III. A new reading of relativity

Prof. Frederick Slate

To cite this article: Prof. Frederick Slate (1920) III. A new reading of relativity , Philosophical Magazine Series 6, 40:235, 31-49, DOI: [10.1080/14786440708636095](https://doi.org/10.1080/14786440708636095)

To link to this article: <http://dx.doi.org/10.1080/14786440708636095>

Published online: 08 Apr 2009.

Submit your article to this journal [↗](#)

Article views: 4

View related articles [↗](#)

increment of energy being due to Marcelin and Rice *, the present theory independently arrives at the same conclusion, and, moreover, shows that the critical increment is equal to a whole number of quanta at the infra-red fundamental. In a series of papers, Lewis † has made certain calculations of the critical increments necessary in certain reactions and the frequencies at which this energy is absorbed. He writes the expression given above in the form

$$\text{Observed heat of reaction} = E_0 - E_1,$$

where E_1 is the sum of the critical increments of the reactant molecules and E_0 that of the resultant molecules. When the energy is absorbed or evolved at the phase frequency, this expression will only be true if the reactant molecules are completely dissociated into atoms. It is evident from the present hypothesis that the critical increment is a whole number of quanta at the infra-red fundamental—an amount of energy which may be very much smaller than one quantum at the phase frequency. In such a case impossible results will be obtained if the frequencies are calculated from the critical increments derived from the heats of dissociation of the molecules into atoms.

III. *A New Reading of Relativity.*

By Prof. FREDERICK SLATE ‡.

A RECENT paper § has broken ground for a rectified comparison between Einstein's kinematics and Newton's methods by insisting first upon restoring the broader connexion of force with "Quantity of motion." New tribute is paid to the hypnotic power of a current view to evade challenge, in the delay of this conclusive step towards juster correlation, after the disadvantage within the range of electromagnetics had been conceded against the narrowed form of Newton's equations. But an undeniable habit has inclined overmuch toward some particulars of rigid dynamics, however aware one is latently that the literal condition of variable mass is often directly natural, and that the devices lie close at hand which fit for parallel treatment the enlarged (or

* Rep. Brit. Assoc. p. 397 (1915).

† Trans. Chem. Soc. 1914 to 1919, *passim*.

‡ Communicated by the Author.

§ Slate, Phil. Mag., April 1920, p. 433.

more figurative) concept of inertia *. Else why Minkowski's words? "Die Gleichungen der Newtonschen Mechanik zeigen eine zweifache Invarianz. Einmal bleibt ihre Form unverändert wenn man das zugrunde gelegte räumliche Koordinatensystem einer beliebigen Lagenveränderung unterwirft, zweitens, wenn man es in seinem Bewegungszustande verändert, nämlich ihm irgendeine gleichförmige Translation aufprägt" †. In the origins of relativity, the equivalent of equations (1, 2, 3) below seems nowhere drawn into consideration. Einstein and Lorentz do not break away radically from the former's equation :

"Massenzahl $\times$ Beschleunigungszahl = Kraftzahl" ‡.

This type of relation is in evidence unaltered, even where a variable "transverse mass" enters. For the Newtonian procedure, a force thus expressed implies an inertia held stationary at its epoch-value ; the derivative of momentum is partial. We shall meet an important instance presently §.

The more thoroughgoing ascription of variable inertia to electrons, as a starting-point either for Newtonian dynamics or for relativity, carries with it then nothing forced, nor subtle, nor unique into the equations of motion, activity, and work. Grant freely how much a dense ignorance about detail in atomic and electronic systems compels us to leave vague, beyond the proof of inertia through observed partitions of energy. But neither Newton nor Einstein escapes that

* Careful elementary comment upon Newton's laws will not neglect this side. See Tait and Steele (1878), 'Dynamics of a Particle,' p. 330, etc. ; Slate (1900), 'Principles of Mechanics,' p. 220, etc. Passing over to moment of inertia, it may change with the rotation-axis. The idea of "effective inertia," expanded fruitfully by Sir J. J. Thomson, includes variableness of that quantity, especially where undetected (or ignored) forces are in play. This latter phase, in its application to electrons, I have indicated earlier: 'Science' (1908), vol. xxviii. p. 180.

† Raum und Zeit. Conveniently accessible in the collection of foundation-laying papers (cited here as "Sammlung") : Lorentz—Einstein—Minkowski; *Das Relativitätsprinzip*, Teubner (1913). See p. 56 ; and p. 72.

‡ Sammlung, p. 51. See also (p. 28) : "Die zu entwickelnde Theorie stützt sich . . . auf die Kinematik des starren Körpers." The position taken by Lorentz shows on pp. 17, 78, etc. Expository writers like Silberstein conform : see his 'Theory of Relativity,' pp. 15, 17 ; Chapter VII, *passim* ; etc. This is an eminently temperate summary. The terms of all such statements are at best misleading. It would be unfair to pin down to them every recent exponent of relativity ; shading of attitude there cannot fail to be. Yet they still muster heavy enough backing to call for definite abatement. The change of "rest-mass" with internal energy is set off as a distinct question.

§ Equation (10), below.

indeterminateness now ; while the fullest employment of their united suggestions must mark the sanest course in adjudicating questions there, or in regard to gravitation. In that interest, this paper turns solely to detecting simple links of interrelation between the two methods. Whatever yokes standard dynamics and relativity for joint service based on complete reciprocal consistency—this the new line of approach seeks. Preferential choice may nevertheless remain open to individual opinion ; or indeed it may prove to shift with the class of problem in hand. Some resettlement of rating in this quarter is scarcely avoidable, because the extension to variable inertia certainly upsets at least one allegation. The fact looms up at once that the rigid dynamics (of Einstein's theory) becomes now a convergence-point of two more general forms, under approximations through $dm/dt=0$; $c=\infty$; respectively. It is not tenable that broader Newtonian dynamics itself reduces directly from relativity by neglecting in the latter terms that contain powers of $(1/c)$ *.

