A Proof of the Exactness of Cayley's Number of Seminvariants of a given Type. By E. B. ELLIOTT. Received and read June 9th, 1892.

1. The general gradient, or rational integral homogeneous isobaric function, of order i and weight w in the n quantities $a_0, a_1, a_2, \dots a_n$, is a sum of arbitrary multiples of $(w;\, i, n)$ products, where $(w;\, i, n)$ denotes the number of partitions of w into i or fewer numbers, none exceeding n.

Let us use the common notation Ω for the differential operator

$$a_0\frac{d}{da_1}+2a_1\frac{d}{da_2}+3a_2\frac{d}{da_3}+\dots+na_{n-1}\frac{d}{da_n}.$$

The result of operating with Ω on our supposed gradient of type w, i, n is a gradient of type $w-1, i, n$, *i.e.*, a sum of multiples of all or some of $(w-1;\, i, n)$ products. The vanishing of this, in case all the $(w-1;\, i, n)$ products occur, and have linearly independent coefficients, requires the satisfaction of $(w-1;\, i, n)$ conditions in the coefficients of the original gradient. In such a case then the number of linearly independent seminvariants—gradients annihilated by Ω—of type w, i, n is the difference $(w;\, i, n)-(w-1;\, i, n)$.

Should this difference be negative, the $(w-1;\, i, n)$ coefficients in the derived gradient of type $w-1, i, n$ cannot of course be all independent, being definite linear functions of a smaller number of original coefficients. If, however, the difference is zero or positive, it is not *a priori* clear whether all must be independent or not. Were there r linear relations connecting them, the number of linearly independent seminvariants of type w, i, n would be

$$(w;\, i, n)-(w-1;\, i, n)+r,$$

provided this be zero or positive, and not $(w;\, i, n)-(w-1;\, i, n)$,

The well-known facts are that, if $in \not< 2w$, the number of linearly independent seminvariants of type w, i, n is exactly

$$(w;\, i, n)-(w-1;\, i, n),$$

and that if on the other hand $in < 2w$, there are no seminvariants of the type. These facts, stated much earlier by Cayley, were first demonstrated by Sylvester (*Phil. Mag.*, 1878), whose proof is reprinted in revised form in his "Lectures on Reciprocants" (*American Journal*, Vol. IX., Lecture 11).

Sylvester has also shown in the course of a second proof (Lecture 27) that, when $in-2w=-1$ in the general gradient ϕ of type w, i, n, no linear relation connects the coefficients in $\Omega\phi$, so that in this case also, as well as when $in-2w \nless 0$, the number $(w\,;\,i,n)-(w-1\,;\,i,n)$ of seminvariants is exact. There is no inconsistency with the general result in this, for the case is one in which $(w\,;\,i,n)=(w-1\,;\,i,n)$.

In the present paper, no attempt is made to improve on Sylvester's proof of the part of the theorem which asserts that when $in-2w<0$ there are no seminvariants of type w, i, n; nor is his quite simple demonstration of the fact reproduced. What is aimed at is so to use his fundamental lemmas as to obtain the other part of the theorem with no less facility, and thus to obviate the profound piece of reasoning by which he has established it.

In fact, that, if $in-2w \nless 0$ (or indeed $\nless -1$), the number of seminvariants of type w, i, n is precisely $(w\,;\,i,n)-(w-1\,;\,i,n)$, must follow if it can be shown that, when $in-2w \nless -1$, the result of operating with Ω on the most general gradient of type w, i, n is the production of the most general gradient of type $w-1, i, n$; and this must be the case if every gradient of type $w-1, i, n$, and in particular every separate product of that type, can be obtained by operating with Ω on some gradient or other of type w, i, n. Now that this is so will be found immediately to underlie Sylvester's lemmas.

For convenience, I will use $w+1$ instead of w.

2. Besides Ω take, as usual, the companion operator

$$O = na_1\frac{d}{da_0}+(n-1)\,a_2\frac{d}{da_1}+(n-3)\,a_3\frac{d}{da_2}+\ldots+a_n\frac{d}{da_{n-1}},$$

its designation O being Sylvester's.

What will be proved is that u, any gradient of type w, i, n, is the result of operating with Ω on the gradient of type $w+1, i, n$,

$$\frac{1}{\eta}\left\{O-\frac{1}{2(\eta+1)}\,O^2\Omega+\frac{1}{2.3\,(\eta+1)(\eta+2)}\,O^3\Omega^2 \right.$$
$$\left. -\frac{1}{2.3.4\,(\eta+1)(\eta+2)(\eta+3)}\,O^4\Omega^3+\ldots\right\}u,$$

where η denotes $in-2w$, and is supposed positive.

Corresponding to the characteristics i, w, η in u, we have i, $w-1$, $\eta+2$ in Ωu, and i, $w+1$, $\eta-2$ in Ou, and consequently i, $w-r+s-t$, $\eta+2(r-s+t)$ in $\Omega^r O^s \Omega^t u$.

Now Cayley and Sylvester have both used the readily proved identity

$$(\Omega O - O\Omega)\,u = \eta u,$$

and Sylvester has based his method of dealing with the present question on the more general equivalence

$$(\Omega O^r - O^r\Omega)\,u = r(\eta-r+1)\,O^{r-1}u,$$

which is obtained by noticing that

$$\begin{aligned}(O^s\Omega O^{r-s} - O^{s+1}\Omega O^{r-s-1})\,u &= O^s(\Omega O - O\Omega)\,O^{r-s-1}u \\ &= O^s(\eta-2r+2s+2)\,O^{r-s-1}u \\ &= (\eta-2r+2s+2)\,O^{r-1}u,\end{aligned}$$

and summing from $s=0$ to $s=r-1$.

