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XX. *The Striated Electrical Discharge.*
By J. H. JEANS*.

PART I.

§ 1. **T**HE present paper is principally concerned with the appearance of light which accompanies the discharge of electricity through a rarefied gas. I have attempted to show that Prof. Thomson's "Theory of the Conduction of Electricity through Gases by Charged Ions"† is capable of supplying an explanation of the striations, and of most of the other phenomena observed in connexion with the discharge through a vacuum-tube.

Considering the case of a one-dimensional discharge between two parallel and infinite plates, Thomson has arrived at a differential equation connecting y , the semi-square of the electric intensity at a point, with x , a coordinate measured along the line of discharge. It is impossible to solve this equation in finite terms, but Thomson has drawn a graph, giving an approximate value of y for every value of x , this graph satisfying the differential equation at every point, and also the two terminal conditions at the electrodes.

In the same paper Thomson has also given a second graph, in which are plotted out the result of some measurements by Graham on the potential gradient in a vacuum-tube. The comparison of the graph thus deduced from experiment with

* Communicated by Prof. J. J. Thomson, F.R.S.

† J. J. Thomson, *Phil. Mag.* xlvii, p. 253 (1899).

the graph calculated from theory is therefore of interest, and it is at once obvious that there are considerable differences between the two curves.

In particular the theoretical graph is convex to the axis of x at every point, while the experimental graph shows concavity over two regions of the axis.

Now when a graph for y is drawn from measurements actually taken from a tube through which a discharge is passing, a comparison of the graph so obtained with the glowing tube shows that the discharge is accompanied by luminosity at those parts of the tube at which the graph is concave to the axis, and at no others. There are also, as Thomson points out (*l. c. ante*, p. 267), theoretical reasons which lead us to suppose that luminosity is accompanied by concavity of the graph.

Thus the graph at which Thomson arrives, being convex at every point, precludes the possibility of luminosity in any part of the tube; it shows the distribution of electric force in a non-luminous discharge.

Further, it is well-known that under suitable conditions the gas in a discharge-tube presents a striated appearance, light and dark bands occurring alternately and at regular intervals, except in the neighbourhood of the electrodes. Thomson has obtained a discharge through a tube fifty feet in length, and observed that the whole tube, with the exception of a few inches near the cathode, was filled with these striations.

It appears therefore as if the true system of electrical equations, account being taken of the volume equations only, without reference to the boundary conditions, ought to lead to an infinite system of striations; in fact, the graph for y should be a regular succession of waves, and y itself a periodic function of x .

It must be borne in mind that Thomson's graph is only one out of an infinite series of curves, all of which are represented by the same differential equations, and this leads us to consider firstly, whether any other member of this family of curves bears any resemblance to Graham's experimental curve; and, secondly, whether any member can consist of a series of regular waves, and so correspond to a periodic solution.

Now it will appear in the course of the present paper that there is considerable probability that both of these questions must be answered in the negative. If this is so, it follows that the complete explanation cannot be implicitly contained in Thomson's differential equation; in order to arrive at it,

new physical considerations will have to be introduced, which were not allowed for in forming this equation.

§ 2. Our analysis seems to indicate that there is only one explanation possible which is consistent with the main facts of the theory of conduction by ions, and this, stated briefly, is somewhat as follows :—

Thomson's equation gives a solution for the case of steady motion which not only is mathematically possible, but is the only solution which satisfies the assumed conditions throughout. This steady motion, however, is, under certain circumstances, unstable, just as the steady motion to which the hydrodynamical equations lead for the case of liquid flowing through an orifice is unstable. And, just as the liquid is known in nature to break up into a succession of small drops, so the discharge in a vacuum-tube breaks up, under suitable conditions, into a succession of small discharges, each discharge forming a single striation.

A mathematical explanation of this may be given in the following way :—

If we suppose that Thomson's equations are absolutely true over all ranges of the variables, then there is only one form of discharge possible, and this is non-luminous. When, however, the volume-density of the ions becomes indefinitely great, it is obvious that these equations can no longer hold without modification. It is impossible to get an infinite volume-density of electricity in nature, and this fact is not contained in the equations. We are therefore compelled to admit the existence of new forces which were not allowed for in the original equations ; *these forces are negligible until the volume-density of the ions reaches a certain large limit, but they then come into action, and are sufficient to prevent the volume-density from ever actually becoming infinite.* These forces may be of the nature of a repulsion between pairs of ions which have approached indefinitely near to each other, or, if we imagine the ions to be bodies of finite size, they will consist of the forces of elasticity which prevent a collection of such bodies from being compressed into an indefinitely small space.

