VII. *On the Crispations of Fluid resting upon a Vibrating Support. By* Lord RAYLEIGH, *D.C.L., F.R.S., Cavendish Professor of Physics in the University of Cambridge**.

IF a glass plate, held horizontally, and made to vibrate as for the production of Chladni's figures, be covered with a thin layer of water or other mobile liquid, the phenomena in question may be readily observed. Over those parts of the plate which vibrate sensibly the surface of the liquid is ruffled by minute waves, the degree of fineness increasing with the frequency of vibration. The same crispations are observed on the surface of liquid in a large wine-glass or finger-glass which is caused to vibrate in the usual manner by carrying the moistened finger round the circumference. All that is essential to the production of crispations is that a body of liquid with a free surface be constrained to execute a vertical vibration. It is indifferent whether the origin of the motion be at the bottom, as in the first case, or, as in the second, be due to the alternate advance and retreat of a lateral boundary, to accommodate itself to which the neighbouring surface must rise and fall.

More than fifty years ago the nature of these vibrations was examined by Faraday with great ingenuity and success. His results are recorded in an Appendix to a paper on a Peculiar Class of Acoustical Figures†, headed "On the Forms and States assumed by Fluids in Contact with vibrating Elastic Surfaces." In more recent times Dr. L. Matthiessen has travelled over the same ground‡, and on one very important point has recorded an opinion in opposition to that of Faraday. In order more completely to satisfy myself, I have lately repeated most of Faraday's experiments, in some cases with improved appliances, and have been able to add some further observations in support of the views adopted.

The phenomenon to be examined is evidently presented in its simplest form when the motion of the vibrating horizontal plate on which the liquid is spread is a simple up-and-down motion without rotation. To secure this, Faraday attached the plate to the centre of a strip of glass or lath of deal, supported at the nodes, and caused to vibrate by friction. In my experiments an iron bar was used about 1 metre long and ·0064 metre thick (in the plane of vibration). The bar was supported horizontally at the nodes; and to its centre a glass plate was attached by gutta-percha and carefully levelled.

* Communicated by the Author.
† Phil. Trans. 1831.
‡ Pogg. *Ann.* t. cxxxiv. 1868, t. cxli. 1870.

The vibrations of the bar were maintained electromagnetically, as in tuning-fork interrupters, with the aid of an electromagnet placed under the centre, the circuit being made and broken at a mercury-cup by a dipper carried at one end of the bar. By calculation from the dimensions*, and without allowance for the load at the centre, the frequency of (complete) vibration is 33. Comparisons with a standard tuning-fork gave more accurately for the actually loaded bar a frequency of 31.

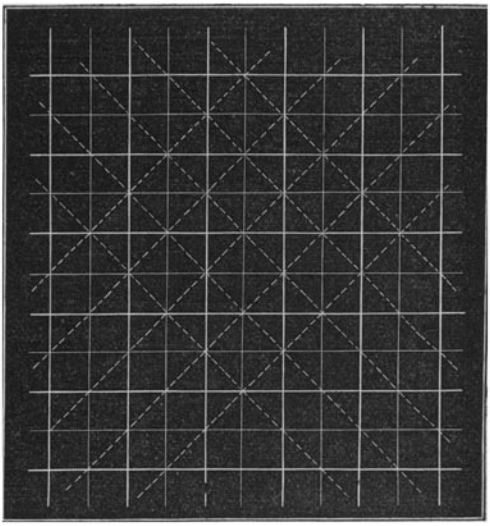
The vibrating liquid standing upon the plate presents appearances which at first are rather difficult to interpret, and which vary a good deal with the nature of the liquid in respect of transparency or opacity, and with the incidence of the light. The vibrations of the liquid, whether at the rate of 31 per second, or, as in fact, at the rate of $15\frac{1}{2}$ per second, are too quick to be followed by the eye; and thus the effect observed is an average, due to the superposition of an indefinite number of components corresponding to the various phases of vibration.