Those expressions for tangential force, activity, and work laid down in the previous paper lie then at the heart of the matter. For convenience we quote them, with minor changes :—

$$T_o \equiv \frac{d}{dt}(Q_o) \equiv \frac{d}{dt}(mv_o) = m \frac{dr_o}{dt} + v_o \frac{dm}{dt} ; \quad . \quad . \quad (1)$$

$$A_o \equiv v_o T_o = \frac{d}{dt} \left(\frac{mv_o^2}{2} \right) + \frac{1}{2} v_o^2 \frac{dm}{dt} ; \quad . \quad . \quad . \quad (2)$$

$$W_o \equiv \int_0^t v_o T_o dt = \left[\frac{E}{v_o} \right]_{v_o=0}^{v_o=t} + \frac{1}{2} \int_0^t v_o^2 \frac{dm}{dt} dt. \quad . \quad . \quad (3)$$

The subscript letter (*o*) tags a quantity, here and elsewhere, as “observed,” or as calculated without artifice from observations. The equations are written for one standard frame (F), always for standard signs and in C.G.S. measure. Plainly (Q_o) symbolizes momentum, and (E_o) kinetic energy. So long as the dynamical process mirrors quantitatively one of pure mechanics, nothing hinders a mixed reading for (m) ; it can cover narrow weight-mass and inclusive inertia, with (Q_o , E_o) read to correspond.

In equations (2, 3), the “Principle of *vis viva*” has gone by the board, along with an exclusive measurement of force

* Cf. Silberstein, p. 115 ; Laue, *Das Relativitätsprinzip*, p. 157 ; and others.

by a product of inertia and acceleration*. Equation (1) does not calculate the same value of (T_o) from the second member, when (v_o) alone is altered there. Neither are values of the acceleration equal for differing (v_o) but equal assigned (m, T_o) . This equation of motion is not fully invariant thus in either aspect of its double function. But even under other conditions that do satisfy the "Newtonian transformation," the freedom due to its force-invariance should not be overrated. Not only do kinetic energy, momentum, activity, change with velocity-values, but since the relative path is affected, so also is the distribution of the same total acceleration into its tangential and normal parts. Therefore entire continuity across transfers among "legitimate frames" will require some composite scheme. The fuller scope of invariance for relatively fixed frames can be worked in here; at times with less than explicit credit for its help, as will be seen later.

Inquiry is still limited to progressive rectilinear motion of an electronic particle (m), in part because this case makes vital contacts and gives cues for less particular suppositions. To lay out its province for relativity gradually by induction runs truer to scientific form in any event; we are not declaring a creed.

Adopting the "deformable electron" brought forward by Lorentz and supported by experimental data, the rule is taken over:

$$\frac{dm}{dv_o} = \frac{mv_o}{c^2 - v_o^2}; \quad \frac{dm}{dt} = \frac{mv_o}{c^2 - v_o^2}; \quad m = m_0 \frac{c}{\sqrt{c^2 - v_o^2}}. \quad (4)$$

Retain:

$$\gamma(z) \equiv \left(1 - \frac{z^2}{c^2}\right)^{-\frac{1}{2}}; \quad m_0 \equiv [m]_{v_o=0}; \quad c \equiv \text{light-speed in (F)}.$$

Then all the work may go into field-structure, as we noted, and equations (1, 2, 3) become, if $v_o = (v)_0$ at $t=0$,

$$T_o = m \left(1 + \frac{v_o^2}{c^2 - v_o^2}\right) \frac{dv_o}{dt} = m \gamma^2(v_o) \frac{dv_o}{dt} = m_0 \gamma^3(v_o) \frac{dv_o}{dt}; \quad (5)$$

$$A_o = c^2 \frac{dm}{dt}; \quad W_o = c^2 m_0 (\gamma(v_o) - \gamma(v)_0). \quad . \quad . \quad . \quad . \quad (6)$$

Whatever physical particulars record themselves through

* Silberstein, p. 195; Laue, p. 5; illustrating the need of some such remark.

equation (4) are of necessity carried with it into all combinations that accept it. With its aid, standard dynamics and C.G.S. measure have led straight to these results for the frame (F). They are formally identical with what relativity would allow, for parallel conditions; though demanding for them also wider validity. Emphasize that (m_0) being a special value here, its identification with "Miukowski mass" requires constancy in the latter*. Modifications of equation (4) on this score or on others can be introduced, but they are not of immediate concern. It can remain plastically true, if held open to revision by further experimental success. The equivalence of consolidations in equation (5) through addition and through multiplication is a feature with notable consequences†.

What proves to be one main thread of the subsequent analysis joins on to an elementary problem, whose results we quote in part. Consider a straight path, constant propelling force (P) and mass (m_1), and a resistance (R) proportional to (v^2). There is a terminal speed (v_1). With notation and detail

$$\frac{P}{m_1} \equiv a; \quad \frac{R}{m_1} \equiv kv^2; \quad v_1^2 = \frac{a}{k};$$

the equation of motion

$$P - R = m_1 \frac{dv}{dt}; \quad \frac{P - R}{m_1} = \frac{dv}{dt}; \quad \dots \quad (7)$$

by simple recasting yields the forms

$$P = \frac{m_1 v_1^2}{v_1^2 - v^2} \frac{dv}{dt}; \quad R = \frac{m_1 v^2}{v_1^2 - v^2} \frac{dv}{dt}; \quad \dots \quad (8)$$

$$P = m_1 \frac{dv}{dt} + \frac{m_1 v^2}{v_1^2 - v^2} \frac{dv}{dt}; \quad \frac{P}{m_1} = \frac{dv}{dt} + \frac{v^2}{v_1^2 - v^2} \frac{dv}{dt} \quad \dots \quad (9)$$

But equations (7) apply legitimately to net field-action combining a constant propelling field (a), and an automatically excited resisting field (kv^2), the acceleration being then characteristically independent of the "body coefficient" (m_1). And any actual physical linkage of (a , k), when (k) changes

* See note †, p. 32; and cf. Silberstein, p. 194.