But, without making any assumptions about the nature of these forces beyond those laid down in the sentence in italics, it can be shown that the existence of such forces makes possible a second type of solution, which is essentially different from that considered by Thomson, and that this solution, interpreted physically, predicts the existence of a system of striations in the middle of the tube, which may be expected

to give place to other phenomena in the region of the electrodes.

§ 3. The present section consists of an explanation of the notation to be used, and a summary of those parts of Thomson's paper which are essential to the theory. We are considering a discharge in perfectly pure gas, between two parallel and infinite plates.

Let x be a coordinate measured along the line of discharge, the positive axis of x being drawn from the anode to the cathode. The electrical intensity at any point will be denoted by X . The positive and negative ions are supposed to move in opposite directions with velocities proportional to X (Thomson, *l. c. ante*, p. 253). These velocities will be denoted by k_1X and k_2X respectively, the former being parallel to the positive axis of x . The number of positive ions per unit volume is n_1 , and of negative ions n_2 . Of these a certain number recombine, and this number is supposed to be $\alpha n_1 n_2$, since, other things being equal, it will be proportional to the product $n_1 n_2$. In the same unit volume, a certain number of molecules become dissociated, and these produce q ions of each sort per unit time.

No assumption is as yet made as to the way in which q and α depend on other quantities; but from their definition they must be positive quantities at every point of the line of discharge,

If e be the charge carried by each ion, the fundamental equations for the case of steady motion are

$$\frac{dX}{dx} = 4\pi e(n_1 - n_2), \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{d}{dx}(k_1 n_1 X) = q - \alpha n_1 n_2, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$-\frac{d}{dx}(k_2 n_2 X) = q - \alpha n_1 n_2. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Of these the first is the fundamental equation of electrostatics; the two others contain the fact that the number of ions of either kind in a fixed element of volume remains constant; they are therefore the mathematical expression of the fact that the motion is steady.

From equations (2) and (3) it appears that $(k_1 n_1 + k_2 n_2)X$ is independent of x , and its value is easily seen to be i/e , where i is the current across any unit area perpendicular to the axis. Thus

$$(k_1 n_1 + k_2 n_2)X e = i. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Solving equations (1) and (4) for n_1 and n_2 , we have

$$n_1 e = \frac{1}{k_1 + k_2} \left\{ \frac{i}{\bar{X}} + \frac{k_2}{4\pi} \frac{d\bar{X}}{dx} \right\}, \quad \dots \quad (5)$$

$$n_2 e = \frac{1}{k_1 + k_2} \left\{ \frac{i}{\bar{X}} - \frac{k_1}{4\pi} \frac{d\bar{X}}{dx} \right\}; \quad \dots \quad (6)$$

and the substitution of these values in either of the remaining equations gives

$$\begin{aligned} & \frac{1}{4\pi e} \frac{k_1 k_2}{k_1 + k_2} \frac{d}{dx} \left(\bar{X} \frac{d\bar{X}}{dx} \right) \\ &= q - \frac{\alpha}{\bar{X}^2 e^2 (k_1 + k_2)^2} \left(i + \frac{k_2}{4\pi} \bar{X} \frac{d\bar{X}}{dx} \right) \left(i - \frac{k_1}{4\pi} \bar{X} \frac{d\bar{X}}{dx} \right). \quad (7) \end{aligned}$$

Writing $2y$ for $\bar{X}^2$ and p for $\frac{dy}{dx}$, this becomes

$$\frac{1}{4\pi e} \frac{k_1 k_2}{k_1 + k_2} \frac{d^2 y}{dx^2} = q - \frac{\alpha}{2ye^2 (k_1 + k_2)^2} \left(i + \frac{k_2}{4\pi} p \right) \left(i - \frac{k_1}{4\pi} p \right), \quad (8)$$

and this is Thomson's equation for y .

The boundary conditions are taken to be

$$n_1 = 0$$

at the anode, and

$$n_2 = 0$$

at the cathode.

