shell

$$\begin{aligned} Fr = & \Sigma \left[p_i \left(y \frac{\partial \chi_i}{\partial z} - z \frac{\partial \chi_i}{\partial y} \right) + a_i \frac{\partial \omega_i}{\partial x} + b_i \frac{\partial}{\partial x} \left(\frac{\omega_i}{r^{2i+1}} \right) + c_i \frac{\partial \phi_i}{\partial x} + d_i \frac{\partial}{\partial x} \left(\frac{\phi_i}{r^{2i+1}} \right) \right] \\ & + \Sigma \left[P_i \left(y \frac{\partial \chi_i}{\partial z} - z \frac{\partial \chi_i}{\partial y} \right) + A_i \frac{\partial \Omega_i}{\partial x} + B_i \frac{\partial}{\partial x} \left(\frac{\Omega_i}{r^{2i+1}} \right) + C_i \frac{\partial \Phi_i}{\partial x} + D_i \frac{\partial}{\partial x} \left(\frac{\Phi_i}{r^{2i+1}} \right) \right] \\ & \dots\dots\dots(17), \end{aligned}$$

where the P, A, B, C, D are to be obtained from the p, a, b, c, d by writing Ψ s everywhere in place of ψ s.

6. The values of p, a, b, c, d , as given by Professor Lamb, are equivalent to

$$\left. \begin{aligned} p_i &= n \left[(i-1) \psi_i(kr) + kr \psi'_i(kr) \right] \\ a_i &= -\frac{n}{h^3} \left[2(i-1) \psi_{i-1}(hr) - \frac{k^3 r^2}{2i+1} \psi_i(hr) \right] \\ b_i &= -\frac{nk^3}{h^3} \frac{r^{2i+3}}{2i+1} \left[\psi_i(hr) + \frac{2(i+2)}{k^3 r^3} hr \psi'_i(hr) \right] \\ c_i &= n \left[2(i-1) \psi_{i-1}(kr) - \frac{k^3 r^2}{2i+1} \psi_i(kr) \right] \\ d_i &= -\frac{i}{i+1} n \frac{k^3 r^{2i+3}}{2i+1} \left[\psi_i(kr) + \frac{2(i+2)}{k^3 r^3} kr \psi'_i(kr) \right] \end{aligned} \right\} \dots\dots(18),$$

where $\psi'_i(kr) = \frac{d}{d(hr)} \psi_i(hr), \quad \psi'_i(kr) = \frac{d}{d(kr)} \psi_i(kr).$

7. We consider two cases, (1) when the shell is perfectly free, and (2) when it contains a given mass of liquid.

Let the radii of the external and internal surfaces be a and b , and suppose both surfaces free from stress, then F, G, H vanish when

$r = a$ and when $r = b$. In future we shall denote by $p_i, a_i \dots$ the values taken by the quantities given in (18) when $r = a$, and by $p'_i, a'_i \dots$ their values when $r = b$, and similarly for $P_i, A_i \dots P'_i \dots$. The surface conditions are six, of the form

$$\Sigma \left\{ \begin{aligned} & p_i \left(y \frac{\partial X_i}{\partial z} - z \frac{\partial X_i}{\partial y} \right) + a_i \frac{\partial \omega_i}{\partial x} + b_i \frac{\partial}{\partial x} \left(\frac{\omega_i}{r^{2i+1}} \right) + c_i \frac{\partial \phi_i}{\partial x} + d_i \frac{\partial}{\partial x} \left(\frac{\phi_i}{r^{2i+1}} \right) \\ & + P_i \left(y \frac{\partial X_i}{\partial z} - z \frac{\partial X_i}{\partial y} \right) + A_i \frac{\partial \Omega_i}{\partial x} + B_i \frac{\partial}{\partial x} \left(\frac{\Omega_i}{r^{2i+1}} \right) + C_i \frac{\partial \Phi_i}{\partial x} + D_i \frac{\partial}{\partial x} \left(\frac{\Phi_i}{r^{2i+1}} \right) \end{aligned} \right\} = 0 \dots\dots\dots (19),$$

viz., two others are to be got by writing $y, z \dots$ for $x \dots$ in cyclical order, and three by changing $p_i, a_i \dots P_i \dots$ into $p'_i, a'_i \dots P'_i \dots$.

Now, we may show that these equations require that, for all values of i ,

$$\left. \begin{aligned} p_i \left(y \frac{\partial X_i}{\partial z} - z \frac{\partial X_i}{\partial y} \right) + P_i \left(y \frac{\partial X_i}{\partial z} - z \frac{\partial X_i}{\partial y} \right) &= 0 \\ a_i \omega_i + c_i \phi_i + A_i \Omega_i + C_i \Phi_i &= 0 \\ b_i \omega_i + d_i \phi_i + B_i \Omega_i + D_i \Phi_i &= 0 \end{aligned} \right\} \dots\dots\dots (20),$$

and similar equations with accented letters.

For, take the three equations such as (19), which hold at the surface $r = a$, multiply them by x, y, z , and add; thus

$$i (a_i \omega_i + c_i \phi_i + A_i \Omega_i + C_i \Phi_i) - \frac{(i+1)}{a^{2i+1}} (b_i \omega_i + d_i \phi_i + B_i \Omega_i + D_i \Phi_i) = 0 \dots\dots\dots (21).$$

Again, consider the function

$$\Sigma \left\{ \begin{aligned} & p_i \left(y \frac{\partial X_i}{\partial z} - z \frac{\partial X_i}{\partial y} \right) + P_i \left(y \frac{\partial X_i}{\partial z} - z \frac{\partial X_i}{\partial y} \right) + a_i \frac{\partial \omega_i}{\partial x} + A_i \frac{\partial \Omega_i}{\partial x} + C_i \frac{\partial \Phi_i}{\partial x} + c_i \frac{\partial \phi_i}{\partial x} \\ & + \left(\frac{r}{a} \right)^{2i+3} \left\{ b_i \frac{\partial}{\partial x} \left(\frac{\omega_i}{r^{2i+1}} \right) + B_i \frac{\partial}{\partial x} \left(\frac{\Omega_i}{r^{2i+1}} \right) + d_i \frac{\partial}{\partial x} \left(\frac{\phi_i}{r^{2i+1}} \right) + D_i \frac{\partial}{\partial x} \left(\frac{\Phi_i}{r^{2i+1}} \right) \right\} \end{aligned} \right\}$$

This is finite and continuous, and satisfies $\nabla^2 U = 0$ within the sphere $r = a$, and $= 0$ at its surface; hence it $= 0$ throughout the sphere. The same is true of the functions V, W got from this by cyclical interchange of the letters x, y, z . Hence, differentiating U, V, W with respect to x, y, z respectively, and adding, we find

$$(2i+3)(i+1) [b_i \omega_i + d_i \phi_i + B_i \Omega_i + D_i \Phi_i] = 0 \dots\dots\dots (22).$$

From (19), (21), and (22) we at once obtain (20).

Observe, now, that precisely the same reasoning applies to the equations derived from (19) by accenting the letters. And so, considering what happens within a sphere $r = b$, using functions formed from U, V, W by accenting the letters $p, a, \dots$, and writing $\left(\frac{r}{b}\right)^{2i+3}$ in the b, d terms, we see that the equations (20) hold also at the surface $r = b$ if we accent the letters. Further, it is to be noticed that all the harmonics occurring in any one of the equations (20), and the similar set we have just considered, are of the same order i ; and therefore, dividing all the equations (20) by a^i , and all the similar equations with accented letters by b^i , we shall be left with only surface harmonics which are the same in the two sets of equations, multiplied by constants which are different in the two sets of equations; *i.e.*, we have

$$\left. \begin{aligned} p_i \frac{1}{r^i} \left(y \frac{\partial X_i}{\partial z} - z \frac{\partial X_i}{\partial y} \right) + P_i \frac{1}{r^i} \left(y \frac{\partial X_i}{\partial z} - z \frac{\partial X_i}{\partial y} \right) &= 0 \\ p'_i \frac{1}{r^i} \left(y \frac{\partial X_i}{\partial z} - z \frac{\partial X_i}{\partial y} \right) + P'_i \frac{1}{r^i} \left(y \frac{\partial X_i}{\partial z} - z \frac{\partial X_i}{\partial y} \right) &= 0 \end{aligned} \right\} \dots\dots\dots (23),$$

$$\text{and} \quad \left. \begin{aligned} \frac{1}{r^i} (a_i \omega_i + c_i \phi_i + A_i \Omega_i + C_i \Phi_i) &= 0 \\ \frac{1}{r^i} (b_i \omega_i + d_i \phi_i + B_i \Omega_i + D_i \Phi_i) &= 0 \\ \frac{1}{r^i} (a'_i \omega_i + c'_i \phi_i + A'_i \Omega_i + C'_i \Phi_i) &= 0 \\ \frac{1}{r^i} (b'_i \omega_i + d'_i \phi_i + B'_i \Omega_i + D'_i \Phi_i) &= 0 \end{aligned} \right\} \dots\dots\dots (24)$$

In each of the equations in each of these sets the same harmonic functions occur, and, on eliminating them, we find the equations for the frequency, *viz.*,

$$p_i P'_i - p'_i P_i = 0 \dots\dots\dots (25),$$

and

$$\begin{vmatrix} a_i & c_i & A_i & C_i \\ b_i & d_i & B_i & D_i \\ a'_i & c'_i & A'_i & C'_i \\ b'_i & d'_i & B'_i & D'_i \end{vmatrix} = 0 \dots\dots\dots (26).$$

Just as in the case of the sphere, the modes of vibration are of two types, one purely transverse depending on the χ, X , and with a frequency equation (25), which involves only the modulus of

rigidity, and the other in which the displacement is partly longitudinal (radial), with a frequency equation (26), which involves both the rigidity and the resistance to compression.

8. When the shell contains liquid, the equations (21) will still hold for the outer surface, but at the inner we shall have

$$Pr = p_1 x, \quad Gr = p_1 y, \quad Hr = p_1 z,$$

where p_1 is the fluid pressure.

The condition of constancy of volume of the liquid is fulfilled, since the normal displacement at any sphere is proportional to a spherical surface harmonic.

The normal velocity at any sphere is

$$\frac{\partial}{\partial t} \left(\frac{ux + vy + wz}{r} \right),$$

and we find

$$ux + vy + wz = -\frac{1}{h^2} r \frac{\partial \theta}{\partial r} + \sum [i\psi_i(kr) \phi_i + i\Psi_i(kr) \Phi_i]$$

[to obtain this we have only to multiply (16) by x, y, z , add, and use the difference equation for the ψ, Ψ],

$$\begin{aligned} \text{also} \quad r \frac{\partial \theta}{\partial r} &= \left\{ r \frac{\partial \psi_i(hr)}{\partial r} + i \right\} \omega_i + \left\{ r \frac{\partial \Psi_i(hr)}{\partial r} + i \right\} \Omega_i \\ &= \{ (2i+1) \psi_{i-1}(hr) - (i+1) \psi_i(hr) \} \omega_i \\ &\quad + \{ (2i+1) \Psi_{i-1}(hr) - (i+1) \Psi_i(hr) \} \Omega_i. \end{aligned}$$

$$\text{Hence} \quad -\frac{1}{h^2} \frac{\partial \theta}{\partial r} = \sum [e_i \omega_i + E_i \Omega_i] \frac{a^i}{r^i}, \text{ when } r = a,$$

$$\text{and} \quad = \sum (e'_i \omega_i + E'_i \Omega_i) \frac{b^i}{r^i}, \text{ when } r = b,$$

$$\begin{aligned} \text{where} \quad c_i &= \frac{1}{h^2 a} \{ (i+1) \psi_i(ha) - (2i+1) \psi_{i-1}(ha) \} \\ E_i &= \frac{1}{h^2 a} \{ (i+1) \Psi_i(ha) - (2i+1) \Psi_{i-1}(ha) \} \end{aligned} \quad \dots\dots\dots(27),$$

and e'_i, E'_i are what these become when in them we put b for a .