The motion of the liquid consists of two sets of stationary vibrations superposed, the ridges and furrows of the two sets being perpendicular to one another, and usually parallel to the edges of the (rectangular) plate. Confining our attention for the moment to one set of stationary waves, let us consider what appearance it may be expected to present. At one moment the ridges form a set of parallel and equidistant lines, the interval being the wave-length. Midway between these are the lines which represent at that moment the position of the furrows. After the lapse of $\frac{1}{4}$ period, the surface is flat; after another $\frac{1}{4}$ period, the ridges and furrows are again at their maximum development, but the *positions are exchanged*. Now, since only an average effect can be perceived, it is clear that no distinction can be recognized between the ridges and the furrows, and that the observed effect must be periodic within a distance equal to *half* a wave-length of the real motion. If the liquid on the plate be rendered moderately opaque by addition of aniline blue, and be seen by diffused transmitted light, the lines of ridge and furrow will appear bright in comparison with the intermediate nodal lines where the normal depth is preserved throughout the vibration. The gain of light when the thickness is small will, in accordance with the law of absorption, outweigh the loss of light which occurs half a period later when the furrow is replaced by a ridge.

The actual phenomenon is more complicated in consequence

* 'Theory of Sound,' § 171.

of the coexistence of the two sets of ridges and furrows in perpendicular directions (x, y). In the adjoining figure the thick



lines represent the ridges, and the thin lines the furrows, of the two systems at a moment of maximum excursion. One quarter period later the surface is flat, and one half a period later the ridges and furrows are interchanged. The places of maximum elevation and depression are the intersections of the thick lines and of the thin lines, not distinguishable by ordinary vision; and these regions will appear like holes in the sheet of colour. The nodal lines, where the normal depth of colour is preserved, are shown dotted; they are inclined at 45°, and pass through the intersections of the thick lines with the thin lines. The pattern is recurrent in the directions of both x and y, and in each case with an interval equal to the real wave-length (λ). The distance between the bright spots measured parallel to x or y is thus λ; but the shortest distance between these spots is in directions inclined at 45°, and is equal to $\frac{1}{2}\sqrt{2}\,.\,\lambda$.

In order to determine the relation of the frequency of the liquid vibrations to that of the bar, an apparatus was fitted up capable of giving an intermittent view of the vibrating system. This consisted of a blackened paper disk pierced with three sets of holes, mounted upon an axle, and maintained in rotation by a small electromagnetic engine of Apps's construction. The whole was fastened to one base-board, and could be moved about freely, the leading wires from the battery being flexible.

The current was somewhat in excess; so that the desired speed could be attained by the application of moderate friction. At a certain speed of rotation the appearances were as follows. Through the set of four holes (giving four views for each rotation of the disk) the bar was seen double. Through the set of two holes the bar was seen single, and the water-waves were seen double. Through the single hole the bar was seen single, and the waves also were seen single. From this it follows that the water vibrations are not, as Matthiessen contends, synchronous with those of the bar, but that there are two complete vibrations of the support for each complete vibration of the water, in accordance with Faraday's original statement.

An attempt was made to calculate the frequency of liquid vibration from measurements of the wave-length and of the depth. The depth (h), deduced from the area of the plate and the whole quantity of liquid, was ·0681 centim.; and by direct measurement $\lambda=\cdot 848$ centim. Sir W. Thomson's formula connecting the velocity of propagation with the wave-length, when the effect of surface-tension is included, is

$$v^2=\frac{\lambda^2}{\tau^2}=982\left(\frac{\lambda}{2\pi}+\frac{\cdot 074\times 2\pi}{\lambda}\right)\times\frac{e^\alpha-e^{-\alpha}}{e^\alpha+e^{-\alpha}}, \quad . \quad . \quad (\text{A})$$

where $\alpha=2\pi h/\lambda$. With the above data we find for the frequency of vibration (τ^{-1}) 20·8. This should have been 15·5; and the discrepancy is probably to be attributed to friction, whose influence must be to diminish the efficient depth, and may easily rise to importance when the total depth is so small.

