three-bar motion, not capable of being so described in a finite manner, are yet given by the following construction:—

If a constant angle moves with a fixed point of one side on a circle, while the other side touches another circle, the locus of the apex is a sextic curve with three finite nodes. The circular points are triple, and at each two tangents are coincident. The two singular foci lie on the circle of nodes. Thus several modes of mechanical description, seemingly far removed from each other, are closely associated, since, analytically, the curves described are included in the equation of three-bar motion curves. I cannot do more here than allude to this result. (See *Proc.*, Vol. II., § 10.)

Values of Certain Infinite Products, with an application to the Summation of the Geometrical Series of the n^{th} Order as a Definite Integral. By J. W. L. GLAISHER, M.A., F.R.S.

[Read November 11th, 1875.]

Starting with the identity

$$1+x^n = 1-\omega x \cdot 1-\omega^3 x \dots 1-\omega^{2n-1} x,$$

in which $\omega = \cos \frac{\pi}{n} + i \sin \frac{\pi}{n}$ (or any other prime $2n^{\text{th}}$ root of unity),

and writing for x successively x^2 , $(\frac{1}{2}x)^2$, $(\frac{1}{3}x)^2$, &c., we have

$$1+x^{2n} = 1-(\zeta x)^2 \cdot 1-(\zeta^3 x)^2 \dots 1-(\zeta^{2n-1} x)^2,$$

$$1 + \left(\frac{x}{2}\right)^{2n} = 1 - \left(\frac{\zeta x}{2}\right)^2 \cdot 1 - \left(\frac{\zeta^3 x}{2}\right)^2 \dots 1 - \left(\frac{\zeta^{2n-1} x}{2}\right)^2,$$

$$\dots \dots \dots \dots \dots \dots$$

and
$$x^n = (-)^{1^n} \cdot \zeta x \cdot \zeta^3 x \dots \zeta^{2n-1} x,$$

$$\zeta \text{ being } = \sqrt{\omega} = \cos \frac{\pi}{2n} + i \sin \frac{\pi}{2n};$$

whence

$$x^n \cdot 1+x^n \cdot 1 + \frac{x^{2n}}{2^{2n}} \cdot 1 + \frac{x^{2n}}{3^{2n}} \dots = (-)^{1^n} \frac{1}{\pi^n} \sin \pi \zeta x \sin \pi \zeta^3 x \dots \sin \pi \zeta^{2n-1} x$$

$$= (-)^{1^n} \frac{1}{\pi^n} \sin \pi (\beta_1 + i\alpha_1) x \sin \pi (\beta_3 + i\alpha_3) x \dots \sin \pi (\beta_{2n-1} + i\alpha_{2n-1}) x;$$

whence, pairing the first and last factors, the second and the last but one, &c., and observing that

$$\sin (a+bi) \sin (-a+bi) = -\frac{1}{2} (\cosh 2b - \cos 2a)$$

and that, if n be uneven, the middle factor $\sin \pi \zeta^n x = i \sinh \pi x$, we obtain—

$$\begin{aligned} \text{I. } x^n \cdot 1+x^n \cdot 1 + \frac{x^{2n}}{2^{2n}} \cdot 1 + \frac{x^{2n}}{3^{2n}} \dots &= \frac{1}{2^{1^n} \pi^n} (\cosh 2\pi\alpha_1 x - \cos 2\pi\beta_1 x) \\ &\times (\cosh 2\pi\alpha_3 x - \cos 2\pi\beta_3 x) \dots (\cosh 2\pi\alpha_{n-1} x - \cos 2\pi\beta_{n-1} x); \end{aligned}$$

c 2

$$\text{or, say,} \quad = \frac{1}{2^{1n} \pi^n} \Pi_1^{n-1} (\cosh 2\pi\alpha x - \cos 2\pi\beta x), \text{ if } n \text{ be even;}$$

$$\text{and} \quad = \frac{1}{2^{1(n-1)} \pi^n} \sinh \pi x \Pi_1^{n-2} (\cosh 2\pi\alpha x - \cos 2\pi\beta x),$$

if n be uneven;

$$\text{where} \quad \alpha_r = \sin \frac{r\pi}{2n}, \quad \beta_r = \cos \frac{r\pi}{2n}$$

(so that $\alpha_r = \beta_{n-r}$).