Hence the normal velocity at the surface $r = b$ is

$$p e^{i\omega t} \sum \left\{ i b^{i-1} \left[\psi_i(kb) \frac{\phi_i}{r^i} + \Psi_i(kb) \frac{\Phi_i}{r^i} \right] + b^i \left[e'_i \frac{\omega_i}{r^i} + E'_i \frac{\Omega_i}{r^i} \right] \right\} \dots\dots\dots(28),$$

where the parts in [ ] are surface harmonics of order i .

The velocity-potential P for the motion of the liquid is

$$P = \rho e^{pt} \Sigma \left[\psi_i(kb) \phi_i + \Psi_i(kb) \Phi_i + \frac{b}{i} (e'_i \omega_i + E'_i \Omega_i) \right].$$

Hence, if σ be the density of the fluid, the pressure p_1 is

$$p_1 = \sigma p^2 \Sigma \left[\psi_i(kb) \phi_i + \Psi_i(kb) \Phi_i + \frac{b}{i} (e'_i \omega_i + E'_i \Omega_i) \right],$$

omitting the terms of the second order arising from the squares of the velocities, and constants which do not depend upon the strain.

Hence, at the surface $r = b$,

$$\begin{aligned} Fr = \sigma p^2 \Sigma & \left[\psi_i(kb) \frac{b^2}{2i+1} \left\{ \frac{\partial \phi_i}{\partial x} - b^{2i+1} \frac{\partial}{\partial x} \left(\frac{\phi_i}{r^{2i+1}} \right) \right\} \right. \\ & \left. + \Psi_i(kb) \frac{b^2}{2i+1} \left\{ \frac{\partial \Phi_i}{\partial x} - b^{2i+1} \frac{\partial}{\partial x} \left(\frac{\Phi_i}{r^{2i+1}} \right) \right\} \right] \\ & + \sigma p^2 \Sigma \frac{b^3}{i \cdot 2i+1} \left[e'_i \left\{ \frac{\partial \omega_i}{\partial x} - b^{2i+1} \frac{\partial}{\partial x} \left(\frac{\omega_i}{r^{2i+1}} \right) \right\} + E'_i \left\{ \frac{\partial \Omega_i}{\partial x} - b^{2i+1} \frac{\partial}{\partial x} \left(\frac{\Omega_i}{r^{2i+1}} \right) \right\} \right]. \end{aligned}$$

The equations which hold at $r = b$ are thus of the same form as in the case of the free vibrations, and we deduce from them that equations (23) still hold, leading to the frequency equation (25); but, instead of the two equations of (24) which contain accented letters, we have the two following equations:—

$$\left\{ \begin{aligned} & \left(a_i - \frac{\sigma p^2 b^3}{i \cdot 2i+1} e'_i \right) \frac{\omega_i}{r^i} + \left(c'_i - \frac{\sigma p^2 b^3}{2i+1} \psi_i(kb) \right) \frac{\phi_i}{r^i} \\ & + \left(A'_i - \frac{\sigma p^2 b^3}{i \cdot 2i+1} E'_i \right) \frac{\Omega_i}{r^i} + \left(C'_i - \frac{\sigma p^2 b^3}{2i+1} \Psi_i(kb) \right) \frac{\Phi_i}{r^i} \end{aligned} \right\} = 0$$

$$\left\{ \begin{aligned} & \left(b'_i + \frac{\sigma p^2 b^{2i+4}}{i \cdot 2i+1} e'_i \right) \frac{\omega_i}{r^i} + \left(d'_i + \frac{\sigma p^2 b^{2i+3}}{2i+1} \psi_i(kb) \right) \frac{\phi_i}{r^i} \\ & + \left(B'_i + \frac{\sigma p^2 b^{2i+4}}{i \cdot 2i+1} E'_i \right) \frac{\Omega_i}{r^i} + \left(D'_i + \frac{\sigma p^2 b^{2i+3}}{2i+1} \Psi_i(kb) \right) \frac{\Phi_i}{r^i} \end{aligned} \right\} = 0$$

....(29).

We may write these for shortness

$$\left. \begin{aligned} & \frac{1}{r^i} (a''_i \omega_i + c''_i \phi_i + A''_i \Omega_i + C''_i \Phi_i) = 0 \\ \text{and} \quad & \frac{1}{r^i} (b''_i \omega_i + d''_i \phi_i + B''_i \Omega_i + D''_i \Phi_i) = 0 \end{aligned} \right\},$$

and the frequency equation for vibrations of this type becomes

$$\begin{vmatrix} a_i & c_i'' & A_i'' & C_i'' \\ b_i'' & d_i'' & B_i'' & D_i'' \\ a_i & c_i & A_i & C_i \\ b_i & d_i & B_i & D_i \end{vmatrix} = 0 \dots\dots\dots(30),$$

which differs from the equation (26), which holds when there is no liquid, only in the addition of certain terms to the quantities

$$a_i', b_i', \dots A_i', \dots$$

9. We proceed now to the consideration of the forced vibrations. We suppose the matter composing the shell and the contained fluid to be subject to gravitation and to the attraction of some external disturbing bodies, or generally to external disturbing forces which have a potential expressible in spherical harmonic series. Also, for a particular example, we consider the whole mass to be in rotation with an angular velocity ω , which is such that $\frac{\omega^2 a}{g}$ is small, g being the acceleration due to gravity at the surface. Then there will be certain symmetrical displacements which are the same at all points of any concentric sphere and which may be taken to be zero at the internal surface. Superposed on these will be unsymmetrical displacements expressible at any spherical surface by series of spherical harmonics; thus there will be bodily forces depending on the attractions of the harmonic inequalities. There will also be certain surface-tractions due to the attraction exerted by the mass on the harmonic inequalities, since the surface-tractions actually fulfil conditions given at the deformed surfaces and not at the initial spherical surfaces.

10. The bodily forces acting at any point of the shell are now—

(1) Forces arising from the attraction of external matter having a potential W_i .

(2) Forces arising from the attraction of the shell and fluid on itself, and symmetrical with respect to angular space.

(3) Forces arising from the attractions of the harmonic inequalities, and having a potential which may be written $U_i + V_i/r^{2i+1}$. Then, if W_i is periodic with period $2\pi/p$, so also are U_i and V_i .

(4) To these must be added the “centrifugal force” when the shell rotates. This is partly symmetrical, and partly has a potential expressed by a zonal harmonic term of order 2. The symmetrical part of this may be treated along with the forces (2).

In our present problem, of the vibrating elastic shell, the forces (2) and the centrifugal forces can only be of importance as producing certain surface-tractions. We shall throughout suppose the rotation so slow that the square of the angular velocity multiplied by the amplitude of any of the forced harmonic inequalities (i.e., by the height of any tide) is negligible. Then in the surface-tractions the terms which depend on the centrifugal force will disappear. Unless we made this supposition, the ellipticity of the figure of relative equilibrium would not be small enough to allow of our using the solutions of the differential equations within a space bounded by concentric spheres, and we should require the solutions for spheroids. When we are considering the liquid, however, the rotation must be treated differently, as it will appear that the ratio of the period of rotation to that of the disturbing force must be taken into account.

Write
$$U_i + V_i/r^{2i+1} + W_i = Y_i, \dots\dots\dots (31).$$

The equations of forced vibration are three such as

$$\begin{aligned} m \frac{\partial \theta}{\partial x} + n \nabla^2 u &= \rho \frac{\partial^2 u}{\partial t^2} - \rho \frac{\partial Y_i}{\partial x} \\ \text{or} \quad m \frac{\partial \theta}{\partial x} + n \nabla^2 u + \rho p^2 u &= -\rho \frac{\partial Y_i}{\partial x} \dots\dots\dots (32) \end{aligned}$$

To get a particular integral, suppose

$$u dx + v dy + w dz = d\phi,$$

then $\theta = \nabla^2 \phi$, and $\frac{\partial}{\partial x} [(m+n) \nabla^2 \phi + p^2 \rho \phi + \rho Y_i] = 0$,

so that a particular integral is $\phi = -Y_i/p^2$.

Thus
$$u = -\frac{1}{p^2} \frac{\partial Y_i}{\partial x}, \quad v = -\frac{1}{p^2} \frac{\partial Y_i}{\partial y}, \quad w = -\frac{1}{p^2} \frac{\partial Y_i}{\partial z} \dots (33).$$

For the complementary functions, we must take the general solution of the equations when $Y_i = 0$; these are equations (6), and the solutions are those given in (16), the period being $2\pi/p$, that of the forced vibrations. It is plain that we may omit the consideration of the χ , X terms, and indeed these will not be forced by the kind of action here considered.

11. The surface-tractions due to the complementary functions are given by the equations (17). We have here to consider the parts due to the particular integrals (33).

We have $\theta = \nabla^2 \phi = -\frac{1}{p^3} \nabla^2 Y_i = 0$.

Hence the component surface-traction parallel to x is

$$\begin{aligned} & -\frac{n}{p^3} \left(r \frac{\partial}{\partial r} - 1 \right) \left(\frac{\partial U_i}{\partial x} + \frac{\partial W_i}{\partial x} \right) - \frac{n}{p^3} \left(r \frac{\partial}{\partial r} - 1 \right) \frac{\partial}{\partial x} \left(\frac{V_i}{r^{2i+1}} \right) \\ & - \frac{n}{p^3} \frac{\partial}{\partial x} \left[i (U_i + W_i) - (i+1) \frac{V_i}{r^{2i+1}} \right] \\ & = -\frac{n}{p^3} \left[\frac{\partial}{\partial x} (i-2) (U_i + W_i) - (i+3) \frac{\partial}{\partial x} \left(\frac{V_i}{r^{2i+1}} \right) \right] \\ & - \frac{n}{p^3} \left[i \frac{\partial}{\partial x} (U_i + W_i) - (i+1) \frac{\partial}{\partial x} \left(\frac{V_i}{r^{2i+1}} \right) \right] \\ & = -\frac{2n}{p^3} \left[(i-1) \left(\frac{\partial U_i}{\partial x} + \frac{\partial W_i}{\partial x} \right) - (i+2) \frac{\partial}{\partial x} \left(\frac{V_i}{r^{2i+1}} \right) \right] \dots\dots\dots (34). \end{aligned}$$

12. To find U_i , V_i . Let σ be the density of the contained fluid, then U_i = potential within a sphere of radius a of a surface distribution of density $\rho \left(\frac{ux+vy+wz}{r} \right)_a$, and $\frac{V_i}{r^{2i+1}}$ is the potential outside a sphere of radius b of a surface distribution of density

$$(\sigma - \rho) \left(\frac{ux+vy+wz}{r} \right)_b.$$

The surface-density for U_i is

$$\frac{\rho}{a} \left\{ \frac{i+1}{p^2} \frac{V_i}{r^i} \frac{1}{a^{i+1}} - \frac{i}{p^3} \frac{U_i + W_i}{r^i} a^i + \frac{i}{a} \left(\psi_i(ka) \frac{\Phi_i}{r^i} a^i + \Psi_i(ka) \frac{\Phi_i}{r^i} a^i \right) \right. \\ \left. \left(e_i \frac{\omega_i}{r^i} + E_i \frac{\Omega_i}{r^i} \right) a^i \right\}$$

Hence $U_i = \frac{4\pi\gamma}{a^{i-1} \cdot 2i+1}$ * times this, so that

$$\begin{aligned} & U_i \left(1 + \frac{4\pi\gamma\rho}{p^3} \frac{i}{2i+1} \right) - V_i \frac{4\pi\gamma\rho}{p^2} \frac{i+1}{2i+1} \cdot \frac{1}{a^{2i+1}} \\ & = \frac{4\pi\gamma\rho}{2i+1} \left[\frac{i}{a} \{ \psi_i(ka) \phi_i + \Psi_i(ka) \Phi_i \} - (e_i \omega_i + E_i \Omega_i) - \frac{i}{p^3} W_i \right]. \end{aligned}$$

The surface-density for $\frac{V_i}{r^{2i+1}}$ is

$$\begin{aligned} \frac{\sigma - \rho}{b} \left\{ \frac{i+1}{p^2} \frac{V_i}{r^i} \frac{1}{b^{i+1}} - \frac{i}{p^3} \frac{U_i + W_i}{r^i} b^i + \frac{i}{b} \left(\psi_i(kb) \frac{\Phi_i}{r^i} + \Psi_i(kb) \frac{\Phi_i}{r^i} \right) b^i \right. \\ \left. - \left(e_i \frac{\omega_i}{r^i} + E_i \frac{\Omega_i}{r^i} \right) b^i \right\} \end{aligned}$$

..... (35).