† Among others this: it favours grafting forms due to variable inertia upon a root-idea of constant (m_0). Relativity's "complex" of force has the magnitude of (T_0): e.g., for the tensor of its quaternion; which the so-called "rest-system" then renders coincident with a "Newtonian" force. Cf. Silberstein, pp. 193-4. The same thought applies at equation (11) below.

with (*a*), that entails constancy of the ratio (*a/k*), would evidently fix also a common value of the terminal speed, whatever strength (*a*) has. Supposing this condition met, and with (*v*₁=*c*) given, equation (9) corresponds visibly to equation (5) save in one essential respect: the type of the original equation (7) makes (*m*₁) necessarily constant. It is true, the elimination of (*m*₁) would permit assigning any series of epoch-values to that quantity, without influencing the kinematics through the consequent readjustments of force. They might differ arbitrarily or in conformity with a rule laid down. They might follow, for example, the range of (*m*), as it varies under the rule of equation (4). Yet each such momentary value of (*m*₁) must be rated constant as a parameter, because equation (7) has required that.

The proposition is defensible that light-speed is also limiting speed for all electrons under any electromagnetic field-forces. It can borrow some confirmation from forms like equation (6), which somehow standardize in terms of (*c*), and it is plausibly consistent with an idea impressed upon relativity regarding (*c*) as an unexceeded value. That this critical speed should coincide with the wave-speed does not look unreasonable. Only one of these rôles for (*c*), then, would be common to light and gravitation, since the critical speed alone is known in the latter; it separates the ellipse as an orbit from the two other conics. The analogies of these two field-actions might go astray on such an ambiguity. As affording a provisional background for some electronic dynamics, we shall write the above assumption about (*c*) into our expressions relative to (F).

Note next how the first of equations (8) in effect suppresses (R), which is *passive* in the sense of not contributing to the total energy-flux, but merely diverting part of it. It is clearly immaterial whether the diverted energy is reversible or not. Read the second member as a product of the true acceleration by an effective inertia*. Since that inertia-factor is here variable, the equation in the form

$$P = m_2 \frac{dv}{dt} = (m_1 \gamma^2(v)) \frac{dv}{dt} \quad . \quad . \quad . \quad (10)$$

must be incomplete, as judged by equation (1) which becomes

* Effective inertia merits wider use perhaps; it is so variously adaptable to bridge gaps in knowledge or to secure compacter statement. The pure translation that equation (5) supposes may thus replace a screw-motion of a rigid solid having constant mass and moment of inertia. The pitch of the screw would vary.

its obligatory type. The artificial infinity of (m_2) for $(v=c)$ is requisite to preserve the zero-acceleration at $(P=R)$ with finite (m_1) . Within the freedom to utilize epoch-values for (m_1) , choose a series that gives perpetual equality of (m, m_1) . That device amounts to completing equation (10), and establishing an exact step-by-step agreement in force-magnitude with equation (5). Yet the two plans do not quite coalesce; (dm) can never be primarily an increment of any constant (m_1) , also residual divergence from equations (3, 6) must continue. The Principle of *vis viva* is differentially valid; each constant (m_1) absorbs a work-element wholly into change of kinetic energy. Remark finally that assignment of permanent value (m_0) to (m_1) brings in a significant instantaneous relation

$$T_o = \gamma(v)P \quad [m_1 = m_0] \dots \dots (11)$$

This line of possible connexion has been dwelt upon as preliminary at more length than its simple thought would justify, did it not strike centrally at the reasons—often left elusively subtle—which necessitate certain transitions peculiar to relativity. Its plain bearing will not be overlooked, it can be hoped, upon the duplicated forces of “Minkowski type” and of “Newton type”; nor upon the introduction of “proper time” as well as “local time.” Both of which discriminations are given effect differentially, on the model of equation (11)*.

Associated with the standard frame (F) are a group (U) of comparison frames (“legitimate frames”) having un-accelerated translations (u) relative to it, (u, v_o) being colinear, and for standard sign codirectional. Transfers between (F) and (U) disturb fundamental expressions; but we shall test there what degree of approach can be disclosed nevertheless to that permanence of type which is relativity’s victory. Velocity and tangential acceleration belonging to (F) will show in any (U) the magnitudes

$$v' \equiv v_o - u; \quad \frac{dv'}{dt} = \frac{dv_o}{dt}. \quad [(u) \text{ is vector-constant.}]$$

* Relativity has invented manipulation of factors like $(\gamma(v))$, where it needs to add or to suppress them, with admirable success. But no more will be expected here than to indicate clues at important turns of the discussion, after which calculative detail is read with new meanings in mere routine. Moreover, the points made can appeal to those only who judge the argument with an interest due to knowledge of the subject.

Accordingly an "apparent" force (T_a') estimated in that (U) is given by

$$T_a' \equiv m \frac{dv_o}{dt} + (v_o - u) \frac{dm}{dt} = \frac{d}{dt}(mv') = T_o - u \frac{dm}{dt} \dots (12)$$

Add this explicit agreement: Effective inertia is inferred from physical relations in (F), and passes unchanged through such purely kinematical revisions. Direct combination of equations (4, 12) under their condition of equal accelerations shows

$$T_a' = \left(1 - \frac{uv_o}{c^2}\right) T_o. \dots (13)$$

The "reduction factor" $(1 - uv_o/c^2)$ connects the forces manifest in (U, F) through using (m) with the respective kinematical data. Or equation (12) is evidently applicable to compare forces in (F), for the same inertia and acceleration, at differing speeds.