For, considering any small element of volume dx , terminated in one direction by the anode, it is clear that no positive ions can occur inside this element except those which have actually been produced by dissociation inside it. Now positive ions are produced at a rate qdx , and the average time during which each ion remains in this element after its production

is of the same order of quantities as $\frac{dx}{k_1 \bar{X}}$. Thus the total number of positive ions in a length dx measured from the anode is of the order of $(dx)^2$, so that at the anode $n_1 = 0$.

Similarly, $n_2 = 0$ at the cathode.

§ 4. With a view to simplifying the discussion of the differential equation, an assumption will now be made to the effect that the quantities q and α depend only on the electric force at the point at which they are measured. It will be seen in a subsequent section that there is very little probability that this assumption is a legitimate one; but in the same section I shall endeavour to show that the limitations caused by its introduction can easily be removed.

In virtue of this assumption g and α will now be single-valued functions of y , and they must, as has already been noticed, be positive for all values of y .

If $\frac{d^2y}{dx^2}$ is replaced by $p \frac{dp}{dy}$, equation (8) will reduce to an equation which involves p and y only, and which may be written

$$\frac{1}{4\pi e} \frac{k_1 k_2}{k_1 + k_2} p \frac{dp}{dy} = q + \frac{\alpha k_1 k_2}{32\pi^2 e^2 (k_1 + k_2)^2 y} \left(p^2 - 4\pi i \left(\frac{1}{k_1} - \frac{1}{k_2} \right) p - \frac{16\pi^2 i^2}{k_1 k_2} \right). \quad (9)$$

Taking p and y as coordinates, a graph may be constructed showing the relation between p and y which is implied by the above equation, and this graph will consist of a singly-infinite system of curves.

Since y, n_1, n_2 must all be positive, the only parts of this graph which are of any importance in connexion with the present problem lie in that part of the p, y plane which is bounded by the two lines $p = \frac{4\pi i}{k_1}$ and $p = -\frac{4\pi i}{k_2}$, and is on the positive side of the axis of p .

For any given values of p and y equation (9) gives a single value for $\frac{dp}{dy}$, so that through every point in the p, y plane one, and only one, curve passes. An exception to this occurs at the intersections of the lines $p = \frac{4\pi i}{k_1}$ and $p = -\frac{4\pi i}{k_2}$ with the line $y=0$. At these two points $\frac{dp}{dy}$ becomes indeterminate, and an infinite number of curves branch out from these points. Moreover, the axis of p satisfies equation (9) at every point, for at all points on it $y=0$ and $\frac{dp}{dy} = \infty$. This line is therefore a curve of the system, and hence no other curve can meet this axis except in the two singular points mentioned above.

The points on the system of curves at which $\frac{dp}{dy} = 0$ are, by equation (9), given by

$$\left\{ p - 2\pi i \left(\frac{1}{k_1} - \frac{1}{k_2} \right) \right\}^2 = 4\pi^2 i^2 \left(\frac{1}{k_1} + \frac{1}{k_2} \right)^2 - \frac{32\pi^2 e^2 (k_1 + k_2) q y}{\alpha k_1 k_2}. \quad (10)$$

If the curve of which this is the equation be drawn in the plane of p , y , it will meet the curves of the graph in the points at which the tangents to the latter are parallel to the axis of y . This curve will in future be spoken of as the curve (α), and the two singular points in which the axis of p meets the lines $p = \frac{4\pi i}{k_1}$ and $p = -\frac{4\pi i}{k_2}$ will be denoted by A and B. It is clear that the curve (α) passes through the two points A and B.

§ 5. In the simplest case, in which q and α are constants, this curve is a parabola whose axis is parallel to the axis of y , and whose concavity is turned downwards (fig. 1). Considering only those parts of the plane for which p is positive, equation (9) shows that for curves of the original system $\frac{dp}{dy}$ is positive at all points outside the parabola, and negative at all points inside; the reverse is true when p is negative.

Fig. 1.

Hence in fig. 1 the value of $\frac{dp}{dy}$ at any point must have the same sign as the value of $\frac{dp}{dy}$ for a point moving in the direction indicated by the lines of the shading at that point. Every curve of the system cuts the line $p=0$ at right angles, for at points on this line $\frac{dy}{dx}=0$, and therefore y passes through a maximum or minimum value.

These considerations enable us to form some idea of the disposition of the curves in that part of the plane of p, y with which we are concerned.