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* γ is the "constant of gravitation."

And $\frac{V_i}{r^{2i+1}} = \frac{4\pi\gamma b^{i+2}}{r^{2i+1}(2i+1)}$ times this, so that

$$V_i \left(1 - \frac{4\pi\gamma(\sigma-\rho)}{p^3} \frac{i+1}{2i+1} \right) + U_i \frac{4\pi\gamma(\sigma-\rho)}{p^3} \frac{i}{2i+1} b^{2i+1} \\ = \frac{4\pi\gamma(\sigma-\rho) b^{2i+1}}{2i+1} \left[\frac{i}{b} \{ \psi_i(kb) \phi_i + \Psi_i(kb) \Phi_i \} - (e'_i \omega_i + E'_i \Omega_i) - \frac{i}{p^3} W_i \right].$$

From (35) it appears that the radial velocity at $r = b$ is

$$\frac{ip}{b} \left[\frac{i+1}{p^3} \frac{V_i}{r^i} \frac{1}{b^{i+1}} - \frac{i}{p^3} \frac{U_i + W_i}{r^i} b^i \right. \\ \left. + \frac{i}{b} \left(\psi_i(kb) \frac{\phi_i}{r^i} + \Psi_i(kb) \frac{\Phi_i}{r^i} \right) b^i - \left(e'_i \frac{\omega_i}{r^i} + E'_i \frac{\Omega_i}{r^i} \right) b^i \right] \dots\dots (36).$$

This will be useful when we come to treat the motion of the liquid.

Solving for U_i , V_i , we have

$$U_i \left[1 + \frac{4\pi\gamma}{p^3} \left(\rho + \frac{i+1}{2i+1} \sigma \right) - \frac{4\pi\gamma\rho \cdot 4\pi\gamma(\sigma-\rho)}{p^4} \frac{i(i+1)}{(2i+1)^3} \left(1 - \frac{b^{2i+1}}{a^{2i+1}} \right) \right] \\ = \frac{4\pi\gamma i}{2i+1} \left[\frac{\rho}{a} \left(1 - \frac{4\pi\gamma(\sigma-\rho)}{p^3} \frac{i+1}{2i+1} \right) \{ \psi_i(ka) \phi_i + \Psi_i(ka) \Phi_i \} \right. \\ \left. + \frac{\sigma-\rho}{b} \frac{4\pi\gamma\rho}{p^3} \frac{i+1}{2i+1} \frac{b^{2i+1}}{a^{2i+1}} \{ \psi_i(kb) \phi_i + \Psi_i(kb) \Phi_i \} \right] \\ - \frac{4\pi\gamma}{2i+1} \left[\rho \left(1 - \frac{4\pi\gamma(\sigma-\rho)}{p^3} \frac{i+1}{2i+1} \right) (e_i \omega_i + E_i \Omega_i) \right. \\ \left. + (\sigma-\rho) \frac{4\pi\gamma\rho}{p^3} \frac{i+1}{2i+1} \frac{b^{2i+1}}{a^{2i+1}} (e'_i \omega_i + E'_i \Omega_i) \right] \\ - \frac{4\pi\gamma\rho}{p^3} \frac{i}{2i+1} \left[1 - \frac{4\pi\gamma(\sigma-\rho)}{p^3} \frac{i+1}{2i+1} \left(1 - \frac{b^{2i+1}}{a^{2i+1}} \right) \right] W_i \dots\dots\dots (37),$$

and

$$V_i \left[1 + \frac{4\pi\gamma}{p^3} \left(\rho + \frac{i+1}{2i+1} \sigma \right) - \frac{4\pi\gamma\rho \cdot 4\pi\gamma(\sigma-\rho)}{p^4} \frac{i(i+1)}{(2i+1)^3} \left(1 - \frac{b^{2i+1}}{a^{2i+1}} \right) \right] \frac{1}{b^{2i+1}} \\ = \frac{4\pi\gamma i}{2i+1} \left[\frac{\sigma-\rho}{b} \left(1 + \frac{4\pi\gamma\rho}{p^3} \frac{i}{2i+1} \right) \{ \psi_i(kb) \phi_i + \Psi_i(kb) \Phi_i \} \right. \\ \left. - \frac{\rho}{a} \left(\frac{4\pi\gamma(\sigma-\rho)}{p^3} \frac{i}{2i+1} \right) \{ \psi_i(ka) \phi_i + \Psi_i(ka) \Phi_i \} \right]$$

$$\begin{aligned}
 & -\frac{4\pi\gamma}{2i+1} \left[(\sigma-\rho) \left(1 + \frac{4\pi\gamma\rho}{p^3} \frac{i}{2i+1} \right) (e'_i\omega_i + E'_i\Omega_i) \right. \\
 & \quad \left. - \rho \frac{4\pi\gamma(\sigma-\rho)}{p^3} \frac{i}{2i+1} (e_i\omega_i + E_i\Omega_i) \right] \\
 & - \frac{4\pi\gamma(\sigma-\rho)}{p^3} \frac{i}{2i+1} W_i \dots\dots\dots (38).
 \end{aligned}$$

Write these

$$\begin{aligned}
 U_i &= g_i\phi_i + h_i\omega_i + G_i\Phi_i + H_i\Omega_i + \alpha_i W_i \} \dots\dots\dots (39). \\
 V_i &= g'_i\phi_i + h'_i\omega_i + G'_i\Phi_i + H'_i\Omega_i + \beta_i W_i \}
 \end{aligned}$$

Then (34) contributes to the surface-tractions parallel to π

$$\begin{aligned}
 & -\frac{2n}{p^3} (i-1) \left[(\alpha_i+1) \frac{\partial W_i}{\partial x} + g_i \frac{\partial \phi_i}{\partial x} + h_i \frac{\partial \omega_i}{\partial x} + G_i \frac{\partial \Phi_i}{\partial x} + H_i \frac{\partial \Omega_i}{\partial x} \right] \\
 & + \frac{2n}{p^3} (i+2) \left[\beta_i \frac{\partial}{\partial x} \left(\frac{W_i}{r^{2i+1}} \right) + g'_i \frac{\partial}{\partial x} \left(\frac{\phi_i}{r^{2i+1}} \right) + h'_i \frac{\partial}{\partial x} \left(\frac{\omega_i}{r^{2i+1}} \right) \right. \\
 & \quad \left. + G'_i \frac{\partial}{\partial x} \left(\frac{\Phi_i}{r^{2i+1}} \right) + H'_i \frac{\partial}{\partial x} \left(\frac{\Omega_i}{r^{2i+1}} \right) \right] \left. \right\} \dots (40)
 \end{aligned}$$

With the exception of the W_i terms, these are of the same form as the terms of (17), and it is to be noticed that the surface harmonics which occur in these formulæ are the same at both bounding surfaces. The parts of the surface-tractions which depend on the complementary functions are of the same forms as those given by (17) with the χ , X omitted.

13. We have now to consider the surface-tractions arising from the symmetrical parts of the forces acting.

$$\text{These are } \frac{2}{3}\omega^2 r, \text{ and } -\frac{4}{3}\pi\gamma \left(\rho \frac{r^3-b^3}{r^2} + \sigma \frac{b^3}{r^2} \right)$$

The equation giving the radial displacement U' is

$$(m+n) \left(\frac{d^2 U'}{dr^2} + \frac{2}{r} \frac{dU'}{dr} - \frac{2U'}{r^2} \right) + \rho \left[\frac{2}{3}\omega^2 r - \frac{4}{3}\pi\gamma \left(\rho \frac{r^3-b^3}{r^2} + \sigma \frac{b^3}{r^2} \right) \right] = 0, *$$

This may be written

$$(m+n) \left(r \frac{d}{dr} - 1 \right) \left(r \frac{d}{dr} + 2 \right) U' + r^2 \rho \left[\frac{2}{3}\omega^2 r - \frac{4}{3}\pi\gamma \left(\rho \frac{r^3-b^3}{r^2} + \sigma \frac{b^3}{r^2} \right) \right] = 0,$$

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* Webb, "Stress and Strain in Cylindrical and Polar Coordinates" (*Messenger*, 1882).

and the solution is

$$U' = Ar + \frac{B}{r^2} + \frac{1}{10} \frac{r^3 \rho}{m+n} \left(\frac{4}{3} \pi \gamma \rho - \frac{2}{3} \omega^2 \right) + \frac{2}{3} \pi \gamma (\sigma - \rho) \frac{b^3 \rho}{m+n},$$

where A and B are arbitrary constants.

Write this for shortness

$$U' = Ar + \frac{B}{r^2} + Hr^3 + K \dots \dots \dots (41),$$

where A and B are arbitrary, H and K known constants. The corresponding u, v, w are given by multiplying this by $x/r, y/r, z/r$ respectively, and then for the cubical dilatation and the 6 strains we have

$$\theta = 5Hr^2 + 3A = e + f + g,$$

$$\left. \begin{aligned} e &= H(r^2 + 2x^2) + K \left(\frac{1}{r} - \frac{x^2}{r^3} \right) + A + B \left(\frac{1}{r^3} - \frac{3x^2}{r^5} \right) \\ f &= H(r^2 + 2y^2) + K \left(\frac{1}{r} - \frac{y^2}{r^3} \right) + A + B \left(\frac{1}{r^3} - \frac{3y^2}{r^5} \right) \\ g &= H(r^2 + 2z^2) + K \left(\frac{1}{r} - \frac{z^2}{r^3} \right) + A + B \left(\frac{1}{r^3} - \frac{3z^2}{r^5} \right) \\ a &= 4Hyz - 2K \frac{yz}{r^3} - 6B \frac{yz}{r^5} \\ b &= 4Hxz - 2K \frac{zx}{r^3} - 6B \frac{zx}{r^5} \\ c &= 4Hxy - 2K \frac{xy}{r^3} - 6B \frac{xy}{r^5} \end{aligned} \right\}.$$

The stresses P, Q, R, S, T, U are then given by

$$P = (m-n)\theta + 2ne, \dots S = na, \dots$$

14. Let

$$\left. \begin{aligned} r &= a + \beta_0 Q_i \text{ be the external surface,} \\ r &= b + \beta_1 Q_i \text{ be the internal surface,} \end{aligned} \right\} Q_i \text{ being a solid harmonic;}$$

then it is at these surfaces that we require the surface-tractions. When the shell contains a given mass of liquid, the condition at $r=b$ is $U'=0$. At a free surface we must make the surface-tractions vanish. Let l', m', n' be the direction-cosines of the outward-drawn normal to the surface $r = a + \beta Q_i$; then parallel to x the component

surface-traction is

$$(l'P + m'U + n'T) = \left(1 + \frac{\beta i Q_i}{r}\right) \left(\frac{x}{r} P + \frac{y}{r} U + \frac{z}{r} T\right) - \beta \left(P \frac{\partial Q_i}{\partial x} + U \frac{\partial Q_i}{\partial y} + T \frac{\partial Q_i}{\partial z}\right).$$

Here $P \frac{x}{r} + U \frac{y}{r} + T \frac{z}{r} = \left(Hr^3 (5m+n) + A (3m-n) - \frac{4n}{r^3} B\right) \frac{x}{r},$

and
$$P \frac{\partial Q_i}{\partial x} + U \frac{\partial Q_i}{\partial y} + T \frac{\partial Q_i}{\partial z} = Hr^3 (5m-3n) \frac{\partial Q_i}{\partial x} + 4nHixQ_i + (3m-n) A \frac{\partial Q_i}{\partial x} + 2n \frac{K}{r} \frac{\partial Q_i}{\partial x} - \frac{2nKx}{r^3} iQ_i + \frac{2nB}{r^3} \frac{\partial Q_i}{\partial x} - \frac{6nBx}{r^3} iQ_i.$$

These and the condition $U' = 0$ when $r = b$ give, for A and B ,

$$\left. \begin{aligned} Ab + \frac{B}{b^3} + Hb^3 + K &= 0 \\ (5m+n) Ha^3 + (3m-n) A - \frac{4nB}{a^3} &= 0 \end{aligned} \right\} \dots\dots\dots(42),$$

where a is the mean outer radius after the strain is effected, *i.e.*, this is the a of our previous work.

