The Normal Law of Error

Author(s): C. S. Jackson

Source: *The Mathematical Gazette*, Vol. 3, No. 54 (Dec., 1905), pp. 241-244

Published by: The Mathematical Association

Stable URL: <http://www.jstor.org/stable/3603862>

Accessed: 10-05-2016 20:40 UTC

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at
<http://about.jstor.org/terms>

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

The Mathematical Association is collaborating with JSTOR to digitize, preserve and extend access to *The Mathematical Gazette*

THE MATHEMATICAL GAZETTE.

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.; PROF. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.;
E. T. WHITTAKER, M.A., F.R.S.; W. E. HARTLEY, B.A.

LONDON :
GEORGE BELL & SONS, PORTUGAL STREET, LINCOLN'S INN,
AND BOMBAY.

VOL. III.

DECEMBER, 1905.

No. 54.

THE NORMAL LAW OF ERROR.

1. The investigations of Professor Karl Pearson and his coadjutors on the applications of the theory of errors to statistics have greatly increased the importance and extent of the subject.

It is hoped that the following version of one of the methods of arriving at the so-called normal law of error may be of interest from an elementary point of view : affording, as it does, a simple application of the binomial theorem and some interesting graphical work.

Let each single observation of a quantity, for instance the measurement of a length, be subject to $2n$ errors, each of amount z and each equally likely, in any one case, to be positive or negative.

If $n+r$ errors are positive and $n-r$ are negative, the total, or resultant error will be $2rz$. Now the number of permutations taken all together of $(n+r) +$ signs and $(n-r) -$ signs is

$\frac{2n!}{(n+r)!(n-r)!}$, and each permutation represents a mode of occurrence of the set of errors, the chance of that particular permutation occurring being $\frac{1}{2^{2n}}$.

Hence the chance of a total error $2rz = \frac{2n}{n+r!n-r!} \frac{1}{2^{2n}}$. In other words, the chances of the various total errors

$$2nz, 2n-1z \dots, 2rz \dots, -2nz,$$

are given by the successive terms in the expansion of $\left(\frac{1}{2} + \frac{1}{2}\right)^{2n}$.

L

2. Let us examine the mean value, or expectation, say σ^2 , of the square* of the total error.

Multiplying the chance of each total error by the square of its amount, and summing the results we have

$$\begin{aligned}\sigma^2 &= \sum_{r=-n}^{r=n} \frac{2n!}{(n+r)!(n-r)!} \cdot \frac{1}{2^{2n}} 4r^2 z^2 \\ &= \sum_{r=-n}^{r=n} \frac{2n!}{(n+r)!(n-r)!} \cdot \frac{1}{2^{2n}} [n^2 - (n^2 - r^2)] 4z^2 \\ &= \frac{4n^2 z^2}{2^{2n}} \sum_{r=-n}^{r=n} \frac{2n!}{(n+r)!(n-r)!} - \sum_{r=-n}^{r=n} \frac{2n!}{(n+r-1)!(n-r-1)!} \cdot \frac{1}{2^{2n}} \cdot 4z^2 \\ &= \frac{4n^2 z^2}{2^{2n}} (1+1)^{2n} - \frac{2n \cdot 2n-1}{2^{2n}} 4z^2 (1+1)^{2n-2} \\ &= 4n^2 z^2 - 2n \cdot (2n-1) z^2 \\ &= 2n z^2.\end{aligned}$$

If then n and z are connected by the equation

$$2n z^2 = \text{constant} = \sigma^2,$$

the "standard deviation" σ will be the same for all values of n .

3. It is interesting to represent the chances of errors of given amount graphically, taking different values of n and z , and keeping σ constant.

Thus we may take $\sigma = 4$ inches,
and the cases $n = 8, 32, 18,$
 $z = 1, \frac{1}{2}, \frac{2}{3}.$

Taking the first case the maximum ordinate is proportional to $\frac{16!}{8!8!}$ and taking 10 inches to represent this, we have, if $r=0$, say $y_0 = 10$ inches.

If $r=1$, y_1 is proportional to $\frac{16!}{7!9!}$ and therefore $y_1 = \frac{8}{9} y_0 = 8.89$, y_1 corresponding to a total error x_1 of 2 inches.

If $r=2$, y_2 is proportional to $\frac{16!}{6!10!}$ and therefore $y_2 = \frac{7}{10} y_1 = 6.22$ and so on. Using a slide rule the successive ordinates are thus rapidly worked out, and the total error x , and the corresponding chance, y , plotted.

Taking in the same way one of the other cases, we find that a fair curve through the second set of points passes very close to the first set also. In other words, provided σ is constant and n

* It is natural to ask why the square should be taken. Prof. Pearson, *Grammar of Science*, p. 386, cites the analogy of the radius of gyration k : k^2 being the mean square of the distance. Of course the simple mean value or expectation of the total error is zero, for positive and negative errors of equal amounts are equally likely.

is fairly large, the chance of getting an error of a given amount at a single trial is, on the above hypothesis, almost independent of the value of n .