Any curve starting from A and meeting OG (fig. 1) must clearly be an arched curve terminating in B. For the conditions already laid down prevent it from meeting either the line OB or arc GEB of the parabola, or again from bending back and recrossing OG. The limiting curves of this system are on the one hand the axis of p , and on the other hand the curve which actually passes through G. At no point on any curve of this system can y reach a value greater than OG; y , therefore, is always less than a critical value, y_0 , given by

$$y_0 = OG = \frac{i^2 \alpha}{2ge^2(k_1 + k_2)^2} \quad \dots \quad (11)$$

There is a second system, consisting of curves which pass through A but do not meet OG. These must therefore meet AG, be bent away from the axis of y after crossing AG, and finally meet AA'.

It can be shown that these curves cannot attain to an infinite value of y before meeting the line AA'. For we can find the shape of the curves which fill up that part of the p, y plane which lies between the lines AA' and BB', and for which y is infinite. The differential equation of these curves is seen by equation (9) to be

$$\frac{1}{4\pi e} \frac{k_1 k_2}{k_1 + k_2} p \frac{dp}{dy} = q,$$

and this represents a system of parabolas which are convex to the axis of p . Since these parabolas cannot be members of that system of curves which we are now discussing, we conclude that there must be some point A', in the line AA', beyond which this system never reaches. As the curves of a system cannot cross one another, it must be the curve that actually passes through G which reaches this limiting point A'.

There is therefore a maximum limit to the value of y on curves of the second class, and this is given by

$$y_1 = AA'.$$

There is an exactly similar system of curves passing through B and meeting BG. For this system also there is a maximum limit to the value of y , given by

$$y_2 = BB'.$$

The system of curves which pass through B, and do not meet BG, must consist of the curves which meet OG. This

system is therefore identical with the first system which we considered, namely, the series of arches which join the points A, B.

The remainder of the part of the p, y plane with which we are dealing is filled up by a system of curves which meet OG above G, each curve having its lowest point on OG. For curves of this type there is an inferior limit to the value of y , and this is the value y_0 already found in equation (11).

The curves in the p, y plane are thus seen to fall into four classes. These are represented in their essential features by

Fig. 2.

the curves in fig. 2, and the four classes will in future be referred to by the numbers in this diagram.

The limiting curves are the two curves AGB' , $A'GB$, which are drawn in thick lines in the diagram. These curves intersect in G and divide the plane into four regions, in each of which lie all the curves in one of the four classes.

§ 6. Let us now examine to what extent the arrangement of curves found in the last section will be altered if we replace the assumption that q and α are constant by the more general assumption that they are positive quantities which vary with y .

From the equations of § 4, it follows at once that :

(i.) The curve (α) is no longer a parabola, but is still a curve which passes through the points A and B, and is symmetrical about the same axis as before.

(ii.) At all points adjacent to the line $y=0$, the sign to be given to $\frac{dp}{dy}$ in drawing the system of curves remains the same as before. The same is true for points at which y is infinite.

(iii.) Hence, since $\frac{dp}{dy}$ must change sign in crossing either the curve (α) or the axis of y , it follows that the curve (α) must cross the positive axis of y an odd number of times.

(iv.) The curve (α) cannot meet the lines AA' , BB' in any points except A and B.

From this we deduce that the new curve (α) consists primarily of an arch through A and B, and secondly, of any number of closed curves either inside or outside the arch.

If, then, a shaded diagram is drawn similar to fig. 2, the directions of the shading adjacent to the arch through AB, as also the direction of the shading at infinity, must remain unaltered.

The four types of curves of which the existence has already been discovered will therefore continue to exist in the new graph in the p, y plane. The general appearance of the new graph will, however, depend on whether the curve (α) cuts the axis of y once or more than once.

For the effect of introducing into this locus new closed curves which do not meet the axis of y , is simply a twisting of the lines already in the graph, and this introduces no essentially different type of curve. But the effect of admitting into this locus new closed curves which do cut the axis of y , or of allowing the original arch to meet this axis more than once, will be the introduction of an entirely new type of curve into the p, y graph, and this will consist of a series of closed curves. Fig. 3 will, I hope, make the truth of these statements clearer*.

§ 7. The presence of closed curves in the p, y graph immediately makes possible a periodic solution of the original equation. For as we pass round such a curve time after time the same values of p and y recur indefinitely; so that the graph of y in terms of x will be a regular succession of waves and furrows.