In particular at ($u = v_o$),

$$T_a' = m \frac{dv_o}{dt} \quad [u = v_o] \dots (14)$$

This introduces the familiar "rest-system" of relativity. And since we may write

$$T_a' = \left[\left(1 - \frac{uv_o}{c^2}\right) m_o \gamma^3(v_o) \right] \frac{dv_o}{dt} \equiv m' \frac{dv_o}{dt},$$

under the suggestion of rigid dynamics, it would follow that

$$\frac{c^2 - v_o^2}{c^2 - uv_o} T_a' = \left(\frac{c^2 - v_o^2}{c^2 - uv_o} m' \right) \frac{dv_o}{dt} = m \frac{dv_o}{dt} \dots (15)$$

Identification of "Fresnel's coefficient" (κ) in this second member fits the idea of its connexion with "inertia-drag" in (U) *.

Einstein's "Addition theorem for velocities" presents the same reduction factor as equation (13). But there it would reduce the velocity of (m) relative to (O') the origin of (U) to dependence upon specifying frames. Newton makes that a constant difference of any pair of corresponding values. The intrinsic meaning of the theorem must go deeper, since its characteristic factor offers itself unconstrainedly in a dynamic relation of the two frames. This must quicken the surmise that what underlies the "Lorentz transformation"

* Cf. Silberstein, pp. 172-3.

is not exhausted when redefining simultaneousness [Einstein] is added to the other avowed purpose of gaining in formal symmetry [Lorentz]. The same thought is continued, if we recall too how gravitation-force (G_1) is "apparent" in weight (W_1), with reduction factor that in the simplest case takes either form,

$$W_1 = G_1 \left(1 - \frac{\omega v}{G}\right); \quad G_1 = W_1 \left(1 + \frac{\omega v}{g}\right). \quad (16)$$

This essential parallelism of equations (13, 16) throws new light on reading (u) in terms of imaginary rotation *.

A second standard use of equations of motion adopts force as physically determinate (given) and predicts its kinematical effects by calculation. This would impose (T_o) invariantly upon (U, F): one stock instance is a convected compressed spring. Then we should pair with equation (12)

$$T_o = m \frac{dv'}{dt} + v' \frac{dm}{dt}, \quad . \quad . \quad . \quad (17)$$

where now the accelerations ($dv'/dt, dv_o/dt$) fall away from equality, when the given values accompanying the same (m) are

$$T_o \leq T_a'; \quad v' = v_o - u.$$

Evidently also, the last relation would not be a permanent adjustment. What devices may look towards reconciling the divergent plans of equations (13, 17) is the next natural question.

The answer will begin at considering the activity ($v'T_o$) developed by the force (T_o) of equation (17) at the working-speed (v'). In this aspect, both factors belong immediately to (U). In view of equation (13) that activity is equivalently

$$v'T_o = v' \frac{T_a'}{1 - \frac{uv_o}{c^2}} = \left(\frac{v'}{1 - \frac{uv_o}{c^2}} \right) T_a' \equiv v_c' T_a', \quad . \quad (18)$$

(v_c') being a calculated (auxiliary) speed, but without departure from C.G.S. measure. Thus the activity within (U) of a force (T_o) implying unequal accelerations in (U, F) can be equalized with that of a force (T_a') implying equal accelerations, if at the same time the working-speed be

* Minkowski; Sommerfeld. It is arithmetic to adjust a "length-contraction" to producing numerical equality of the fields (G, g). The difference that these replace (c^2) is provably superficial.

modified from the observed value ($v_o' = v_o - u$) as equation (18) denotes. Put into other words: Einstein's rule yields a distorted speed in (U) which compensates exactly the distortion of force into (T_a') from (T_o). What may well be called a process internal to (U) is made in so far indifferent about the supposition of equal accelerations. Plainly the same rule holds between (U) and any frame (F') fixed relatively to the original (F). But observe how our derivation locates that compensation within the experimentally verified scope of equation (4). As a matter of algebra, we obtain the four forms :

$$v_c' \equiv \frac{v_o - u}{1 - \frac{uv_o}{c^2}}; \quad v_o = \frac{v_c' + u}{1 + \frac{uv_c'}{c^2}}; \quad \gamma^2(u)(v_c' + u) = \frac{v_o}{1 - \frac{uv_o}{c^2}};$$

$$\frac{dv_o}{dt} \gamma^2(v_o) = \frac{dv_c'}{dt} \gamma^2(v_c'). \quad (19)$$

They all connect some Newtonian velocity of (*m*) relative to (O, O') the origins of (F, U) with Einstein's specification of it. None of these alternative aspects is released, however, from its source in a strictly conventional expedient to the same end, which is here just nakedly announced, but which the Lorentz transformation has masked under the Einstein variables.

The completer activity ($v_o T_o$) in (F), expressed also by means of (v_c' , T_a') is

$$A_0 \equiv v_o T_o = \gamma^2(u) T_a' (v_c' + u). \quad (20)$$

Therefore

$$T_o = \gamma^2(u) T_a' \left(1 + \frac{uv_c'}{c^2}\right); \quad \gamma(u) T_a' = m_0 \frac{dv_c'}{dt} \gamma^3(v_c'). \quad (21)$$

Equations (13, 21) being demonstrated corollaries of equation (5), they are pivotal relations between the "observing-frame" (F), and the auxiliary quantities (appearances) in a comparison frame (U).