Also the surface-tractions at $r = a + \beta Q_i$ which depend on Q_i are

$$\begin{aligned} &\left[2Ha\beta Q_i (5m+n) - \frac{12nB}{a^4} \beta Q_i\right] \frac{x}{a} \\ &+ \left(-\frac{\beta Q_i}{a} + \frac{i\beta Q_i}{a}\right) \left(Ha^3 (5m+n) + (3m-n) A - \frac{4n}{a^3} B\right) \frac{x}{a} \\ &- \beta \frac{\partial Q_i}{\partial x} \left(Ha^3 (5m-3n) + (3m-n) A + \frac{2nK}{a} + \frac{2nB}{a^3}\right) \\ &- 2n\beta xiQ_i \left(2H - \frac{K}{a^3} - \frac{3B}{a^5}\right), \end{aligned}$$

parallel to x , with two similar expressions for those parallel to y and z .

The parts contributed by these to $F.r$ are, at the external surface,

$$\left. \begin{aligned} &x \left[2Ha\beta_0 Q_i (5m+n) - \frac{12nB}{a^4} \beta_0 Q_i\right] \\ &- \beta_0 \frac{\partial Q_i}{\partial x} \left[\frac{2nK}{a} + \frac{6nB}{a^3} - 4nHa^3\right] a \\ &- 2ni\beta_0 xQ_i \left[2H - \frac{K}{a^3} - \frac{3B}{a^5}\right] a \end{aligned} \right\} \dots\dots\dots(43),$$

and at the internal surface,

$$\left. \begin{aligned} & x\beta_1 Q_i \left[2Hb (5m+n) - \frac{12nB}{b^4} \right] \\ & + ix\beta_1 Q_i \left[Hb (5m+n) + (3m-n) \frac{A}{b} - \frac{4n}{b^4} B \right] \\ & - \beta_1 \frac{\partial Q_i}{\partial x} \left[Hb^3 (5m-3n) + (3m-n) Ab + 2nK + \frac{2nB}{b^3} \right] \\ & - 2ni\beta_1 xQ_i b \left[2H - \frac{K}{b^3} - \frac{3B}{b^5} \right] \end{aligned} \right\}.$$

Of these, the part at the external surface is

$$\left. \begin{aligned} & 2\beta_0 \frac{a^3}{2i+1} \frac{\partial Q_i}{\partial x} \left[Ha \{ 5m + (2i+3) n \} - \frac{i+1}{a^3} nK + \frac{3n(i+3)}{a^4} B \right] \\ & - 2\beta_0 \frac{a^{2i+3}}{2i+1} \frac{\partial}{\partial x} \left(\frac{Q_i}{r^{2i+1}} \right) \left[Ha \{ 5m - (2i-1) n \} + \frac{2i}{a^3} nK + \frac{3n(i-2)}{a^4} B \right] \end{aligned} \right\} \dots\dots\dots (44),$$

and at the internal surface,

$$\left. \begin{aligned} & \beta_1 \frac{b^3}{2i+1} \frac{\partial Q_i}{\partial x} \left[\{ n (5+3i) - 5m (i-1) \} Hb \right. \\ & \quad \left. - 2 (i+1) n \frac{K}{b^3} - (3m-n)(i+1) \frac{A}{b} + (6i-10) \frac{nB}{b^4} \right] \\ & - \beta_1 \frac{b^{2i+3}}{2i+1} \frac{\partial}{\partial x} \left(\frac{Q_i}{r^{2i+1}} \right) \left[\{ 5m (i+2) - n (3i-2) \} Hb \right. \\ & \quad \left. + 2in \frac{K}{b^3} + (3m-n) i \frac{A}{b} + (2i-12) \frac{nB}{b^4} \right] \end{aligned} \right\} \dots\dots\dots (45).$$

In these $\beta_0 Q_i$ is

$$\left. \begin{aligned} & \frac{1}{a} \left[\frac{i+1}{p^3} \frac{V_i}{a^{2i+1}} - \frac{i}{p^3} (U_i + W_i) \right. \\ & \quad \left. + \frac{i}{a} \{ \psi_i (ka) \phi_i + \Psi_i (ka) \Phi_i \} - (e_i \omega_i + E_i \Omega_i) \right] \end{aligned} \right\} \dots (46).$$

and $\beta_1 Q_i$ is

$$\left. \begin{aligned} & \frac{1}{b} \left[\frac{i+1}{p^3} \frac{V_i}{b^{2i+1}} - \frac{i}{p^3} (U_i + W_i) \right. \\ & \quad \left. + \frac{i}{b} \{ \psi_i (kb) \phi_i + \Psi_i (kb) \Phi_i \} - (e'_i \omega_i + E'_i \Omega_i) \right] \end{aligned} \right\}$$

As $\omega^3\beta$ is of the order neglected, we may suppose A , B , H in (45) simplified by the omission of all terms in ω^3 , so that

$$\begin{aligned} H &= \left(\frac{2}{3}\pi\gamma\rho\right) \frac{1}{5} \frac{\rho}{m+n}, \quad K = \frac{2}{3}\pi\gamma (\sigma - \rho) b^3 \frac{\rho}{m+n}, \\ &\quad A \left(\frac{3m-n}{b^3} + \frac{4n}{a^3} \right) \\ &= - \left(\frac{2}{3}\pi\gamma\rho\right) \frac{\rho}{m+n} \left[\frac{1}{5} \frac{a^3}{b^3} (5m+n) + \left(\frac{\sigma}{\rho} - \frac{4}{5} \right) \frac{b^3}{a^3} 4n \right], \\ &\quad B \left(\frac{3m-n}{b^3} + \frac{4n}{a^3} \right) \\ &= - \left(\frac{2}{3}\pi\gamma\rho\right) \frac{\rho}{m+n} \left[-\frac{1}{5} a^3 (5m+n) + \left(\frac{\sigma}{\rho} - \frac{4}{5} \right) b^3 (3m-n) \right] \end{aligned}$$

15. As a verification consider what these become in case there is no cavity, but the sphere is complete up to the centre and incompressible; in this case we know that the stresses should reduce to a normal stress equal to the weight of the harmonic inequality.*

The constant B is now zero, and A is given by

$$(5m+n) Ha^3 + (3m-n) A = 0.$$

K is also zero, and

$$H = \frac{1}{10} \left(\frac{4}{3} \pi \rho \right) \frac{\rho}{m+n}.$$

Supposing m very great compared with n , we have

$$A = -\frac{4}{3} Ha^3,$$

and we need only consider terms in which H or A is multiplied by a factor containing m ; thus by (43) the normal stress is

$$\begin{aligned} &\frac{1}{r} \left(Fr \frac{x}{r} + Gr \frac{y}{r} + Hr \frac{z}{r} \right) \\ &= a 2H\beta_0 Q_i (5m+n) = \frac{4}{3} (\pi\gamma\rho a) (\rho\beta_0 Q_i) = g\rho\beta_0 Q_i, \end{aligned}$$

where $g = \frac{4}{3} (\pi\gamma\rho a)$ is the value of gravity at the surface.

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* Darwin on the "Bodily Tides of Viscous and semi-elastic Spheroids" (*Phil. Trans.*, 1879, p. 6).

Also the only component stresses which do not vanish are

$$\frac{x}{r} 2Ha Q_i \beta_0 (5m+n), \quad \frac{y}{r} 2Ha Q_i \beta_0 (5m+n), \quad \frac{z}{r} 2Ha Q_i \beta_0 (5m+n),$$

which give no tangential stress.

These calculations are correct to the first order of the small quantity β_0 , and we know that the tangential stress is in the case supposed small of the second order. This is no longer true when the sphere is not supposed incompressible.

16. To calculate the fluid pressure we must take account of the rotation. Suppose we know a fluid motion which satisfies the equations of Hydrodynamics and gives no normal flow at a surface S ; then if u, v, w be the components of this motion, these satisfy (a) three such equations as

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = \frac{\partial}{\partial x} \left(V' - \frac{p_1}{\sigma} \right),$$

where V is the potential of the acting forces, also (b) the equation of continuity $\partial u / \partial x + \partial v / \partial y + \partial w / \partial z = 0$, and (c) the boundary condition $lu + mv + nw = 0$ at S , (l, m, n) being the direction-cosines of the normal to S drawn outwards.

We are going to take for the known motion a steady motion of rotation with angular velocity ω about the axis of z , so that

$$u = -\omega y, \quad v = \omega x, \quad w = 0,$$

and to superpose a small oscillatory motion, consisting of an irrotational oscillation combined with periodic changes in the rate and axis of rotation; and we are going to make the boundary change form without changing its volume; *i.e.*, we take for the components of the disturbed motion $u + u', v + v', w + w'$, where u', v', w' are small, and

$$u' = \frac{\partial \phi}{\partial x} - \omega_2 y + \omega_3 z, \quad v' = \frac{\partial \phi}{\partial y} - \omega_1 z + \omega_3 x, \quad w' = \frac{\partial \phi}{\partial z} - \omega_2 x + \omega_1 y,$$

and suppose ϕ satisfies $\nabla^2 \phi = 0$, while $\omega_1, \omega_2, \omega_3$ are functions of the time. We suppose that the bounding surface is a sphere deformed into an harmonic spheroid, so that the boundary condition at the surface $r = b$ is of the form

$$r \frac{\partial \phi}{\partial r} = lu' + mv' + nw' = \Sigma S_i,$$

where S_i is a surface harmonic; we shall show that for such a motion to be possible the surface harmonic must be of order 2.

The Helmholtz equations of vortex motion referred to the moving axes* are such as

$$\frac{\partial \xi}{\partial t} - \eta \omega + u' \frac{\partial \xi}{\partial x} + v' \frac{\partial \xi}{\partial y} + w' \frac{\partial \xi}{\partial z} = \xi \frac{\partial (u+u')}{\partial x} + \eta \frac{\partial (u+u')}{\partial y} + \zeta \frac{\partial (u+u')}{\partial z},$$

where $\xi = \omega_1$, $\eta = \omega_2$, $\zeta = \omega + \omega_3$.