A solution of this type, interpreted physically, would give a striated solution as regards the middle of the tube; but the whole solution breaks down when we attempt to satisfy the

* The three closed curves which are symmetrical about the central axis, together with the arch through AB, constitute the curve (α). The closed curves in the p, y graph are seen surrounding the point Z on the axis of y .

terminal conditions. To satisfy these, the closed curve would have to meet the lines AA' , BB' . Now if it crosses either of these lines it cannot give a solution at all, and if it is to touch either of them the curve (α) must pass through the point of contact, and this has been seen to be impossible.

Fig. 3.

It might, however, be urged that near the electrodes the volume-equations of § 3 give place to other and different volume-equations; and if this were so, there is nothing to prevent all the conditions of the problem from being satisfied.

Under these circumstances, the examination of whether closed curves can or can not occur in the p, y graph becomes a matter of importance. The necessary and sufficient condition that closed curves shall be an impossibility is that the curve (α) shall only meet the axis of y in one point, and this condition will in future be spoken of as condition A.

We have seen that if condition A is fulfilled all the curves which constitute the p, y graph come under one or other of the four classes represented in fig. 2.

We shall in the following sections confine ourselves to the consideration of these four types of curves, and it will appear that if these four types alone exist, there is only one type of solution possible in addition to that already arrived at by Thomson.

Even if condition A is not satisfied this alternative type of solution will still be possible, but, as has already been pointed out, we are no longer justified in asserting that it is the only alternative. A second alternative becomes possible on assuming that the gas near the electrodes is in an abnormal state.

§ 8. From our knowledge of the graph in the p, y plane it is easy to deduce the corresponding graph in the plane of x, y . An intersection of a p, y curve with the axis of p will give rise to a maximum or minimum value for y on the corresponding x, y curve, and an intersection with the curve (α) will give rise to a point of inflexion. The new graph, in strictness, consists of a doubly infinite system of curves, but it will be sufficient to consider the system as singly infinite, a single curve corresponding to each curve in the p, y plane. A second infinity is then obtained by moving each curve parallel to itself along the axis of x .

The curves in the plane of x, y fall into four classes corresponding to the four classes of curves in the p, y plane.

The curves of the first class will each give rise to a single arch in the plane of x, y ; this arch stands on the axis of x , and is on the positive side of this axis. Since we are now considering the case in which q and α are no longer constants, this arch may or may not have points of inflexion. Every curve of this system leaves the axis of x at an angle $\tan^{-1}\left(\frac{4\pi i}{k_1}\right)$, and meets it again at an angle $\tan^{-1}\left(\frac{4\pi i}{k_2}\right)$.

The limiting curves of this system are :—

(i.) An arch which shrinks to a point on the axis of x . This corresponds to the limiting curve AFB in fig. 2.

(ii.) An arch of infinite span which touches asymptotically the line $y=y_0$ (see equation 11). This corresponds to the curve AGB in fig. 2, and y_0 must now be defined as the smallest (or, if condition A is satisfied, the only) root of the equation

$$y = \frac{i^2 \alpha}{2qe^2(k_1 + k_2)^2} \cdot \cdot \cdot \cdot \cdot \quad (12)$$

The curves of the second class are single slant lines, which leave the axis of x , making with it an angle $\tan^{-1}\left(\frac{4\pi i}{k_2}\right)$ in the negative direction. After passing through at least one point of inflexion the tangent again makes an angle $\tan^{-1}\left(\frac{4\pi i}{k_2}\right)$ with the negative axis of x , and at this point, in so far as they are of interest in connexion with the present problem, they terminate abruptly. There is a superior limit y_1 (see § 5) to the value of y at this point of termination.

The limiting curves of this class are :—

(i.) An indefinitely short straight line close to the axis of x , and making with it an angle $\tan^{-1}\left(\frac{4\pi i}{k_2}\right)$.

(ii.) A curve which approaches the line $y=y_0$ asymptotically, has a point of inflexion on this line, and again recedes from it asymptotically. The length on both sides of the point of inflexion is therefore infinite.

There is no difference between curves of the third class and those of the second, except that the slant is in the opposite direction, and that $\frac{4\pi i}{k_1}$ must now be substituted for $\frac{4\pi i}{k_2}$. There is again a superior limit y_2 to the value of y .