Continuing, we may undertake to render equation (13) more fully reciprocal between the frames (F, U). Instead of depending upon observations in (F), let them be primarily taken in (U), the distinctive notation being (dv_o'/dt , v_o' , T_o'). Correlate the force denoted by (T_o') on the one hand with its associated kinematics and on the other with a supposititious partner (T_a) in (F). The plan of equation (18) gives

symmetrically, for equivalent activities now "internal" to (F),

$$(v_o' + u) T_o' = v_c T_a; \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where the factors in the second member are first any pair that satisfy the equality. Then equation (20) repeats in the form

$$v_o' T_o' = \gamma^2(u) T_a(v_c - u). \quad . \quad . \quad . \quad . \quad (23)$$

Since (T_a) is a force in (F) at the speed (v_c) , equation (5) holds; so

$$T_a = m_o \gamma^3(v_c) \frac{dv_c}{dt}. \quad . \quad . \quad . \quad . \quad (24)$$

Both equations (22, 23) are satisfied when (v_c) is determined by the conversion rule that pursues the track of relativity:

$$v_c = \frac{v_o' + u}{1 + \frac{uv_o'}{c^2}}; \quad v_o' = \frac{v_c - u}{1 - \frac{uv_c}{c^2}}; \quad \gamma^2(u)(v_c - u) = \frac{v_o'}{1 + \frac{uv_o'}{c^2}};$$

$$\frac{dv_o'}{dt} \gamma^2(v_o') = \frac{dv_c}{dt} \gamma^2(v_c) * . \quad . \quad (25)$$

And provable corollaries of this assumed value are

$$T_a = \left(1 + \frac{uv_o'}{c^2}\right) T_o'; \quad T_o' = \gamma^2(u) T_a \left(1 - \frac{uv_c}{c^2}\right);$$

$$T_o' = m_o \gamma(u) \gamma^3(v_o') \frac{dv_o'}{dt}. \quad . \quad . \quad . \quad . \quad (26)$$

The mathematics fixes upon equation (23) as committing us already to the consequences in equation (26), by requiring for consistency that (v_c) which equation (25) calculates. The symmetry of (F, U) postulated by relativity is in fact seen to lurk in equation (23). The logical sequence may be reversed and begin by superposing the third of equations (26) upon equation (5). Whatever is hypothetical in equation (23) is reflected in equation (26), and *vice versa*.

It would not be overbold analogy, as the outcome thus far shows, to anticipate through the second form of equation (16) the factor that "reduces" from (U) to (F). In the view of local action, (T_o, T_o', T_a, T_a') of an electromagnetic statement would involve effective strengths of a "motionless field" (F-space, medium), and of a "convected field" (U-space, medium). But whereas physics has long labelled (W_1) without hesitation "apparent," the notation here need not be stressed hastily with similar decision. When *relations* are ascertained clearly between mathematical sequences whose

* Connect equation (10) with this invariance, and one similar in equations (19).

correspondence has been more obscure, and a way is opened to standardize in one frame by systematic reduction, the final issues in "true physics *versus* correct mathematics" are not prejudiced by some postponement. Indeed, the closer scrutiny is revealing multiplied chances for illusion and encouraging suspended judgment.

Certain critical values of the auxiliary velocities (v_c' , v_c) deserve mention. We find

$$\left. \begin{aligned} v_o &= 0, v_c' = -u; & v_o' &= 0, v_c = u; \\ v_o &= c, v_c' = c; & v_o' &= c, v_c = c; \\ \frac{v_c'}{v_o - u} &= \frac{0}{0} = \gamma^2(u) [v_o = u]; & \frac{v_o'}{v_c - u} &= \frac{0}{0} = \gamma^2(u) [v_c = u]. \end{aligned} \right\} \dots (27)$$

These make plain first that the limiting speed (c), if affirmed for (v_o), applies also to (v_c'). The two series (v_o , v_c') in (F, U) respectively, and each within the range (0, c), are so telescoped relatively that the initial overlap (u) is gradually annulled. Secondly, the relative translation ($\pm u$) of (F, U) links all aspects of the combinations as an unchanged magnitude. There is just so much demand as can be read into these special ratios, for the "practical infinity of (c);" or for the "axiomatic necessity" that light-speed relatively to all frames (U) stands invariant at (c)^{*}.

Equations (13, 21, 26) have connected force-values by four reduction factors; with algebraically equal values shown by

$$\left. \begin{aligned} 1 - \frac{uv_o}{c^2} &= \frac{v_o - u}{v_c'} = \frac{\gamma(v_c')}{\gamma(u)\gamma(v_o)}; \\ 1 + \frac{uv_c'}{c^2} &= \frac{v_c' + u}{v_o} = \frac{\gamma(v_o)}{\gamma(u)\gamma(v_c')}; \\ 1 + \frac{uv_o'}{c^2} &= \frac{v_o' + u}{v_c} = \frac{\gamma(v_c)}{\gamma(u)\gamma(v_o')}; \\ 1 - \frac{uv_c}{c^2} &= \frac{v_c - u}{v_o'} = \frac{\gamma(v_o')}{\gamma(u)\gamma(v_c)}. \end{aligned} \right\} \dots (28)$$

* The comment on equation (9) would lead to a finite activity of (P) expending work upon (It) during unlimited time of asymptotic approach to finite (c). Equation (6) also presents gross work (of (T_o)), with whose result relativity agrees. Finding $W_o = \infty$; $v_o = c$; constant (m_o); does not confirm the practical infinity of (c). Not even after redefining (W_o) as that "change of kinetic energy" whose disappearance from it we have traced. Attentive sifting of infinity-values is seldom misplaced in physics.