II. From these formulæ we derive

$$1 + x^{2n} \cdot 1 + \frac{x^{2n}}{3^{2n}} \cdot 1 + \frac{x^{2n}}{5^{2n}} \dots = \frac{1}{2^{1n}} \Pi_1^{n-1} (\cosh \pi\alpha x + \cos \pi\beta x), \text{ if } n \text{ be even;}$$

$$\text{and} \quad = \frac{1}{2^{1(n-1)}} \cosh \frac{1}{2} \pi x \Pi_1^{n-2} (\cosh \pi\alpha x + \cos \pi\beta x),$$

if n be uneven.

III. We have also

$$\frac{2^{2n} + x^{2n} \cdot 4^{2n} + x^{2n} \cdot 6^{2n} + x^{2n} \dots}{1^{2n} + x^{2n} \cdot 3^{2n} + x^{2n} \cdot 5^{2n} + x^{2n} \dots} = \frac{1}{x^n} \Pi_1^{n-1} \frac{\cosh \pi\alpha x - \cos \pi\beta x}{\cosh \pi\alpha x + \cos \pi\beta x}, \text{ if } n \text{ be even;}$$

$$\text{and} \quad = \frac{1}{x^n} \tanh \frac{1}{2} \pi x \Pi_1^{n-2} \frac{\cosh \pi\alpha x - \cos \pi\beta x}{\cosh \pi\alpha x + \cos \pi\beta x},$$

if n be uneven;

where in the expression on the left-hand side it is supposed that there is one more factor in the denominator than in the numerator, and that the additional factor is under the square root sign. We thus have a generalization of Wallis's formula for $\frac{1}{2}\pi$.

IV. By differentiating the results in I. and II., it follows that

$$\frac{1}{1^{2n} + x^{2n}} + \frac{1}{2^{2n} + x^{2n}} + \frac{1}{3^{2n} + x^{2n}} + \&c.$$

$$= \frac{\pi}{nx^{2n-1}} \Sigma_1^{n-1} \frac{\alpha \sinh 2\pi\alpha x + \beta \sin 2\pi\beta x}{\cosh 2\pi\alpha x - \cos 2\pi\beta x} - \frac{1}{2x^{2n}}, \text{ if } n \text{ be even;}$$

$$\text{and} \quad = \frac{\pi}{2nx^{2n-1}} \coth \pi x + \frac{\pi}{nx^{2n-1}} \Sigma_1^{n-2} \frac{\alpha \sinh 2\pi\alpha x + \beta \sin 2\pi\beta x}{\cosh 2\pi\alpha x - \cos 2\pi\beta x} - \frac{1}{2x^{2n}},$$

if n be uneven.

$$\frac{1}{1^{2n} + x^{2n}} + \frac{1}{3^{2n} + x^{2n}} + \frac{1}{5^{2n} + x^{2n}} + \&c.$$

$$= \frac{\pi}{2nx^{2n-1}} \Sigma_1^{n-1} \frac{\alpha \sinh \pi\alpha x - \beta \sin \pi\beta x}{\cosh \pi\alpha x + \cos \pi\beta x}, \text{ if } n \text{ be even;}$$

$$\text{and} \quad = \frac{\pi}{4nx^{2n-1}} \tanh \frac{1}{2} \pi x + \frac{\pi}{2nx^{2n-1}} \Sigma_1^{n-2} \frac{\alpha \sinh \pi\alpha x - \beta \sin \pi\beta x}{\cosh \pi\alpha x + \cos \pi\beta x},$$

if n be uneven

(Σ of course, here and throughout, only including uneven value sof the suffixes).

V. From the integral $e^{-ab} = \frac{2}{\pi} \int_0^\infty \frac{x \sin ax}{b^2 + x^2} dx,$

we derive $q^n = \frac{2n}{\pi} \int_0^\infty \frac{x^{2n-1} \sin(ax^n)}{x^{2n} + x^2} dx, \quad \left(a = \log \frac{1}{q}\right);$

whence $\frac{1}{2} + q^{1^n} + q^{2^n} + q^{3^n} + \&c.$

$$= \frac{1}{\pi} \int_0^\infty dx \sin(ax^n) \frac{d}{dx} \log \left\{ x^n \cdot 1 + x^{2^n} \cdot 1 + \frac{x^{2^{2n}}}{2^{2n}} \dots \right\}$$

$$= 2 \int_0^\infty \Sigma_1^{n-1} \frac{\alpha \sinh 2\pi\alpha x + \beta \sin 2\pi\beta x}{\cosh 2\pi\alpha x - \cos 2\pi\beta x} \sin(ax^n) dx, \text{ if } n \text{ be even};$$

and $= \int_0^\infty \left\{ \coth \pi x + 2 \Sigma_1^{n-2} \frac{\alpha \sinh 2\pi\alpha x + \beta \sin 2\pi\beta x}{\cosh 2\pi\alpha x - \cos 2\pi\beta x} \right\} \sin(ax^n) dx,$
if n be uneven.