Curves of the fourth class are inverted arches, which terminate abruptly in two points α , β . At α the tangent makes an angle $\tan^{-1}\left(\frac{4\pi i}{k_2}\right)$ with the negative axis of x , and after the value of y has passed through some minimum χ , the curve terminates in β , at which point its tangent makes an angle $\tan^{-1}\left(\frac{4\pi i}{k_1}\right)$ with the positive axis of x . The value of y at α has an inferior limit y_1 (see p. 252), and the value at β an inferior limit y_2 . Also χ has an inferior limit, y_0 (equation 12).

The limiting curves are :—

(i.) A curve which approaches $y=y_0$ asymptotically in both directions.

(ii.) A finite arc of a parabola, at every point of which y is infinite, the latus rectum of the parabola being finite (see p. 252).

A curve of each of the four classes is shown in fig. 4. The dotted lines show the three critical values for y , which have been denoted by y_0 , y_1 , y_2 .

At an anode the terminal condition has been seen to be $n_1=0$ or $\frac{dy}{dx} = -\frac{4\pi i}{k_2}$. It is therefore possible for an anode to be at a point such as α or α' in fig. 4, but nowhere else.

Fig. 4.

And similarly a cathode can only exist at a point such as β or β' . If then it is possible to satisfy the condition for an anode and the condition for a cathode at two points on the same curve, that curve must be of the fourth class.

§ 9. The question which now suggests itself is the following:— If we plot out the actual graph for y along any discharge, does the curve so obtained necessarily coincide with a single curve of the infinite series typified in fig. 4?

If this question must be answered in the affirmative, the graph must, as we have seen in the last section, be a curve of the fourth class, terminated by the points α and β . This is in fact the only solution which satisfies equation (8) at every point, satisfies the assumed boundary conditions at the electrodes, and makes n_1 , n_2 , and y positive at every point. It is a solution of this type that is arrived at in Thomson's paper.

The only alternative to this solution would lie in making up a graph discontinuously out of a number of the curves or parts of curves shown in fig. 4. We are therefore led to inquire whether it is ever possible to pass from one curve to another without violating physical conditions of continuity.

Since the velocity of the ions must be continuous, x , and

therefore y , must be continuous. If we suppose that p also is to be continuous we are reduced to keeping on the same curve, for two curves can never touch each other, since by equation (8) the curvature depends only on y and p .

The effect of making p discontinuous is to make $\frac{d^2y}{dx^2}$ become infinite. If, then, equation (8) is not to be violated, y must be zero at that point.

An alternative solution can therefore be obtained by building up the graph of y out of pieces from the curves in fig. 4, care being taken to only pass from one curve to another on the axis. The solution so obtained is such that the electric force and velocities of the ions are continuous, and it satisfies all the mathematical equations.

It is, moreover, obvious that there is only one type of solution of this kind possible. This consists of any number of arched curves of the first class placed end to end and terminated by a curve of the second class at the anode, and a curve of the third class at the cathode; the points α' and β' of these curves (fig. 4) being the anode and cathode respectively.

Fig. 5.

A solution of each of the two types that have been found to be mathematically possible is shown in fig. 5.

§ 10. When we attempt to interpret this second solution, we are immediately confronted with the fact that it is impossible, for physical reasons, that y should vanish. At a point such

as B, C, or D in fig. 5, we should not only have an infinite volume-density of ions, but also all the ions would be at rest.

It has, however, been already pointed out (§ 2) that as y becomes indefinitely small, the equation from which our solution has been derived ceases to hold. New forces come into play, with the result that the graph representing the actual values of y ceases any longer to satisfy the differential equation which holds elsewhere. And if these forces are forces which prevent the volume-density of ions from becoming infinite (as must be the case if the ions are of finite size), the change in the differential equation will be of such a nature as to prevent the graph from ever meeting the axis.

Let us imagine that η is some small value of y , such that these forces do not produce any effect till y falls below a certain value η , but that for values of y less than η the equations of § 3 cease to hold.

In fig. 6 let A be the anode, A B C D E the axis along which $y=0$, $l m n \dots$ the line $y=\eta$.

Fig. 6.

Starting from the point α' in the anode, such that the value of y is less than the critical value y_1 , let the graph be drawn as in fig. 5 until it meets the line $y=\eta$ in some point m . And after the graph has reached this line, let it be continued so as to satisfy the true differential equation which holds when y is less than η .