4. Let us now prepare to suppose n to become infinite and z infinitely small in such a manner that $2nz^2$ retains the constant value σ , and r to become infinite in such a manner that $2rz = x$.

Let y denote the relative * chance of a total error $x = 2rz$,
and $y + \Delta y$ " " " $x + \Delta x = (2r + 2)z$.

Thus as before

$$y = \frac{2n!}{(n+r)!(n-r)!} \frac{1}{2^{2n}},$$

$$y + \Delta y = \frac{2n!}{(n+r+1)!(n-r-1)!} \frac{1}{2^{2n}},$$

whence, by division,

$$1 + \frac{\Delta y}{y} = \frac{n-r}{n+r+1} \quad \text{or} \quad \frac{\Delta y}{y} = \frac{-2r-1}{n+r+1} = \frac{-x-z}{nz + \frac{1}{2}x + z},$$

or, since

$$\Delta x = 2z,$$

$$\frac{1}{y} \frac{\Delta y}{\Delta x} = \frac{-x-z}{2nz^2 + xz + 2z^2} = \frac{-x-z}{\sigma^2 + xz + 2z^2}.$$

Now proceed to the limit as above indicated and we have

$$\frac{1}{y} \frac{dy}{dx} = -\frac{x}{\sigma^2};$$

or, integrating, $\log y = -\frac{x^2}{2\sigma^2} + C,$

or $y = y_0 e^{-\frac{x^2}{2\sigma^2}}.$

Taking

$$y_0 = 10 \text{ inches,}$$

$$\sigma = 4 \text{ inches,}$$

and plotting the curve, we find once more that it passes very closely to the points already found by giving n the finite value 32. If we adjust the value of y_0 so that the whole area of the curve is unity, which should be so, for the area represents the chance of obtaining *some* error between $+$ and $-$ infinity (zero included), we find

$$y_0 = \frac{1}{\sigma \cdot \sqrt{2\pi}}.$$

It is now possible to plot, in their proper relative proportions, the curves corresponding to *different* values of σ .

* Of course the *absolute* chance of an error being *exactly* x is zero. The chance that the error lies between x and $x + \Delta x$ is $y\Delta x$ nearly.

Abundant illustrations of the application to statistics of the properties of the Normal Curve of Error

$$y = \frac{1}{\sigma \cdot \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

may be found in Professor Pearson's *Grammar of Science*, and the series of papers in the *Phil. Trans.* already alluded to. Mr. W. F. Sheppard proves many of the properties by elementary methods in a paper in the *Philosophical Transactions*, vol. 192A, p. 101. C. S. JACKSON.

THE POWER SERIES FOR $\sin x$, $\cos x$.

THE hopeless complexity, from the elementary didactic point of view, of the hitherto accepted accurate trigonometrical proofs of the power series for $\sin x$, $\cos x$ has been so fully insisted upon in these pages (this volume, pp. 85, 86), that a more elementary method seems urgently called for. The series in question follow at once from the two theorems

$$\sin x \text{ lies between } S_n, S_{n+1}, \dots \dots \dots (1)$$

$$\cos x \text{ lies between } C_n, C_{n+1}, \dots \dots \dots (2)$$

where

$$S_n = \frac{x}{1} - \frac{x^3}{3} + \dots + (-1)^n \frac{x^{2n+1}}{2n+1},$$

$$C_n = 1 - \frac{x^2}{2} + \dots + (-1)^n \frac{x^{2n}}{2n}.$$

Now it may be shown by elementary methods that (1) follows from (2) for all real values of x . It may also be similarly shown that if (1) be granted, then $\cos x$ lies between C_{n+1} , C_{n+2} provided $-2\pi \leq x \leq +2\pi$. Since then it is known that (2) is true when $n=0$ for all real values of x , we at once obtain an inductive proof of (1), (2) for $-2\pi \leq x \leq +2\pi$.

To show that (1) follows from (2) let there be m angles $\alpha, \beta, \gamma, \dots$, then

$$\cos \alpha \cos \beta \cos \gamma \dots = \frac{1}{2^m} \Sigma \cos (\pm \alpha \pm \beta \pm \gamma \pm \dots).$$

Hence, according as $\cos x \geq a_0 + \Sigma a_r x^{2r}$,
we have $\cos \alpha \cos \beta \cos \gamma \dots \geq a_0 + \Sigma a_r T_r$,

where $T_r = \frac{1}{2^m} \Sigma (\pm \alpha \pm \beta \pm \gamma \pm \dots)^{2r}$.

Now taking $\alpha, \beta, \gamma, \dots$ to be $\frac{x}{2}, \frac{x}{2}, \dots, \frac{x}{2m}$, we have

$$T_r = \frac{2x^{2r}}{2^{m(2r+1)}} t_r,$$