The second members appear as "telescoping ratios" for velocities. The third members may reduce effective inertia conveniently. For instance,

$$T_o' = \frac{T_a'}{1 - \frac{uv_o}{c^2}} = \frac{m}{1 - \frac{uv_o}{c^2}} \frac{dv_o}{dt} + \frac{v_o - u}{1 - \frac{uv_o}{c^2}} \frac{dm}{dt} \\ = m_o \gamma(u) \gamma(v_c') \frac{dv_c'}{dt} + v_c' \frac{dm}{dt}. \quad (28a)$$

The Fresnel coefficient (κ) of equation (15) is given also by

$$\kappa = \frac{\gamma(u) \gamma(v_o)}{\gamma(v_c')} \cdot \frac{1}{\gamma^2(v_o)} = \frac{\gamma(u)}{\gamma(v_c') \gamma(v_o)} \\ = \frac{\gamma(u) \gamma(v_c')}{\gamma(v_o)} \cdot \frac{1}{\gamma^2(v_c')} = \frac{c^2 - v_c'^2}{c^2 + uv_c'};$$

and by

$$\kappa = 1 - \frac{v_o v_c'}{c^2}. \quad (29)$$

Equations (28a, 29) lead directly to the relation, significant like equation (15) of meaning for (κ),

$$[m_o \gamma(u) \gamma(v_c')] \frac{dv_c'}{dt} = T_o' \left(1 - \frac{v_o v_c'}{c^2} \right) = \kappa T_o. \quad (29a)$$

Because

$$\left(1 - \frac{uv_o}{c^2} \right) \left(1 + \frac{uv_c'}{c^2} \right) = \left(1 + \frac{uv_o'}{c^2} \right) \left(1 - \frac{uv_c}{c^2} \right) = \frac{1}{\gamma^2(u)},$$

the product of each factor by $\gamma(u)$ modifies the reductions so that the original magnitude is restored by a combination of direct and reverse reduction between (F, U). Therefore our results reproduce formally the transformations for Minkowski force (K). If we identify $T_o \equiv K_F$; $\gamma(u) T_a' \equiv K_u$;

$$K_u = \gamma(u) \left(1 - \frac{uv_o}{c^2} \right) K_F; \quad K_F = \gamma(u) \left(1 + \frac{uv_c'}{c^2} \right) K_u. \quad (30a)$$

Similarly, for $K_F' \equiv T_a$; $K_u' \equiv \frac{T_o'}{\gamma(u)}$;

$$K_u' = \gamma(u) \left(1 - \frac{uv_c}{c^2} \right) K_F'; \quad K_F' = \gamma(u) \left(1 + \frac{uv_o'}{c^2} \right) K_u'. \quad (30b)$$

On the line of this parallelism, the "Newton force" of relativity and its "local time" do not enter directly. Perhaps they are secondary in a larger sense, as more artificial alignment with Newtonian mechanics in part misread.

It is a striking feature about the forms referring to the frame (U) that $\gamma(u)$ recurs as a sort of "weighting factor" which also designates the particular frame that is associated with (F). Comparing the last forms in equations (21, 26), we may say that $\gamma(u)$ disturbs the symmetry or completes it; we may seek symmetry in the first member or in the second; and one additional step reaches either end. Consideration of the work-equations helps to analyse these connexions further. Direct integration of either extreme member in equation (18), referring to equations (5, 21), gives

$$\int_u^v T_o(v_o - u)dt = \int_o^{v_c'} T_a' v_c' dt = c^2 \frac{m_o}{\gamma(u)} (\gamma(v_c') - 1). \quad (31)$$

From equation (22) the work obtained is, remembering equations (24, 26),

$$\int_o^{v_o'} T_o'(v_o' + u)dt = \int_u^{v_c} T_a v_c dt = c^2 m_o (\gamma(v_c) - \gamma(u)). \quad (32)$$

Complete the set; adding in reliance upon equation (26)

$$\int_o^{v_o'} T_o' v_o' dt = c^2 m_o \gamma(u) (\gamma(v_o') - 1). \quad (33)$$

Equations (6, 32) express work done in (F); equations (31, 33) work done in (U). The scheme of substituted velocities and new limits is symmetrical in all four. But transition from (F) to (U) divides by $\gamma(u)$ in one case and multiplies by it in the other. The deciding fact is patent: Equation (6) presents the original (or physical) quantity as belonging to (F), and makes allowance for its "appearance" as judged in (U); equation (33) takes its start in the physics of (U) and calculates what appears in (F). Substituting for equation (26) the form

$$T_o'' \equiv \left(1 - \frac{uv_c}{c^2}\right) T_a, \quad (34)$$

which would embody the reversed reduction with the same (T_a), gives immediate verification. Finally it is plain that the various reduction factors, in combination with original totals and with fractions of them, constitute a mode of projection*.

The above group of relations brings out the idea that they approach a treatment of motion from rest in (U) as though

* This furnishes a strong hint about their affiliations with imaginary angles, and the natural adaptation of hyperbolic trigonometry to relativity.