Another method by which I succeeded in determining the frequency of these waves requires a little preliminary explanation. If $n=2\pi/\tau$, and $\kappa=2\pi/\lambda$, the stationary waves parallel to y may be expressed as the resultant of opposite progressive waves in the form

$$\cos(\kappa x+nt)+\cos(\kappa x-nt)=2\cos\kappa x\cos nt. \quad . \quad . \quad (1)$$

This represents the state of things referred to an origin fixed in space. But now let us refer it to an origin moving forward with the velocity (n/κ) of the progressive waves, so as to obtain the appearance that would be presented to the eye, or to the photographic camera, carried forward in this manner. Writing $\kappa x'+nt$ for κx, we get

$$\cos(\kappa x'+2nt)+\cos\kappa x'. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now the average effect of the first term is independent of x', so that what is seen is simply that set of progressive waves which moves with the eye. In this way a kind of resolution of the stationary wave into its progressive components may be effected.

In the actual experiment two sets of stationary waves are combined; and the analytical expression is

$$\cos(\kappa x+nt)+\cos(\kappa x-nt)+\cos(\kappa y+nt)+\cos(\kappa y-nt), \quad . \quad (3)$$

which is equal to

$$2\cos\kappa x\cos nt+2\cos\kappa y\cos nt, \quad . \quad . \quad . \quad . \quad (4)$$

or to

$$4\cos\frac{\kappa(x+y)}{2}\cos\frac{\kappa(x-y)}{2}\cos nt. \quad . \quad . \quad . \quad . \quad (5)$$

If, as before, we write $\kappa x'+nt$ for κx, we get

$$\cos(\kappa x'+2nt)+\cos\kappa x'+2\cos\kappa y.\cos nt. \quad . \quad . \quad (6)$$

The eye, travelling forward with the velocity n/κ, sees mainly the corresponding progressive waves, whose appearance, however, usually varies with y, i. e. along the length of a ridge or furrow. If the effect could be supposed to depend upon the *mean* elevation only, this complication would disappear, as we should be left with the term $\cos\kappa x'$ standing alone. With the semi-opaque coloured water the variation along y is evident enough; but the experiment may be modified in such a manner that the ridges and furrows appear sensibly uniform. For this purpose the coloured water may be replaced by milk, lighted from above, but very obliquely. The appearance of a set of (uniform) ridges and furrows varies greatly with the direction of the light. If the light fall upon the plate in a direction nearly parallel to the ridges, the disturbance of the surface becomes almost invisible; but if, on the other hand, the incidence be perpendicular to the line of ridges, the disturbance is brought into strong relief. The application of this principle to the case before us shows that, when the eye is travelling parallel to x, the ridges and furrows will look nearly uniform if the incidence of the light be also nearly parallel to x; but if the incidence of the light be nearly parallel to y, the ridges will show marked variations along their length, and in fact be resolved into a series of detached humps. The former condition of things is the simplest, and the most suitable as the subject of measurement.

In order to see the progressive waves it is not necessary to move the head as a whole, but only to turn the eye as when we look at an ordinary object in motion. To do this without assistance is not at first very easy, especially if the area of the plate be somewhat small. By moving a pointer at various speeds until the right one is found, the eye may be guided to do what is required of it; and after a few successes repetition becomes easy. If we wish not merely to see the progressive waves, but to measure the velocity of propagation with some approach to accuracy, further assistance is required. In my experiments an endless string, passing over pulleys and driven by a small water-engine, travelled at a small distance above the plate so that its length was in the direction of wave-propagation. A piece of wire was held at one end by the fingers, and at the other rested upon the travelling string and was carried forward with it. In this way, by adjusting the water supply, the speed of the string could be made equal to that of wave-propagation; and the former could easily be determined from the whole length of the string, and from the time required by a knot upon it to make a complete circuit. Thus (on February 7) the velocity of propagation was found to be 5·4 inches per second. At the same time, by measurement of the pattern as seen by ordinary vision, $14\lambda = 4\frac{7}{8}$ inches. Hence frequency $= \dfrac{5{\cdot}4}{\lambda} = 15{\cdot}5$ per second; exactly one half the observed frequency of the bar, viz. 31.