Neglecting products of small quantities, these become

$$\left. \begin{aligned} \dot{\omega}_1 - \omega \omega_2 &= \omega_2 (-\omega) + \omega \frac{\partial^2 \phi}{\partial x \partial z} \\ \dot{\omega}_2 + \omega \omega_1 &= \omega_1 \omega + \omega \frac{\partial^2 \phi}{\partial y \partial z} \\ \dot{\omega}_3 &= \omega \frac{\partial^2 \phi}{\partial z^2} \end{aligned} \right\} \dots\dots\dots (47),$$

so that $\dot{\omega}_1 = \omega \frac{\partial^2 \phi}{\partial x \partial z}$, $\dot{\omega}_2 = \omega \frac{\partial^2 \phi}{\partial y \partial z}$, $\dot{\omega}_3 = \omega \frac{\partial^2 \phi}{\partial z^2}$,

and $\frac{\partial \phi}{\partial z}$ is a linear function $(\dot{\omega}_1 x + \dot{\omega}_2 y + \dot{\omega}_3 z) / \omega$ of x, y, z , which shows that ϕ is a solid harmonic of order 2. Hence $r \frac{\partial \phi}{\partial r}$ contains a surface harmonic of order 2, and the sphere must be deformed into a surface harmonic of order 2. Thus it is only in case the disturbing function is of the second order that this method can be applied.

17. Let V' be the potential of all the forces acting on the liquid, not counting the "centrifugal force." V' contains terms depending on (1) the attraction of the shell and fluid on itself, (2) the attraction of the external disturbing bodies, (3) the attraction of the harmonic inequalities. And the harmonic inequalities are of two sorts, one a function of the time depending on the external disturbing bodies, and the other independent of the time, and due to the rotation.

If, as before, p_1 be the pressure, and σ the density of the fluid, the

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* Greenhill, *Proc. Camb. Phil. Soc.*, iv., p. 6, 1882. This method of proving equations (47) was suggested by the referee.

equations may be written—

$$\begin{aligned}\frac{\partial}{\partial x} \left(V' - \frac{p_1}{\sigma} \right) &= \frac{\partial^2 \phi}{\partial x \partial t} - \omega^2 x + \left[- \left\{ \dot{\omega}_3 + \omega \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) \right\} \right] y \\ &\quad + \omega \left(y \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial \phi}{\partial y} \right) - 2\omega\omega_3 x + (\dot{\omega}_3 + \omega\omega_1) z + \omega x \frac{\partial^3 \phi}{\partial x \partial y}, \\ \frac{\partial}{\partial y} \left(V' - \frac{p_1}{\sigma} \right) &= \frac{\partial^2 \phi}{\partial y \partial t} - \omega^2 y + \left[\dot{\omega}_3 + \omega \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) \right] x \\ &\quad - \omega \left(x \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial \phi}{\partial x} \right) - 2\omega\omega_3 y + (\omega\omega_3 - \dot{\omega}_1) z - \omega y \frac{\partial^3 \phi}{\partial x \partial y}, \\ \frac{\partial}{\partial z} \left(V' - \frac{p_1}{\sigma} \right) &= \frac{\partial^2 \phi}{\partial z \partial t} - \left(\dot{\omega}_3 - \omega \frac{\partial^2 \phi}{\partial y \partial z} \right) x + \left(\dot{\omega}_1 - \omega \frac{\partial^2 \phi}{\partial x \partial z} \right) y + \omega\omega_3 y + \omega\omega_1 x.\end{aligned}$$

In these the terms in [ ] vanish, and, since ϕ is homogeneous of order 2,

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= x \frac{\partial^2 \phi}{\partial x \partial y} + y \frac{\partial^2 \phi}{\partial y^2} + z \frac{\partial^2 \phi}{\partial y \partial z}, \\ \frac{\partial \phi}{\partial x} &= x \frac{\partial^2 \phi}{\partial x^2} + y \frac{\partial^2 \phi}{\partial x \partial y} + z \frac{\partial^2 \phi}{\partial x \partial z};\end{aligned}$$

hence, using (47), the only terms which remain when we multiply by dx , dy , dz , add, and integrate, are included in

$$\begin{aligned}V' - \frac{p_1}{\sigma} &= \frac{\partial \phi}{\partial t} + f(t) + \int (-\omega^2 x - 2\omega\omega_3 x + \omega\omega_1 z) dx \\ &\quad + (-\omega^2 y - 2\omega\omega_3 y + \omega\omega_3 z) dy + \omega(\omega_2 y + \omega_1 x) dz,\end{aligned}$$

$$\begin{aligned}\text{so that } V' - \frac{p_1}{\sigma} &= \frac{\partial \phi}{\partial t} + f(t) - \frac{1}{2}\omega^2(x^2 + y^2) - \omega\omega_3(x^2 + y^2) \\ &\quad + \omega\omega_1 xz + \omega\omega_3 yz \dots\dots\dots (48),\end{aligned}$$

where $f(t)$ is an arbitrary function of t .

If ω be not too great, the solutions depending on the terms of V' which are periodic, and those which are independent of the time, remain separate. The new terms introduced by the rotation into the pressure equation (48) are those in $\omega\omega_1$, $\omega\omega_2$, and $\omega\omega_3$, and it is on the relative magnitude of these and ω^2 that the subsequent discussion turns.

In calculating the pressure from equation (48), we see, by (47), that if p be great compared with ω , the products $\omega\omega_1$, $\omega\omega_2$, $\omega\omega_3$ are small compared with ω^2 , and may be neglected; if p be comparable with ω , ω_1 is comparable with β ; and if β be small compared with the ellipticity due to the rotation ω , these terms can still be neglected. The term $\omega^2\beta$ is always negligible, so that, unless p be small compared

with ω , the only way in which ω enters into the equations determining the amplitude is by occurring in H , and consequently in A and B . But H is $\frac{1}{10} \frac{1}{n+n} (\frac{4}{3}\pi\gamma\rho - \frac{2}{3}\omega^2)$, and, $\frac{\omega^2}{2\pi\gamma\rho}$ being very small, we see that a rotation ω which is small of this order, does not sensibly alter the amplitude except in the case of oscillations of very long period.

Considering the Earth as a liquid mass contained within a solid elastic shell whose internal and external surfaces are approximately spherical, we see that the amount by which it would yield to the tidal disturbing force of the sun and moon, regarded as having a diurnal or semi-diurnal period, is not sensibly altered by the angular velocity of rotation, since the ratio $\frac{\omega^2 a}{g} = \frac{1}{289}$. If we consider the fortnightly tides we see that $\left(\frac{p}{\omega}\right) = \frac{1}{14}$ nearly, and the terms in (48) hitherto neglected are of the order $14\omega\beta$, which we may take to be small compared with ω^2 ; so that, with somewhat less accuracy as the period increases, it will be true of the tides of shorter period but not of the semiannual and annual tides, that the "quasi-rigidity" set up in the liquid by the vertex-motion is not such as to interfere sensibly with calculations of the resistance of the Earth to tidal distortion founded on the neglect of the spin. Comparing this result with the investigations of Thomson and Darwin on the rigidity of the Earth, we seem to be justified in concluding that the Earth is not such a shell as is here considered.

18. To consider more generally the problem of the rotating shell, let us refer the motion of the liquid to moving axes which rotate with angular velocity ω about the axis z fixed in space.

The equations of motion are three, such as

$$\frac{\partial u}{\partial t} - \omega v + U \frac{\partial u}{\partial x} + V \frac{\partial u}{\partial y} + W \frac{\partial u}{\partial z} = \frac{\partial}{\partial x} \left(V' - \frac{p_1}{\sigma} \right), *$$

where U, V, W are the rates at which the coordinates of a fluid particle are changing, so that

$$U = u + \omega y, \quad V = v - \omega x, \quad W = w.$$

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* Greenhill, article on "Hydromechanics," in *Encyclopædia Britannica*.

Thus the equations may be written

$$\begin{aligned}\frac{\partial U}{\partial t} - \omega (V + \omega x) + U \frac{\partial U}{\partial x} + V \left(\frac{\partial U}{\partial y} - \omega \right) + w \frac{\partial U}{\partial z} &= \frac{\partial}{\partial x} \left(V' - \frac{p_1}{\sigma} \right), \\ \frac{\partial V}{\partial t} + \omega (U - \omega y) + U \left(\frac{\partial V}{\partial x} + \omega \right) + V \frac{\partial V}{\partial y} + w \frac{\partial V}{\partial z} &= \frac{\partial}{\partial y} \left(V' - \frac{p_1}{\sigma} \right), \\ \frac{\partial w}{\partial t} + U \frac{\partial w}{\partial x} + V \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} &= \frac{\partial}{\partial z} \left(V' - \frac{p_1}{\sigma} \right).\end{aligned}$$

Let us suppose the liquid to rotate as if rigid about the axis z , and, superposed on this rotatory motion, let there be a small oscillatory motion, then U , V , w are all small, and we have

$$\left. \begin{aligned}\frac{\partial U}{\partial t} - 2\omega V &= \frac{\partial}{\partial x} \left[V' - \frac{p_1}{\sigma} + \frac{1}{2}\omega^2 (x^2 + y^2) \right] = \frac{\partial \psi}{\partial x} \text{ say} \\ \frac{\partial V}{\partial t} + 2\omega U &= \frac{\partial \psi}{\partial y} \\ \frac{\partial w}{\partial t} &= \frac{\partial \psi}{\partial z}\end{aligned} \right\} \dots\dots (49)$$

and the equation of continuity is

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\begin{aligned}\text{Thus} \quad \nabla^2 \psi &= -2\omega \frac{\partial V}{\partial x} + 2\omega \frac{\partial U}{\partial y}, \\ \frac{\partial}{\partial t} \nabla^2 \psi &= -2\omega \frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial y} - 2\omega U \right) + 2\omega \frac{\partial}{\partial y} \left(\frac{\partial \psi}{\partial x} + 2\omega V \right) \\ &= 4\omega^2 \left(-\frac{\partial w}{\partial z} \right); \end{aligned}$$

$$\text{hence} \quad \frac{\partial^2}{\partial t^2} (\nabla^2 \psi) = -4\omega^2 \frac{\partial^2 \psi}{\partial z^2} \dots\dots\dots (50).*$$

Let us now suppose U , V , $w \propto e^{\mu t}$, then

$$\nabla^2 \psi - \frac{4\omega^2}{p^2} \frac{\partial^2 \psi}{\partial z^2} = 0,$$

$$\text{or} \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z'^2} = 0 \quad \dots\dots\dots (51),$$

$$\text{where} \quad z'^2 \left(1 - \frac{4\omega^2}{p^2} \right) = z^2.$$

* This equation is due to Poincaré, *Acta Mathematica*, Vol. VII., 1885, p. 356.
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19. We have to solve the differential equation (50) within a sphere

$$x^2 + y^2 + z^2 = b^2.$$

This is the same thing as solving the equation (51) within the quadric

$$\frac{x^2 + y^2}{b^2} + \frac{z^2}{b^2} \left(1 - \frac{4\omega^2}{p^2}\right) = 1.$$

Let $x^2 + y^2 = \rho^2$, and $\tan \phi = \frac{y}{x}$, then the equation is

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \phi^2} = 0 \dots\dots\dots (52).$$

(1) Suppose $1 - \frac{4\omega^2}{p^2}$ positive, then the surface within which we have to solve $0 = \nabla^2 \psi$ is a spheroid of revolution about its greatest axis.

Take conjugate functions, such that

$$z' + i\rho = c \cosh (\alpha + i\beta),$$

where $c^2 = b^2 \frac{4\omega^2}{p^2 - 4\omega^2}$, and write $b_1^2 = \frac{b^2 p^2}{p^2 - 4\omega^2}$.