the initial inertia were $(m_0' = \gamma(u)m_0)$, which continuity of value with our assumptions for the standard (F) would make it. The result that one vital difference hidden under identical kinematics in (F, U) is due to altered inertia affecting work provoked early comment. In one dynamical analysis of relativity, it was shrewdly proposed as a chief postulate*. The accompanying pairs of interchanges between $(v_o' + u, v_o, v_o')$ and $(v_o', v_o, v_o - u)$ must certainly not be ignored. Yet the commonsense reason for those is also one key to Einstein's redefined simultaneousness. The same value of fluxion time (t) as an argument in time functions defines Newtonian simultaneous values of them. In this meaning, the auxiliary (v_o') and the observed (v_o) are simultaneous; also (v_o, v_o') . In addition, simultaneous observations in (F, U) would of necessity be recorded as $(v_o, v_o - u)$, or as $(v_o' + u, v_o')$. It is clear that a pair of such observable data $(v_o' + u, v_o')$ would not realize, for instance, the simplicity of equations (26), if $(v_o' + u)$ were substituted for (v_o) there. But if time-slip (lag or lead) between straightforward C.G.S. observations in (F, U) be regulated under a known law of velocity-change, the paired values (v_o, v_o') requisite for simplicity are also attainable as observed. The same thought coordinates equation (13) [Newton] and equation (21) [Einstein]. Further, because this "propagated simultaneousness" and the Newtonian allowance for "apparent force" establish identical reduction factors; for Newtonian velocity and for Einstein's force-law respectively; it seems fairly proven that the two systems of procedure are effectively alternative—for the electronic case discussed.

The reciprocal relations between the dynamics of the standard frame (F) and that formulated for any one frame of the group (U) are to this extent conclusively settled. It remains to examine how much of such reciprocity persists, when neither of the compared frames is (F). This will confront the conditions where relativity can do best service through its simplifications. Choose then two comparison frames (U', U'') with origins (O', O'') and with translations (u', u'') relative to (F), still colinear with (v_o) of course. Denote observed velocities of (m) now by (v, v', v'') , so that

$$v = v' + u' = v'' + u''; \quad v' - v'' = u'' - u'. \quad (35)$$

A reasonable notation for other quantities will also distinguish them by accents. Thus simultaneously for (m):

$$T_a' = T_o \left(1 - \frac{u'v}{c^2} \right); \quad T_a'' = T_o \left(1 - \frac{u''v}{c^2} \right). \quad (36)$$

* Frank, Wied. Ann. (1912) vol. xxxix. p. 693.

Observations in (F) would show the velocity of (m) relative to (O'')

$$v'' = v' - (u'' - u') \equiv v' - u_o. \quad (37)$$

Equation (19) applied correctly to each part of the third member gives for velocities in (U') auxiliary to (v' , u_o),

$$v_c' \equiv \frac{v - u'}{1 - \frac{u'v}{c^2}}; \quad u_c' \equiv \frac{u'' - u'}{1 - \frac{u'u''}{c^2}}. \quad (38)$$

Then without complications the identity is verified :

$$\gamma(u'') T_{a''} \equiv \gamma(u_c') \left(1 - \frac{u_c' v_c'}{c^2}\right) (\gamma(u') T_{a'}). \quad (39)$$

In verbal form : Direct transition (F, U'') and two successive transitions (F, U'), (U', U''), coincide upon the force finally apparent in (U''). But in terms made consistent with equation (18) as regards all entries under (v''); and only when each apparent force is properly "weighted." The previous developments furnish plain reasons for both conditions.

Begin with (U') as observation-frame, and under transferred guidance of equations (13, 26) construct the provable identity:

$$\gamma(u_o) \left(1 - \frac{u_o v'}{c^2}\right) T_o' \equiv \left[\gamma(u_c) \left(1 - \frac{u_c v_c}{c^2}\right) \right] \gamma(u') \left(1 + \frac{u' v'}{c^2}\right) T_o'. \quad (40)$$

Again results agree exactly for direct transition (U', U'') and for the two-step process, (U' F) followed by (F, U''). In this use, corresponding to equation (38) we must have in their relation to (v) and (U') :

$$v_c \equiv \frac{v' + u'}{1 + \frac{u'v'}{c^2}}; \quad u_c \equiv \frac{u_o + u'}{1 + \frac{u'u_o}{c^2}}. \quad (41)$$

Without repetitive detail, therefore, the conclusion can be put generally : When these discriminations about terms are upheld, the calculated net distortion (into apparent force) depends upon the terminal frames alone. Hence it vanishes if the series of transitions closes at the initial frame; the remark under equation (29) becomes a wider truth*. As bearing upon our immediate purpose, it places all transitions within the group (U) in a comprehensive setting of partial reduction to the same standard (F).

* A consequence that one might borrow from the familiar theorem on superposition of colinear Lorentz transformations, by assigning present meanings to its symbols.

Equation (14) records the special value of (T_a') for that rest-frame to which relativity concedes unique prominence through a coincidence in magnitude for the two force-types ("Minkowski" and "Newton") and a peculiar "rest-acceleration." Analysis of the transition from (F) to a rest-frame (R) in those aspects detects three stages :

- (a) Changing Minskowski-force into Newton-type.
- (b) Cancelling a factor $(\gamma^2(u))$; or $(\gamma^2(v))$ transiently undistinguished from it; with some freedom among assignable reasons.
- (c) Restoring the magnitude $(\gamma^3(v) dv/dt)$; now as acceleration in (R) under the sliding-scale of units for time and length.

A mathematically colourless regrouping of factors led to the important quantities (v_e', v_e) of equations (18, 25); its convenience has wide range. But such fully interchangeable forms are often unequally accommodated to physics. Equation (5) presents in effect this same rest-acceleration as incidental to a (probably fictitious) fusion of force-items. Only an inflexible view about constant inertia can force upon relativity the result of (c) as an acceleration. In comparison, is the latter not artificial? What is worse, it is superficial as well, if it obscures that inclusion of rest-frames in the "shift process" common to many coordinate-systems and describable as a parameter-variation*. In that feature, it enforces the thought attached to equations (10, 11), and prompts a directer interpretation of rest-frame calculations, though the formal results stand uncorrected.