Also $q^{1^n} + q^{2^n} + q^{3^n} + \&c.$

$$= \int_0^\infty \Sigma_1^{n-1} \frac{\alpha \sinh \pi\alpha x + \beta \sin \pi\beta x}{\cosh \pi\alpha x - \cos \pi\beta x} \sin(ax^n) dx, \text{ if } n \text{ be even};$$

and $= \int_0^\infty \left\{ \frac{1}{2} \tanh \frac{1}{2} \pi x + \Sigma_1^{n-2} \frac{\alpha \sinh \pi\alpha x + \beta \sin \pi\beta x}{\cosh \pi\alpha x - \cos \pi\beta x} \right\} \sin(ax^n) dx,$ if n
be uneven.

VI. We have also incidentally the curious results, obtainable at once from the integral written above,

$$(i.) \frac{1}{2} + q^{1^n} + q^{2^n} + q^{3^n} + \&c. = \frac{1}{\pi} \int_{-\infty}^\infty \sin az \, dz,$$

where z is given by the equation

$$z \cdot 1 + z^2 \cdot 1 + \frac{z^2}{2^{2n}} \cdot 1 + \frac{z^2}{3^{2n}} \dots = e^z.$$

$$(ii.) q^{1^n} + q^{2^n} + q^{3^n} + \&c. = \frac{1}{\pi} \int_0^\infty \sin az \, dz,$$

where z is given by the equation

$$1 + z^2 \cdot 1 + \frac{z^2}{3^{2n}} \cdot 1 + \frac{z^2}{5^{2n}} \dots = e^z.$$

The equations in I. and II. are the general formulæ, some cases of which I published in the "Messenger of Mathematics," New Series, t. II. pp. 188—190 (1873); the method there adopted is not so good as that used in this paper, and I did not obtain the values for the general products. In a paper "On the summation by Definite Integrals of geometrical series of the Second and Higher Orders," ["Quarterly Journal of Mathematics," t. XI., pp. 328—343 (1871),] I have employed the integral in V. to sum the series $q + q^4 + q^9 + \&c.$, and

$q + q^3 + q^7 + \&c.$,* remarking (p. 339) that "the results (which I have obtained as far as the fourth order inclusive) become so complicated, and their calculation and reduction is so tedious, that it does not seem worth while to work out any of the integrals for higher orders." The theory is thus completed in V. by the absolute expression of the series

of the n^{th} order $q^{1^n} + q^{2^n} + q^{3^n} + \&c.$

as a definite integral, the process here employed being far preferable to the direct summation of the series

$$\frac{1}{1^{2n} + x^{2n}} + \frac{1}{2^{2n} + x^{2n}} + \&c.$$

which was the method pursued in the paper in the "Quarterly Journal." Kummer ("Crelle," t. XVII., p. 220) and Schlömilch ("Analytische Studien," Pt. I., p. 156) have both summed the geometrical series of the second order, but their methods do not seem applicable when the exponents are higher powers than squares.

I may add, in conclusion, that it really seems of interest to exhibit as integrals the sums of those series which admit of being so expressed, as, ignoring varieties due to the coefficients, and considering only the exponents, the number of distinct series which can be summed in this manner is very restricted.

December 9th, 1875.

Prof. H. J. S. SMITH, F.R.S., President, in the Chair.

The Auditor (Mr. Perigal) reported that he had audited the Accounts of the Society, comparing them with the vouchers, and had found them correct. It was ordered that the Accounts should be printed in the Proceedings.

* The value of $q + q^3 + q^7 + \&c.$ [equation (29)] is wrongly given, that of $q - q^3 + q^7 - \&c.$ having been written twice over by mistake ("Quarterly Journal," t. XI., p. 339). The correct value, viz.,

$$\frac{1}{2} + q + q^3 + q^7 + \&c. = \int_0^{\infty} \left\{ \coth \pi x + \frac{\sinh \pi x + \sqrt{3} \sin(\pi x \sqrt{3})}{\cosh \pi x - \cos(\pi x \sqrt{3})} \right\} \sin(ax^3) dx,$$

is derivable at once from the formula at the bottom of p. 338 or from V. in this paper. Also, in equation (30), on p. 339, the fraction whose denominator is $\cosh(\pi x^{\frac{1}{3}}) - \cos(\pi x^{\frac{1}{3}} \sqrt{3})$ should be multiplied by 2; and, same page, tenth line from bottom, the exponents $1^r, 2^r$ &c. are misprinted $1'', 2'',$ &c.