Then, taking $\psi \propto e^{i\omega t}$, we find, for ψ as a function of α, β , the equation

$$\frac{\partial^2 \psi}{\partial \alpha^2} + \frac{\partial^2 \psi}{\partial \beta^2} + \coth \alpha \frac{\partial \psi}{\partial \alpha} + \cot \beta \frac{\partial \psi}{\partial \beta} - s^2 \psi \left(\frac{1}{\sinh^2 \alpha} + \frac{1}{\sin^2 \beta} \right) = 0.$$

Let $\psi = S \cdot T$, where S is a function of α only, and T of β only; then

$$\frac{d^2 S}{d\alpha^2} + \coth \alpha \frac{dS}{d\alpha} - \frac{s^2}{\sinh^2 \alpha} S - i(i+1) S = 0,$$

and $\frac{d^2 T}{d\beta^2} + \cot \beta \frac{dT}{d\beta} - \frac{s^2}{\sin^2 \beta} T + i(i+1) T = 0,$

where $i(i+1)$ is some constant.

Put $\cosh \alpha = \nu$, $\cos \beta = \mu$, then

$$\left. \begin{aligned} \frac{d}{d\nu} \left[(1-\nu^2) \frac{dS}{d\nu} \right] + \left[i(i+1) - \frac{s^2}{1-\nu^2} \right] S &= 0 \\ \frac{d}{d\mu} \left[(1-\mu^2) \frac{dT}{d\mu} \right] + \left[i(i+1) - \frac{s^2}{1-\mu^2} \right] T &= 0 \end{aligned} \right\} \dots\dots\dots (53),$$

and we can obtain solutions in finite terms by taking i integral. In this case, the equations for S and T are the equations of tesseral

harmonics, the argument ν of S being always finite and greater than unity.

$$\text{Thus} \quad \psi = \sum_{s < i} \sum_i P_i^{(s)}(\nu) P_i^{(s)}(\mu) e^{is\phi} * \dots\dots\dots (54).$$

Since $\mu = \frac{z'}{b_1} = \frac{z}{b}$, the factor $P_i^{(s)}(\mu) e^{is\phi}$ is a spherical surface harmonic on the sphere of radius b . The quantity ν is given at the surface by $c^2(\nu^2 - 1) = b^2$, so that

$$\nu = \frac{p}{2\omega} \text{ at the surface.}$$

(2) Let $1 - \frac{4\omega^2}{p^2}$ be negative.

$$\text{Write} \quad b_1^2 \left(\frac{4\omega^2}{p^2} - 1 \right) = b^2,$$

then the surface into which we project the sphere is the one-sheeted hyperboloid of revolution

$$\rho^2/b^2 - z'^2/b_1^2 = 1,$$

$$\text{and taking} \quad c^2 = b^2 + b_1^2 = 4\omega^2 b^2 / (4\omega^2 - p^2),$$

$$\text{we write} \quad \rho + iz' = c \cosh(\alpha + i\beta),$$

and shall take this to represent a system of conjugate functions. The projection is imaginary, and, in relation to the motion, z' is imaginary. For the purpose of solving equation (50), however, within the real surface of the sphere, we must regard this as solving the equation (51) with z' supposed real for the space on one side of the surface of the hyperboloid also supposed real. We shall then treat the above as a system of conjugate functions, and suppose that we are solving the equation (51) with real variables for the space on one side of this surface.

Supposing $\psi \propto e^{is\phi}$, and multiplying (52) by $\frac{\partial(z', \rho)}{\partial(\alpha, \beta)}$, we have

$$\frac{\partial^2 \psi}{\partial \alpha^2} + \frac{\partial^2 \psi}{\partial \beta^2} + \tanh \alpha \frac{\partial \psi}{\partial \alpha} - \tan \beta \frac{\partial \psi}{\partial \beta} - s^2 \psi (\sec^2 \beta - \operatorname{sech}^2 \alpha) = 0;$$

* I write $P_i^{(s)}(x)$ for the associated function of the first kind, proportional to Ferrers's $T_i^{(s)}(x)$, and to Heine's $P_{(s)}^{(i)}(x)$; the numerical determination is unimportant in this place.

taking $\mu' = \sin \beta$, and $\nu' = \sinh \alpha$, we have $\psi = S \cdot T$, where

$$\begin{aligned} \frac{d}{d\mu'} \left\{ (1 - \mu'^2) \frac{dT}{d\mu'} \right\} + \left\{ i(i+1) - \frac{s^2}{1 - \mu'^2} \right\} T &= 0, \\ \frac{d}{d\nu'} \left\{ (1 + \nu'^2) \frac{dS}{d\nu'} \right\} - \left\{ i(i+1) - \frac{s^2}{1 + \nu'^2} \right\} S &= 0. \end{aligned}$$

The solutions are $P_i^{(s)}(\sin \beta)$, $P_i^{(s)}(\sinh \alpha)$.

These solutions are obtained by supposing α a real variable; but, when z' is imaginary, α becomes imaginary, so that $\sinh \alpha$ is real in the motion, and on the surface

$$\sinh \alpha = \frac{z'}{b_1} = \frac{z}{b} = \mu.$$

Hence, taking μ, μ' new variables defined by the real equations

$$\rho^2 = c^2 (1 - \mu^2)(1 - \mu'^2), \quad z = c\mu\mu',$$

μ will be the ordinary μ at the surface, and the solutions will be

$$T = P_i^{(s)}(\mu'), \quad S = P_i^{(s)}(\mu) \dots \dots \dots (55),$$

which remain finite for the space considered. Thus

$$\psi = \Sigma \Sigma P_i^{(s)}(\mu') P_i^{(s)}(\mu) e^{i\omega t} \dots \dots \dots (56),$$

where at the surface $c^2 (1 - \mu'^2) = b^2$, so that $\mu' = p/2\omega$.

20. The normal surface velocity is given by (36); it is also to be found as follows. We have, from (49),

$$\left. \begin{aligned} Uip - 2\omega V &= \frac{\partial \psi}{\partial x} \\ Vip + 2\omega U &= \frac{\partial \psi}{\partial y} \\ wip &= \frac{\partial \psi}{\partial z} \end{aligned} \right\};$$

hence, if l, m, n be the direction-cosines of the outward-drawn normal to the sphere,

$$\begin{aligned} &(4\omega^2 - p^2)(lU + mV + nw) \\ &= ip \left\{ l \frac{\partial \psi}{\partial x} + m \frac{\partial \psi}{\partial y} + n \left(1 - \frac{4\omega^2}{p^2} \right) \frac{\partial \psi}{\partial z} \right\} + 2\omega \left(l \frac{\partial \psi}{\partial y} - m \frac{\partial \psi}{\partial x} \right) \\ &= \frac{ip}{b} \left(x \frac{\partial \psi}{\partial x} + y \frac{\partial \psi}{\partial y} + z' \frac{b^2}{b_1^2} \frac{\partial \psi}{\partial z} \right) + \frac{2\omega}{b} \left(x \frac{\partial \psi}{\partial y} - y \frac{\partial \psi}{\partial x} \right) \end{aligned}$$

$$\begin{aligned}
 &= \varphi b \left(\frac{x}{b^2} \frac{\partial \psi}{\partial x} + \frac{y}{b^2} \frac{\partial \psi}{\partial y} + \frac{z}{b_1^2} \frac{\partial \psi}{\partial z} \right) + \frac{2\omega}{b} \left(x \frac{\partial \psi}{\partial y} - y \frac{\partial \psi}{\partial x} \right) \\
 &= \varphi b \frac{1}{p'} \frac{\partial \psi}{\partial n'} + \frac{2\omega}{b} \frac{\partial \psi}{\partial \phi} \dots\dots\dots (57),
 \end{aligned}$$

where p' is the perpendicular from the centre on the tangent plane to the surface into which we project the sphere, and $\frac{\partial}{\partial n'}$ denotes differentiation along the normal to this surface drawn outwards. The above holds whether the surface into which we project the sphere be a spheroid or an hyperboloid.

The normal velocity at the surface of the sphere can always be expanded in a sum of spherical surface harmonics.

The requisite normal velocity is that of the inner surface of the shell, which is given by equation (36) with (37) and (38).

(1) Take $1 - \frac{4\omega^2}{p^2}$ positive.

We have $p' dn' = c \cosh \alpha d(c \cosh \alpha) = c^2 \nu d\nu$,

also $lU + mV + nw = \Sigma \Sigma C_i^{(s)} P_i^{(s)}(\mu) e^{i\phi}$,

where $C_i^{(s)}$ is a complex constant. This is to be equated to the expression in (36). Also

$$\psi = \Sigma \Sigma A_i^{(s)} P_i^{(s)}(\nu) P_i^{(s)}(\mu) e^{i\phi};$$

hence, by (57),

$$\left[\frac{\varphi b}{c^2} \frac{1}{\nu} \frac{d}{d\nu} P_i^{(s)}(\nu) + \frac{2i\omega}{b} P_i^{(s)}(\nu) \right] A_i^{(s)} = (4\omega^2 - p^2) C_i^{(s)},$$

where

$$\nu = \frac{p}{2\omega},$$

and thus $\psi = \Sigma \Sigma \frac{(4\omega^2 - p^2) C_i^{(s)} P_i^{(s)}(\nu) P_i^{(s)}(\mu) e^{i\phi} e^{i\phi}}{\left[\frac{\varphi b}{c^2} \frac{1}{\nu} \frac{d}{d\nu} P_i^{(s)}(\nu) + \frac{2i\omega}{b} P_i^{(s)}(\nu) \right]_{\nu=p/2\omega}} \dots\dots\dots (58).$

(2) Take $1 - \frac{4\omega^2}{p^2}$ negative.

Then $p' dn' = -c \cos \beta d(c \cos \beta) = c^2 \mu' d\mu'$,

since μ' increases as we go outwards from the centre of the sphere.

Taking $\psi = \Sigma \Sigma A_i^{(s)} P_i^{(s)}(\mu') P_i^{(s)}(\mu) e^{i\phi}$,

we have $\left[\frac{\varphi b}{c^2} \frac{1}{\mu'} \frac{d}{d\mu'} P_i^{(s)}(\mu') + \frac{2i\omega}{b} P_i^{(s)}(\mu') \right] A_i^{(s)} = (4\omega - p^2) C_i^{(s)},$

where

$$\mu' = \frac{p}{2\omega};$$

$$\text{thus} \quad \psi = \Sigma \Sigma \frac{(4\omega^2 - p^2) C_i^{(s)} P_i^{(s)}(\mu') P_i^{(s)}(\mu) e^{i\phi} e^{i\omega t}}{\left[\frac{\gamma b}{c^2} \frac{1}{\mu'} \frac{d}{d\mu'} P_i^{(s)}(\mu') + \frac{2i s \omega}{b} P_i^{(s)}(\mu') \right]_{\mu' = p/2\omega}} \dots\dots\dots (59),$$

where the summation extends to all positive integral values of s which are not greater than i , and to all positive integral values of i .

21. Equation (59) expresses exactly the effect of the rotation, and gives us the means of finding the fluid motion completely, when the inner surface is deformed periodically into an harmonic spheroid. Since, however, the force necessary to keep the amplitude of the inequalities all small is only provided in case $\omega^2 C_i^{(s)}$ is small, we have to evaluate this expression for the different cases that may arise.