In addition to the phantoms which may be considered to represent the four component progressive waves, others may be observed travelling in directions inclined at 45°. If we take coordinates ξ, η in these directions, (5) may be written

$$4 \cos \frac{\kappa\xi}{\sqrt{2}} \cos \frac{\kappa\eta}{\sqrt{2}} \cos nt\,; \quad . \quad . \quad . \quad . \quad . \quad (7)$$

in which if we put

$$\frac{\kappa\xi}{\sqrt{2}} = \frac{\kappa\xi'}{\sqrt{2}} + nt$$

(*i. e.* if we suppose the eye to travel with velocity $\sqrt{2}\,.\,n/\kappa$), we get

$$2 \cos \frac{\kappa\eta}{\sqrt{2}} \cos \frac{\kappa\xi'}{\sqrt{2}} + \text{ terms in } 2nt.$$

The non-periodic part may be supposed roughly to represent the phenomenon.

In order if possible to settle the question beyond dispute, I made yet another comparison of the frequencies of vibration of the fluid and of the support, using a plan not very different from that originally employed by Faraday. A long plank was supported on trestles at the nodes, and could be tuned within pretty wide limits by shifting weights which rested upon it near the middle and ends. At the centre was placed a beaker $4\frac{1}{4}$ inches in diameter, and containing a little mercury. The plank was set into vibration by properly timed impulses with the hand, and the weights were adjusted until the period corresponded to one mode of free vibration of the pool of mercury. When the adjustment is complete, a very small vibration of the plank throws the mercury into great commotion, and unless the vessel is deep there is risk of the fluid being thrown out. The question now to be decided is whether, or not, the vibrations of the mercury are executed in the same time as those of the plank.

On March 18 the plank was adjusted so as to excite that mode of vibration of the mercury in which there are two nodal diameters. Two other diameters bisecting the angles between these give the places of maximum vertical motion. At one moment the mercury is elevated at *both* ends of one diameter and depressed at both ends of the perpendicular diameter; half a period later the case is reversed. The frequency of the fluid vibrations could be counted by inspection, and was found to be 30 (complete) vibrations in 15 seconds, or exactly two vibrations per second. The vibrations of the plank were counted by allowing it to tap slightly against a pencil held in the hand. In five seconds there were 21 complete vibrations, *i. e.* $4\frac{1}{5}$ vibrations per second, almost exactly twice as many as was found for the mercury. The measurements were repeated several times; and the general result is beyond question.

On another occasion the mode of fluid vibration was that in which there is but one nodal diameter, the fluid being most raised at one end of the perpendicular diameter and most depressed at the other end. The frequency of fluid vibration was $30/22 = 1{\cdot}36$; while that of the plank was $27/10 = 2{\cdot}7$. Here again the fluid vibrations are proved to be only half as quick as those of the support.

The mechanics of the question are considered in a communication to the Philosophical Magazine for April 1883, to which reference must be made. Merely to observe the phenomenon, it is sufficient to take a porcelain evaporating-dish containing a shallow pool of mercury 2 or 3 inches in diameter, and, holding it firmly with both hands, to impose upon it a vertical vibratory motion. After a few trials of various speeds it is possible to excite various modes of vibration, including those referred to in connexion with the plank. The first (with two nodal diameters) is more interesting in itself, and is more certainly due to a vertical as opposed to a horizontal vibration of the support. The gradually shelving bank presented by the dish adds to the beauty of the experiment by its tendency to prevent splashing.