We build upon a more general relation for any frame (U) and specify a momentum (Q_1) determined from a velocity (v_1) and an inertia (m_1) through

$$Q_1 \equiv m_1 \gamma(u) v_1 = (m_1 \gamma(u)) v_1 = m_1 (\gamma(u) v_1) \dots \quad (42)$$

This introduces the same weighting factor as equations (30, 31, 33) and it is written out to suggest a double possibility: associating $(\gamma(u))$ with either (m_1) or (v_1) . Regarding (Q_1) as a function of (u, v_1) , its exact differential and its total

* For the elaboration of this idea see Slate, 'Fundamental Equations of Dynamics' (1918); Index, under "Shift," Shift-rate." Nothing but unformed habit bars embracing in the same scheme continuous change in the unit-magnitudes. Other seed-thoughts of the present paper are to be found at pp. 38-43; p. 211; where the need of equations (1, 2, 3) is emphasized.

time-rate are, when (m_1) is constant,

$$dQ_1 = \frac{\partial Q_1}{\partial u} du + \frac{\partial Q_1}{\partial v_1} dv_1; \\ \frac{dQ_1}{dt} = m_1 v_1 \frac{cu}{(c^2 - u^2)^{3/2}} \frac{du}{dt} + \frac{cm_1}{(c^2 - u^2)^{1/2}} \frac{dv_1}{dt}. \quad (43)$$

Defining $m' \equiv m_1 \gamma(u)$, the last equation is mathematically presentable as a measure of tangential force,

$$\frac{dQ_1}{dt} \equiv T = v_1 \frac{dm'}{dt} + m' \frac{dv_1}{dt} = \frac{d}{dt}(m'v_1). \quad (44)$$

Yet the first term in the third member is spurious for the physics, so long as (m_1) is constant; it has its sole origin in a shift-rate among the frames (U).

In application to a finite interval, the name rest-system at equation (14) is a collective noun, comprising selected frames in the possible group (U); the earmark of a rest-frame is the transient equality of its permanent (u) and the velocity (v) of (m) at that one epoch. Identify (m_1, m_0) and (v_1, v) ; and let the shift-rate be so regulated that $(v=u)$ perpetually. If the spurious force were real (*i. e.*, physical), equation (44) would give (T_0) correctly, on making the substitutions

$$u = v; \quad \frac{du}{dt} = \frac{dv}{dt}; \quad m_1 = m_0. \quad (45)$$

Or it would repeat the essentials of equations (21, 24, 26), on exchanging $(v_c' v_c, v_o')$ for (v) ; while due use of the weighting factor harmonizes the forms completely. Thus an assumed constant inertia, plus illusory (imaginary, quaternionic) force due to a glossed-over shift, can replace with mathematical precision our assumed variable inertia*, through continuous use of the rest-frames for the several epochs.

Since the working-speed (v_o', v_o') becomes zero for a rest-frame, that cannot be self-sufficing for the expression of activity. The directest composite plan contains as well frames fixed relatively to (F)†; but more deviously, relativity reduces the quasi-force $(v_1 dm'/dt)$ by the factor (c/v_1) , coins an unrealized velocity-term (c), attaches it loosely to (v_1) , and arrives correctly at the activity (A_o) of equation (6) essentially thus:

$$A_o = c^2 \frac{dm}{dt} = c \left(c \frac{dm}{dt} \right) = c \left(\frac{c}{v} \right) \left(v \frac{dm}{dt} \right). \quad (46)$$

* It is apposite to quote Sommerfeld's remark that the element $(d\tau)$ of proper time "Ist kein vollständiges Differential."

† Slate, 'Fundamental Equations,' pp. 147-54. That background of standard frame cannot be quite ignored. Lorentz remarks "Bei einer absoluten Genauigkeit wäre auch der Unterschied der 'Eigenzeit' der Erde von der Zeit des Sonnensystems zu berücksichtigen."

Ingenious device contrives a mathematical status for the last term in equation (2), whose recognition at first hand constancy of inertia excludes.

The natural inferences that equation (13) has forced upon attention by restating the situation of equation (16), offer more than a glimpse of a self-selected frame, basic for physical phenomena in this sense: Departures from it, either by rotation or by translation impress distortions of common type upon dynamical magnitudes. To render that deeply-seated symmetry convincing might crown our instinctive search better than inventing for physical laws a formal indifference among base-frames. If such ideal outlook has restricted appeal, there is practical service in coordinating at the same time elements of potential. That a potential is available which gives coherence to all phases of our electron's energy, the work equation indicates through its sole dependence upon terminals of the interval. Equations (39, 40) are another aspect of that thought. Moreover, the initial supposition of equation (5) and its probable connexions at equations (9, 10, 11) fit at sight the more standard form of the Lagrange function (L) as measuring appropriately a difference of energy-states. Write

$$L \equiv -m_0 c^2 \left(\frac{\sqrt{c^2 - v^2}}{c} - 1 \right) = m_0 c^2 - m(c^2 - v^2). \quad (47)$$

Fixing the zero-phase as rest in the frame (F), the last member sets down an expectation then bounded by bringing (m_0) to the limiting speed (c). In the general phase, (m_0) has become (m) by gathered increment; so reading (mv^2) as a fact of realization the last term exhibits what other energy of (m) at that epoch is outstanding for final conversion into kinetic form; at the limit-value (c), whatever (m) may be attained by accretion. The form ($2E - L$) for total energy agrees with equation (6) for work done from rest.

Whatever cumulative pressure from these arguments is felt will carry us toward restoring to fuller proportions the shrunken claim that Lorentz urges for "Wahre Zeit" and for frames "Die im Äther ruhen;" a passage remarkable for its scientific candour*. The complications due to curvilinear path, to radiation, and to other causes do not weaken this primary correlation of Einstein with Newton. Both lines of thought alike must cope with the superposed difficulties.

University of California.

* *Sammlung*, p. 75.