(1) Take ω small compared with p , then $\frac{p}{2\omega}$ approaches to an infinite limit, and we have to find the value of

$$\frac{P_i^{(s)}\left(\frac{p}{2\omega}\right)}{\left[\frac{\gamma b}{c^2} \frac{1}{x} \frac{d}{dx} P_i^{(s)}(x) + \frac{2s\omega}{b} P_i^{(s)}(x) \right]_{x = p/2\omega}} = \gamma_i^{(s)} \text{ say,}$$

where

$$\frac{p}{2\omega} = \infty.$$

We take for determination of $P_i^{(s)}(x)$,

$$P_i^{(s)}(x) = (x^2 - 1)^{\frac{1}{2}s} \frac{d^s}{dx^s} P_i(x).$$

Then this is $\frac{2i!}{2^i \cdot i! \cdot i - s!}$ times Heine's $P_{(s)}^{(i)}(x)$,* whose limit for $x = \infty$ is x^i , and we have

$$\begin{aligned} \frac{d}{dx} P_i^{(s)}(x) &= (x^2 - 1)^{\frac{1}{2}s} \frac{d^{s+1}}{dx^{s+1}} P_i(x) + s x (x^2 - 1)^{\frac{1}{2}s-1} \frac{d^s}{dx^s} P_i(x) \\ &= \frac{1}{\sqrt{x^2 - 1}} P_i^{(s+1)}(x) + \frac{s x}{x^2 - 1} P_i^{(s)}(x). \end{aligned}$$

The limit of this, when x is ∞ , is the same as that of

$$\left(\frac{i-s}{\sqrt{x^2 - 1}} + \frac{s x}{x^2 - 1} \right) P_i^{(s)}(x).$$

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* Heine, *Handbuch der Kugelfunctionen*, Ch. IV., p. 206.

$$\begin{aligned}\text{Thus } [\gamma_i^{(s)}]_{x=\infty} &= \frac{1}{\frac{pb}{b^3}(x^2-1) \frac{1}{x} \left(\frac{i-s}{\sqrt{x^2-1}} + \frac{sx}{x^2-1} \right) + \frac{2s\omega}{b}} \\ &= \frac{1}{\frac{p}{b}i + \frac{2s\omega}{b}} = \frac{b}{ip} \text{ ultimately.}\end{aligned}$$

In this case

$$\psi = \Sigma \Sigma_i \frac{pb}{i} O_i^{(s)} P_i^{(s)}(\mu) e^{i\phi} e^{i\omega t}$$

at the surface $r = b$.

(2) When p and ω are comparable, $(4\omega^2 - p^2) O_i^{(s)}$ is of the order neglected, $P_i^{(s)}\left(\frac{p}{2\omega}\right)$ and its differential coefficients are finite, and the rotation has no effect.

(3) When ω is great compared with p , we have to evaluate

$$\frac{(4\omega^2 - p^2) P_i^{(s)}(x)}{\left(\frac{pb}{c^2} \frac{1}{x} \frac{d}{dx} P_i^{(s)}(x) + \frac{2s\omega}{b} P_i^{(s)}(x)\right)},$$

where $x = \frac{p}{2\omega}$ is to be made very small.

There will be two cases according as $P_i^{(s)}(x)$ does or does not vanish with x . If it does not, then $\frac{d}{dx} P_i^{(s)}$ is proportional to x when $x = 0$; otherwise $P_i^{(s)}(x)$ is proportional to x and $\frac{d}{dx} P_i^{(s)}(x)$ does not vanish for $x = 0$.

We take for determination of $P_i^{(s)}(x)$ the form

$$P_i^{(s)}(x) = (1-x^2)^{is} \frac{d^s}{dx^s} P_i(x).$$

The lowest terms are then

$$(-1)^{i(i-s)} (1-x^2)^{is} \left\{ \frac{2i!}{2^i \cdot i!} \frac{1}{2 \cdot 4 \dots (i-s)(2i-1)(2i-3) \dots (i+s+1)} \right. \\ \left. + \text{term in } x^2 \right\} \text{ when } i-s \text{ is even,}$$

and

$$(-1)^{i(i-s-1)} (1-x^2)^{is} \left\{ \frac{2i!}{2^i \cdot i!} \frac{x}{2 \cdot 4 \dots (i-s-1)(2i-1)(2i-3) \dots (i+s)} \right. \\ \left. + \text{term in } x^3 \right\} \text{ when } i-s \text{ is odd.}$$

Thus, when $i-s$ is even the limit is $\frac{2\omega b}{s}$, and when $i-s$ is odd it is zero, and

$$\psi = \Sigma \Sigma -i \frac{2\omega b}{s} C_i^{(s)} P_i^{(s)}(\mu) e^{is\phi}$$

at the surface $r = b$, when $i-s$ is even.

The case of a zonal harmonic solution requires special attention. If ψ contain a zonal harmonic term, the term of (57) in $\frac{\partial \psi}{\partial \phi}$ will be wanting, and the term containing s in (59) will also vanish; thus

$$\psi = \Sigma \frac{(4\omega^2 - p^2) C_i P_i(\mu') P_i(\mu)}{\left(\frac{ipb}{c^3} \frac{1}{\mu'} \frac{d}{d\mu'} P_i(\mu') \right)_{\mu' = p/2\omega}},$$

and we have to evaluate this when $p/2\omega = 0$.

The only important case is when i is even, as when i is odd $P_i(\mu')$ vanishes with μ' .

$$\text{Taking } i \text{ even, } P_i(\mu') = (-1)^{i/2} \frac{1 \cdot 3 \dots (i-1)}{2 \cdot 4 \dots i} \text{ when } \mu' = 0,$$

$$\text{and } \frac{1}{\mu'} \frac{d}{d\mu'} P_i(\mu') = (-1)^{i/2-1} 2 \frac{3 \cdot 5 \dots (i+1)}{2 \cdot 2 \cdot 4 \dots (i-2)},$$

$$\text{so that } \psi = \Sigma - \frac{4\omega^2 b}{p} \frac{1}{i(i+1)} C_i P_i(\mu) \text{ at the surface,}$$

$$\text{or } \psi = i \frac{2\omega}{p} \Sigma \frac{2\omega b}{i(i+1)} C_i P_i(\mu),$$

and these terms produce an important effect when $\frac{2\omega}{p}$ is great.

22. We are now in a position to calculate the surface-traction on the inner surface from the fluid pressure.

The pressure equation in the fluid is

$$\frac{p_1}{\sigma} = V' + \frac{1}{2}\omega^2(x^2 + y^2) - \psi,$$

where, at the surface $r = b$,

$$V' = \Sigma \left(U_i + W_i + \frac{V_i}{b^{2i+1}} \right) + \text{non-periodic terms.}$$

The periodic terms in $(x^2 + y^2)$ will be proportional to the amplitude of the harmonic inequality, and this multiplied by ω^2 we neglect.

Hence the surface-tractions at the surface $r = b$ are

$$\frac{x}{b} \sigma (V' - \psi), \quad \frac{y}{b} \sigma (V' - \psi), \quad \frac{z}{b} \sigma (V' - \psi) \dots \dots \dots (60),$$

where the ψ term is to be retained in case it becomes important through the smallness or greatness of the ratio p/ω .

The surface values of $V' - \psi$ are in every case surface harmonics, the same as occur in the typical terms representing the harmonic inequalities, so that these surface values are of the form

$$\frac{\sigma}{b^{i+1}} x \cdot r^i (V' - \psi),$$

in which $r^i (V' - \psi)$ is a spherical solid harmonic of degree i . Calling this solid harmonic R_i and treating it by the formula

$$x R_i = \frac{1}{2i+1} \left[r^2 \frac{dR_i}{dx} - r^{2i+3} \frac{d}{dx} \left(\frac{R_i}{r^{2i+1}} \right) \right],$$

in which r is to be put equal to b after differentiation, we find for this surface-traction a series of terms of the same form as those in (40).

The surface conditions are then to be found as follows:— Fr , Gr , Hr at the inner surface $r = b$ have just been found; at the same surface their values are given by adding the terms of (19), (40), and (45), in which we put $r = b$. These values are to be equated. At the surface $r = a$, the terms contributed by (19), (40), (45), in which r is put $= a$, are to be added and equated to zero. We may then, by a process similar to that in Art. 7, obtain from these two sets of equations four equations between the harmonics ω_i , ϕ_i , Ω_i , Φ_i , and W_i .

This will be done by simply dropping the $\frac{\partial}{\partial x}$, ... and powers of r , and taking separately the terms in which occurred $\frac{\partial R_i}{\partial x}$ and those in which occurred $\frac{\partial}{\partial x} \left(\frac{R_i}{r^{2i+1}} \right)$, R_i denoting any one harmonic. These four equations give the ratios of the harmonics ω_i , ϕ_i , Ω_i , Φ_i to W_i , and the process we have gone through shews that any term $P_i^{(s)}(\mu) e^{i\theta}$ in W_i yields a term of the same form in each of the harmonics expressing the solution.

The free vibrations under gravity and "centrifugal force" would be found by putting $W_i = 0$, and the frequency equations would not differ in form from those which hold when there are no acting forces.

23. As an example, let us take vibrations under slow rotation only. Then, ω^2 being small, we need not consider the terms of Art. 14, and we have simply to find the forms which have to be substituted for those of Art. 8.

Let us take each of the harmonics $\phi_i, \Phi_i, \omega_i, \Omega_i$ to consist of a single term proportional to $r^i P_i^{(s)}(\mu) e^{i s t}$, and in future let $\phi_i \dots$ denote the complex constants multiplying this expression in what we before called $\phi_i \dots$. Then

$$\begin{aligned} & (lU + mV + nw) \\ &= ip e^{i s t} \{ i b^{i-1} [\psi_i(kb) \phi_i + \Psi_i(kb) \Phi_i] + b^i [e'_i \omega_i + E'_i \Omega_i] \} \\ &= O_i^{(s)}, \text{ dropping the harmonic and time factors.} \end{aligned}$$

Let $\frac{p}{2\omega} > 1$, and take

$$\frac{(4\omega^2 - p^2) P_i^{(s)} \left(\frac{p}{2\omega} \right) O_i^{(s)}}{\left[\frac{ipb}{c^3} \frac{1}{\nu} \frac{d}{d\nu} P_i^{(s)}(\nu) + \frac{2is\omega}{b} P_i^{(s)}(\nu) \right]_{\nu=p/2\omega}} = \delta_i^{(s)} O_i^{(s)},$$

where $c^2 = b^2 \frac{4\omega^2}{p^2 - 4\omega^2}$; then the component surface-tractions are

$$-\sigma \delta_i^{(s)} \frac{O_i^{(s)}}{b^i} \frac{x}{r} r^i P_i^{(s)}(\mu) e^{i s t}, \dots$$

Calling the solid harmonic Z_i , we have

$$-\sigma \delta_i^{(s)} \frac{O_i^{(s)}}{b^i} \frac{x}{r} Z_i, \dots,$$

where $xZ_i = \frac{b^2}{2i+1} \left[\frac{\partial Z_i}{\partial x} - b^{2i+1} \frac{\partial}{\partial x} \left(\frac{Z_i}{r^{2i+1}} \right) \right];$

therefore $iFr / \sigma p \delta_i^{(s)} =$

$$\begin{aligned} & \left[\frac{i}{b} \psi_i(kb) \frac{b^2}{2i+1} \left\{ \frac{\partial \phi_i}{\partial x} - b^{2i+1} \frac{\partial}{\partial x} \left(\frac{\phi_i}{r^{2i+1}} \right) \right\} \right. \\ & \quad + \frac{i}{b} \Psi_i(kb) \frac{b^2}{2i+1} \left\{ \frac{\partial \Phi_i}{\partial x} - b^{2i+1} \frac{\partial}{\partial x} \left(\frac{\Phi_i}{r^{2i+1}} \right) \right\} \\ & \quad + \frac{b^2}{2i+1} e'_i \left\{ \frac{\partial \omega_i}{\partial x} - b^{2i+1} \frac{\partial}{\partial x} \left(\frac{\omega_i}{r^{2i+1}} \right) \right\} \\ & \quad \left. + \frac{b^2}{2i+1} E'_i \left\{ \frac{\partial \Omega_i}{\partial x} - b^{2i+1} \frac{\partial}{\partial x} \left(\frac{\Omega_i}{r^{2i+1}} \right) \right\} \right], \end{aligned}$$

with similar equations for $G.r$ and $H.r$, where the $\phi_i \dots$ now stand for solid harmonics.