Dr. Matthiessen, in the papers referred to, records a long series of measurements of the wave-lengths of crispations corresponding to various frequencies of vibration, not only in the case of water, but also of mercury, alcohol, and other liquids. He remarks that the nature of the liquid affects the relation in a marked manner, contrary to the theoretical ideas of the time, which recognized gravity only as a "motive" for the vibrations. In the following year Sir W. Thomson gave the complete theory of wave-propagation*, in which it is shown that in the case of wave-lengths so short as most of those experimented upon by Matthiessen, the influence of cohesion, or capillary tension, far outweighs that of gravity. In general, if T be the tension, $\kappa=2\pi/\lambda$, the velocity of propagation (v) is given by

$$v=\sqrt{\left\{\frac{g}{\kappa}+\mathrm{T}\kappa\right\}}; \quad . \quad . \quad . \quad . \quad . \quad (8)$$

or, when λ is small enough,

$$v=\sqrt{(\mathrm{T}\kappa)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Since $\lambda=v\tau$, the relation between τ and λ is, by (9),

$$2\pi\mathrm{T}\tau^2=\lambda^3; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

or, if N be the frequency of vibration,

$$\mathrm{N}^{\frac{3}{2}}\lambda=\text{constant}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Dr. Matthiessen's results agree pretty well with (11), much better in fact than with the formula proposed by himself.

There is another point of some interest on which the views expressed by Matthiessen call for correction. It was observed by Lissajous some years ago, that if two vibrating tuning-forks of slightly different pitch are made to touch the surface of water, the nearly stationary waves formed midway between the sources of disturbance travel slowly towards the graver. We may take as the expression for the two progressive waves

$$\cos(\kappa x-nt)+\cos(\kappa' x+n't),$$

or, which is the same,

$$2\cos\{\tfrac{1}{2}(\kappa+\kappa')x+\tfrac{1}{2}(n'-n)t\}\times\cos\{\tfrac{1}{2}(\kappa'-\kappa)x+\tfrac{1}{2}(n'+n)t\}.$$

The position at any time of the crests of the nearly stationary waves is given by

$$\tfrac{1}{2}(\kappa+\kappa')x+\tfrac{1}{2}(n'-n)t=2m\pi,$$

where m is an integer. The velocity of displacement V is thus

$$\mathrm{V}=\frac{n-n'}{\kappa+\kappa'}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

* Phil. Mag. Nov. 1871.

from which it appears that in every case the shifting is in the direction of propagation of waves of higher pitch, or towards the source of graver pitch.

According to Matthiessen, the shifting takes place with a velocity equal to half the difference of velocities of the component trains, *i. e.*

$$2V = \frac{n}{\kappa} - \frac{n'}{\kappa'}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and in the direction of that component train which moves with greatest velocity. So far as regards the direction merely, the two rules come to the same thing for the range of pitch used by Lissajous and Matthiessen, since over this range the velocity increases with pitch. If, however, we have to deal with waves longer than the critical value (1·7 centim. for water), the two rules are at issue, since now the velocity increases as the pitch diminishes. The following are a few corresponding values, in C.G.S. measure, of wave-length, velocity, and frequency of vibration calculated by Thomson's formula (A).

Wave-length...	·5	1·0	1·7	2·5	3·0	5·0
Velocity	31·48	24·75	23·11	23·94	24·92	29·54
Frequency ...	62·97	24·75	13·60	9·579	8·306	5·908

I have examined the matter experimentally with the aid of vibrators making from 12 to 7 complete vibrations per second, and therefore well below the critical point, with the result that the transference is towards the source of graver pitch, although this is the direction of propagation of the component which travels with the smaller velocity. I reserve for the present a more detailed description of the apparatus, as I propose to apply it to the general verification of Thomson's law of velocities.

VIII. *An Illustration of the Crossing of Rays.* *By* Walter Baily*.

[Plate I.]

WHEN rays of light are passing through a point, the resultant motion of the æther is in general far too complicated to be conceived; but if the light is homogeneous, it can readily be shown that the motion at each point is simply harmonic motion in an ellipse; so that in that case the

* Communicated by the Physical Society; read May 26, 1833.