The graph must either remain in the region in which y is less than η , or else emerge from this region at some point n in $l m n \dots$. If in the limit we assume η to be very small, the former alternative need not be considered. Beyond n

the graph satisfies the known differential equation ; it is therefore an arch of the type occurring in fig. 5 ; and since both p and y are already determined at the point n which is not on the axis $y=0$, this arch is completely determined by the part of the graph already drawn. This arch again enters the region in which y is less than η at some point v , and so on, until it finally emerges at some point s , and takes the shape of a line similar to $D\beta'$ in fig. 6. The point β' on this curve at which the graph satisfies the condition for a cathode may be taken for the actual cathode, and in this case the graph we have arrived at will satisfy the appropriate terminal conditions at each electrode.

The graph so obtained also satisfies the true physical equations at every point. For so long as y is greater than η it satisfies the mathematical equations, with which the physical equations are supposed to coincide, and in the region for which y is less than η it must, from the way in which it has been constructed, satisfy the physical equations.

Further, the smaller η becomes, the more closely will this graph approach to the known graph in fig. 5, so that if η be sufficiently small, the graph in fig. 5 may be supposed to represent the real graph if we allow for the existence of the disturbing forces by rounding off the corners B, C, D.

This effect of disturbing forces finds a parallel in almost every branch of physics. Consider, for example, liquid flowing along the edge of an obstacle. Let us draw a graph giving the reciprocal of the velocity of the fluid at the various points of the edge, the velocity being determined by the hydrodynamical equations for the motion of a non-viscous incompressible fluid. At a point of the graph corresponding to a sharp corner on the obstacle there will be a cusp on the axis exactly similar to the points B, C, D in fig. 5, showing that the theoretical velocity at these points is infinite. In the actual flow the velocity is not infinite on account of the viscosity of the fluid, and the effect on the graph of this disturbing influence is therefore a "rounding-off" of sharp corners.

§ 11. We are now in a position to give a physical interpretation of the solution which is represented by the graph in fig. 6.

As is well known, both theory and experiment show that light may be expected to appear at points for which $q - \alpha n_1 n_2$ is negative; that is to say, wherever the number of ions that recombine is greater than the number produced by dissociation. Equation (8) shows that $q - \alpha n_1 n_2$ is negative or positive

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according to whether $\frac{d^2y}{dx^2}$ is negative or positive. Luminescence may accordingly be expected whenever the graph is concave to the axis, and darkness where it is convex*.

In fig. 6 let u , v be the points of inflexion which are known to exist in $D\beta'$, $\alpha'B$.

Then from β' to u the curve is convex to the axis, and therefore we expect darkness—the Crookes's dark space.

From u to s the curve is concave, and therefore we expect luminosity—the negative glow.

Beyond s we expect alternations of light and darkness—the striæ—until we come to the neighbourhood of the anode, and here the solution leads us to expect phenomena similar to those occurring at the cathode.

What happens in nature is that the appearances presented at the anode vary greatly in different discharges, and, in general, bear very little resemblance to those observed at the cathode. This appears, at first, to be in opposition to our theory, but I hope in the second part of this paper to show that, so far from being in opposition to the theory, it becomes a logical consequence of the theory as soon as we introduce the supposition that the velocity of the negative ions is much greater than that of the positive.

XXI. *On the Coincidence of Refracted Rays of Light in Crystalline Media.* By LOUIS TRENCHARD MORE, *Ph.D.*,
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IT is very generally assumed in treatises on optics that rays of light in doubly-refracting media break up into two parts, unless the light traverses a path coinciding with an optic axis of the crystal, or unless the path coincides with the major or minor axis of the elliptic section of the wave-surface. In the former case, both the velocities and the directions of the rays coincide; in the latter, although the velocities are different, the paths are the same. That these are but two special cases of an infinite number in which the paths of the ordinary and extraordinary rays coincide was first shown by Dr. D. B. Brace‡. The reason that this phenomenon has hitherto escaped experimenters is probably because, under these conditions, the angle of refraction of the rays is, in general, either so large that it is greater than the critical

* J. J. Thomson, *Phil. Mag.* xlvii. p. 267 (1899).

† Communicated by the Author. Read in part to the American Mathematical Society, August 1899.

‡ *Wied. Ann.* Bd. xxvi. p. 576.