Hence, just as in Art. 8, we have for $\phi_i \dots$ the complex constants,

$$\left. \begin{aligned} & \left(a'_i + \sigma_i p \delta_i^{(s)} \frac{b^2}{2i+1} e'_i \right) \omega_i + \left(c'_i + \sigma_i p \delta_i^{(s)} \frac{ib}{2i+1} \psi_i(kb) \right) \phi_i \\ & + \left(A'_i + \sigma_i p \delta_i^{(s)} \frac{b^2}{2i+1} E'_i \right) \Omega_i + \left(C'_i + \sigma_i p \delta_i^{(s)} \frac{ib}{2i+1} \Psi_i(kb) \right) \Phi_i \end{aligned} \right\} = 0,$$

$$\left. \begin{aligned} & \left(b'_i - \sigma_i p \delta_i^{(s)} \frac{b^{2i+3}}{2i+1} e'_i \right) \omega_i + \left(d'_i - \sigma_i p \delta_i^{(s)} \frac{ib^{2i+2}}{2i+1} \psi_i(kb) \right) \phi_i \\ & + \left(B'_i - \sigma_i p \delta_i^{(s)} \frac{b^{2i+3}}{2i+1} E'_i \right) \Omega_i + \left(D'_i - \sigma_i p \delta_i^{(s)} \frac{ib^{2i+2}}{2i+1} \Psi_i(kb) \right) \Phi_i \end{aligned} \right\} = 0,$$

where it is to be observed that $\delta_i^{(s)}$ is a real quantity

$$= \frac{(4\omega^2 - p^2) P_i^{(s)} \left(\frac{p}{2\omega} \right)}{\left[\frac{pb}{c^2} \frac{1}{\nu} \frac{d}{d\nu} P_i^{(s)}(\nu) + \frac{2s\omega}{b} P_i^{(s)}(\nu) \right]_{\nu=p/2\omega}}.$$

The equations just written, together with the first two of equations (24), give the ratios of the amplitudes of the several harmonics ω_i , Ω_i , ϕ_i , Φ_i , and, on elimination of these, give a determinantal equation for the frequency.

The above reduce to the equations (29) when $\omega=0$, for in this case

$$\delta_i^{(s)} = -\frac{1}{i} pb,$$

as we saw before.

Summary.

The main purpose of this paper is to throw some further light on the question of the rigidity of the earth. One test of the earth's rigidity is the existence of tides. If the bodily tides that would take place in a completely liquid earth exceed sensibly those in the actual earth, there must be sensible ocean tides on the latter. In Thomson and Tait's *Natural Philosophy*, Art. 835 sq., we have an estimate of the earth's rigidity founded on an equilibrium theory of the bodily tides, and the conclusion is that the interior of the earth must be solid, and composed of very rigid material. Sir William Thomson, in an address to Section A of the British Association in 1876, pointed out that this is conclusive against that theory of the internal constitution of the earth, which assumes a solid crust to rest on an inner mass of liquid. It will be so unless the "quasi-rigidity" set up in the liquid by the vortex-motion can interfere with the calculation. In other words,

Sir William's conclusion will be established if an equilibrium theory holds.

We have at our disposal mathematical methods which will enable us to construct a dynamical theory of the bodily tides in an earth whose constitution is that just described. Professor Lamb, in his papers in Vols. XIII. and XIV. of the *Proceedings* of this Society, has taught us what are the modes of vibration of the crust supposed free; and M. Poincaré, in his memoir in the *Acta Mathematica*, Vol. VII., has devised a method for treating the oscillations of a rotating mass of liquid. The problem becomes a special case of that stated at the commencement of this paper. The analysis at our command makes it possible to replace the tidal disturbing forces by any system of bodily forces having a potential expressible in spherical harmonic series.

I have recapitulated, in the first place, the theory of the free vibrations of spheres and spherical shells in order to introduce a modification into Professor Lamb's analysis. The effect of this is to make all the spherical harmonics which occur solid harmonics of positive degree, which facilitates the subsequent applications. This occupies Arts. 1 to 8. The shell is supposed filled with liquid in order to shew in a simple case how the surface-tractions on the inner surface of the shell are to be calculated. The frequency-equations are of the same form as when there is no liquid, coming out in the form of determinants. The equation for the purely tangential vibrations is unaltered, but the elements of two of the rows of the determinant which occurs in the frequency-equation for vibrations of other types are all changed by the addition of certain terms, and the amplitudes of the harmonic disturbances have their ratios altered.

In treating the forced vibrations, we suppose that the external disturbing forces have a potential expressible in spherical harmonic series. The bodily forces arise (1) from the gravitation of the shell and liquid, (2) from the attractions of the harmonic inequalities caused by the external disturbing forces, (3) from the "centrifugal force" when the shell rotates, and (4) include also the external forces themselves. We easily find the particular integrals of the equations of vibration which correspond to forces (2) and (4), and the terms in the surface-tractions on any concentric sphere which are contributed by these particular integrals; this is done in Arts. 10—12. The forces of types (1) and (3) are not periodic. The strain produced by the gravitation included in (1) is purely symmetrical about the centre of the sphere, and the same is true of one part of the forces (3). The effect of the unsymmetrical part of (3) I have not calculated, except for the liquid, as for the problem in hand the rotation is supposed

slow. The symmetrical parts of the forces (1) are of importance because they contribute certain terms to the surface-tractions. The attraction of the shell and liquid is great compared with that of the harmonic inequality, or with the external disturbing force. The consequent strains and stresses are in the same way not of comparable magnitudes. It is sufficient to suppose the surface-tractions arising from the smaller forces to fulfil conditions at the surfaces of the mean inner and outer spheres; it is necessary in the case of the greater forces to remember that the conditions are actually fulfilled at the surfaces of the harmonic inequalities. In the well-known case of incompressible material, the condition that the surface of the harmonic inequality is free can be satisfied by applying to the surface of the mean sphere a normal traction equal to the weight of the harmonic inequality. The formulæ for the tractions that must in general be applied to the surfaces of the mean spheres, in order that the conditions may be fulfilled, are given in Arts. 13 and 14, and in Art. 15 it is verified that the known theory for incompressible material is included in these formulæ. It may be remarked that, in general, the *tangential* stresses are of the same order of magnitude as the *normal* stresses.

We have now only to calculate the fluid pressure in order to form the stress-conditions, and these conditions will determine the amplitudes of the inequalities or tides produced.

Before proceeding to apply Poincaré's general method, it seemed worth while to try under what conditions the oscillations of the fluid could be reduced to an irrotational oscillation combined with periodic changes in the rate and axis of rotation; and it was found that it was necessary and sufficient that the disturbing function should be a spherical solid harmonic of order 2. This is precisely the most important case of tidal forced vibrations. In treating the liquid, "centrifugal force" is eliminated, and the rotation treated kinematically. The pressure-equation for the disturbed motion is formed without difficulty, and makes it possible to answer the question as to the effect of the "quasi-rigidity," due to vortex-motion, on the "tidal effective rigidity." It appears that this effect is negligible when the period of the tide is of the same order of magnitude as the period of rotation, but that it becomes of great importance as the ratio of the former to the latter increases; so that it will vitiate an equilibrium theory for long-period tides, but not for short-period tides. On the other hand, it is only in the case of long-period tides that an equilibrium theory of the ocean tides gives a good result. The very long- and very short-period tides are thus beset with difficulties of a different nature: for the first, the vortex-motion of the supposed in-

ternal liquid produces great tidal effective rigidity; for the second, the inertia of the external liquid covering comes into play. Both theories point to the fortnightly tide as the one which alone can settle the question. So many difficulties have occurred in the work of tide-registering that the Tidal Committee of the British Association appears to be still doubtful whether there really is an appreciable fortnightly tide. If there is, we shall be entitled to say that the tidal effective rigidity of the earth is too great to allow us to suppose it to consist of a liquid mass covered with a thin solid crust.

The simple method of treating the liquid just described will only be permissible in case the disturbing function contains only harmonics of order 2, and for the treatment of the general case we must adopt the method of Poincaré's memoir. It was there shown that the oscillations of a mass of fluid about that state of steady motion in which it rotates as if rigid, can be determined by solving Laplace's equation subject to a condition given over a surface which may be described as a projection of the surface of the liquid. In the case before us, the spherical surface of the liquid is to be projected into a quadric of revolution about the axis of rotation, which is a spheroid or an hyperboloid of one sheet according as the period of rotation is greater or less than twice the period of the forced oscillation. The required solution ψ of Laplace's equation can thus be expressed by means of spheroidal harmonics, and the surface-values of these are proportional to spherical surface harmonics on the sphere of which the spheroid or hyperboloid is a projection. The condition to be satisfied by ψ at the surface of the quadric is obtained thus:—the radial velocity at the surface of the sphere can be expressed in terms of differential coefficients of ψ ; it can also be expressed in terms of the harmonics which occur in the complementary and particular solutions of the equations of vibration. These two expressions are to be equated. Now, it appears that if we take one tesseral harmonic term as the typical surface term in ψ , the equation will contain the other tesseral harmonic of the same degree and rank, but no other harmonics. It is, therefore, better to take for the representative harmonic term a complex harmonic containing both the harmonics of the same degree and rank, and after solution retain only the real part of our equations. When this is done it appears that if the disturbing function consist of a series of harmonic terms of the form $r^i P_i^{(n)}(\mu) e^{i n t}$ multiplied by complex constants, then all the harmonics which occur in the solution are of the same form, and each term of this form provokes a motion represented by surface harmonics of this form and of the same degree and rank as the term considered.

When the function ψ is known the whole motion is known and the

pressure. Now, ψ involves linearly the harmonics which occur in the complementary and particular solutions of the equations of vibration, the pressure involves the same harmonics linearly, and the result is the determination of the surface-tractions applied to the inner surface in terms of these harmonics. We have already another expression for these calculated from the strain. Equating the two expressions, we should obtain sufficient equations, with those which hold at the free surface, to express all the unknown harmonics in terms of those occurring in the expression for the disturbing potential.

This completes the analysis for the bodily tides of any order in the system considered.

As an example, I have considered free vibrations of the system supposed rotating slowly, but free from gravitation, and have verified that the solution reduces to that which we already know for free vibrations when the rotation is also annulled.

The results in the general case bear out those arrived at by other methods when the disturbing function is of order 2, shewing how the effect of the rotation in altering an equilibrium solution is null, unless the period of the disturbance is long compared with that of the rotation.

On the Volume generated by a Congruency of Lines.

By R. A. ROBERTS, M.A.

[Read Feb. 9th, 1888.]

In this paper I propose to investigate an expression for the volume generated by a congruency of lines, and to deduce therefrom an extension of Abel's theorem to double integrals.

Taking rectangular coordinates, we may write the equation of a line in the form

$$x = pz + \alpha, \quad y = qz + \beta \dots\dots\dots(1),$$

and then, if the line belongs to a congruency, that is, if it varies subject to two conditions, we may suppose that α, β are given as functions of p, q . Thus for any point in space, drawing through it a line of the system, we may regard x, y as functions of p, q , and z ; so that, if the element of volume $dx dy dz$ be expressed in terms of the