Cr. FINANCIAL REPORT FOR SESSION 1874-5.

Dr.

	£.	s.	d.		£.	s.	d.	
62 Subscriptions, 1874-6	65	2	0	Balance due to Treasurer, brought forward	1	15	6	
16 " 1873-4	15	15	0	Postage and Stationery	10	16	10	
3 " 1872-3	3	3	0	Journals	5	13	10½	
2 " 1871-2	2	2	0	Rent	10	0	0	
3 " 1875-6	3	3	0	Attendance	3	7	0	
5 Entrance Fees"	5	5	0	Coal and Gas	1	10	0	
Jan. Dividend on Lord Rayleigh's Fund	21	11	4	Banker's discount on two Irish Cheques.....	0	1	0	
July Dividend on same	21	11	4	Ditto on Scotch Cheque	0	~	8	
Jan. Dividend on Composition Fund	4	3	8	Proceedings, Printing &c., paid Messrs. Hodgson & Son—				
July Dividend on same	5	3	5	Balance due from Session 1873-4 £20 0 0				
Sale of Proceedings and extra Copies	39	2	9	Account rendered 13 April, 1875 ...	29	12	6	
						49	12	6
				Account rendered Nov. 1875...	87	14	0	
						170	11	4½
				Balance in hand	15	11	1½	
						186	0	0
	£186	2	6		£186	0	0	

Audited and found correct, 30 Nov. 1875.

(Signed) H. PERIGAL.

Cr.

ASSETS AND LIABILITIES.

Dr.

	£.	s.	d.		£.	s.	d.
Balance in hand	15	11	1½	Printing in progress, about	20	0	0
36 Subscriptions, 1874-5	37	16	0	Postage and Stationery	3	10	10
9 " 1873-4	9	9	0	Balance.....	43	19	3½
4 " 1872-3	4	4	0				
	£67	0	1½		£67	0	1½
6 Life Compositions made in Session 1874-5.							
33 " " in all, namely—							
28 at £10	£280	0	0				
5 at £10. 10s.	52	10	0				
	£332	10	0				

Invested in £368. 14s. 11d. Consols.

Major J. R. Campbell and Prof. G. M. Minchin were elected members of the Society.

Prof. Clifford read a paper on "The Transformation of Elliptic Functions." Prof. Cayley gave some results of his paper "On a System of Algebraical Equations." The President spoke (1) on a Problem of Eisenstein's; (2) on the joint Invariants of two Conics or Quadrics; and (3) on the Equation $P \times D = \text{constant}$, of a Geodesic Line on an Ellipsoid. Mr. Tucker communicated a paper by Prof. H. W. Lloyd Tanner, "On the Solution of certain Partial Differential Equations of the Second Order, having more than two Independent Variables."

The following presents were received:—

"Monatsbericht," Juli and Aug. 1875.

"Bulletin de la Société Mathématique de France," tome iii., Oct., No. 6, Paris.

"Illustrations of the Centimètre-Gramme-Second System of Units:" by Prof. Everett (Physical Society).

"Bulletin des Sciences Mathématiques et Astronomiques," Juillet, Août, 1875.

"Reprint of Mathematics from the 'Educational Times,'" vol. xxiii.

"Crelle," 81 Band, erstes Heft.

Kronecker "über die congruenten Transformationen der bilinearen Formen," Berlin, 1874.

Kronecker "über Schaaren von quadratischen und bilinearen Formen," Berlin, 1874.

Kronecker "über die algebraischen Gleichungen von denen die Theilung der elliptischen Functionen abhängt." (Auszug aus dem "Monatsbericht," 19 Juli 1875.)

Kronecker "über quadratische Formen von negativen Determinante."

Kronecker, "Bemerkungen zur Geschichte des Reciprocitäts-Gesetzes. (Auszug aus dem "Monatsbericht," 19 April 1875.)

"Nautical Almanac," for 1879.

"The Repulsion of Solid Bodies referable to Radiation, a speculation concerning Molecular Physics," by Rev. A. J. Stevens, M.A., Nov. 1875: from the Author.

"On a General Method of producing exact Rectilinear Motion by Link-work," by A. B. Kempe, B.A. (Proceedings of Royal Society, No. 163, 1875): from the Author.