Now take in succession 1, 2, 3, 4, ... for r, and operate, not always on u, whose type is w, i, n, but in succession on u, Ωu, $\Omega^2 u$, $\Omega^3 u$, ..., remembering that the η characteristics of these gradients are respectively η, $\eta+2$, $\eta+4$, $\eta+6$, ... We thus find

$$\begin{aligned}(\Omega O - O\Omega)\,u &= \eta u, \\ (\Omega O^2 - O^2\Omega)\,\Omega u &= 2(\eta+1)\,O\Omega u, \\ (\Omega O^3 - O^3\Omega)\,\Omega^2 u &= 3(\eta+2)\,O^2\Omega^2 u, \\ \&c. \qquad & \&c.;\end{aligned}$$

that is to say,

$$\begin{aligned}\eta u + O\Omega u &= \Omega O u, \\ 2(\eta+1)\,O\Omega u + O^2\Omega^2 u &= \Omega O^2 \Omega u, \\ 3(\eta+2)\,O^2\Omega^2 u + O^3\Omega^3 u &= \Omega O^3 \Omega^2 u, \\ \&c. \qquad & \&c.\end{aligned}$$

After a time these equalities become mere identities of zeros, for $\Omega^{w+1}u$, at any rate, if no earlier $\Omega^r u$, vanishes, since its weight would be -1, an impossibility for any gradient or rational integral function.

The deduction is immediate, by addition of suitable multiples, alternately positive and negative, of the above equivalent pairs of gradients, that

$$u = \frac{1}{\eta}\,\Omega\left\{O - \frac{1}{2(\eta+1)}O^2\Omega + \frac{1}{2.3(\eta+1)(\eta+2)}O^3\Omega^2 - \dots\right\}u,$$

the series being continued till its terms vanish, as must be the case from the $(w+2)^{\text{th}}$ onwards at any rate.

Thus a gradient of type $w+1$, i, n, equal to $\Omega^{-1}u$, where u is any gradient of type w, i, n, has been found, provided η or $in-2w$ is positive. As should be the case, there is failure to find any such gradient if η be zero or negative.

If, then, $in-2w > 0$, Ω, operating on the most general gradient of type $w+1$, i, n, can produce nothing less general than the most general gradient of type w, i, n. In this, replace w by $w-1$, and we deduce that, if $in-2w \nless -1$, the result of operating with Ω on the most general gradient of type w, i, n is the most general gradient of type $w-1$, i, n. Consequently, if $in-2w \nless -1$, the number of linearly independent seminvariants of order i, weight w, and extent n or less, is exactly $(w;\ i,\ n)-(w-1;\ i,\ n)$, and no greater number.

3. Our aim is now achieved; but it may not be out of place to append a few remarks in connexion with the method and the theorem.

It occurs, for instance, to consider what theorem replaces that of the determination of the form of $\Omega^{-1}u$ when η or $in-2w$ is zero for u. The answer clearly is that

$$\Omega\left\{O - \frac{1}{1.2}O^2\Omega + \frac{1}{1.2^2.3}O^3\Omega^2 - \frac{1}{1.2^2.3^2.4}O^4\Omega^3 + \dots\right\}$$

annihilates u. Now we need not stop here, for Ω can annihilate no gradient of weight $w+1$ for which $in-2(w+1)$ is negative, as is here the case. Consequently

$$\left\{O - \frac{1}{1.2}O^2\Omega + \frac{1}{1.2^2.3}O^3\Omega^2 - \frac{1}{1.2^2.3^2.4}O^4\Omega^3 + \dots\right\}u = 0,$$

u being any gradient for which $in-2w = 0$.

If $in-2w$ is negative $(=-\eta'$ say), we are in like manner led to two theorems: that

$$\Omega\left\{O+\frac{1}{2(\eta'-1)}O^2\Omega+\frac{1}{2.3(\eta'-1)(\eta'-2)}O^3\Omega^2+\dots\right.$$
$$\left.\dots+\frac{1}{2.3\dots\eta'(\eta'-1)(\eta'-2)\dots1}O^{\eta'}\Omega^{\eta'-1}\right\}u$$
$$=-\eta' u+\frac{1}{2.3\dots\eta'(\eta'-1)(\eta'-2)\dots1}O^{\eta'}\Omega^{\eta'}u,$$

and that

$$\Omega\left\{O^{\eta'+1}\Omega^{\eta'}-\frac{1}{1(\eta'+2)}O^{\eta'+2}\Omega^{\eta'+1}\right.$$
$$\left.+\frac{1}{1.2(\eta'+2)(\eta'+3)}O^{\eta'+3}\Omega^{\eta'+2}-\dots\right\}u=0.$$

Of these two the first has apparently trifling interest. The second tells us that Ω annihilates a gradient for which $in-2w'$ is negative, which must accordingly itself vanish. Thus

$$O^{\eta'+1}\Omega^{\eta'}-\frac{1}{1.(\eta'+2)}O^{\eta'+2}\Omega^{\eta'+1}+\frac{1}{1.2(\eta'+2)(\eta'+3)}O^{\eta'+3}\Omega^{\eta'+2}-\dots$$

is an annihilator of any gradient u for which $in-2w'$ $(=-\eta')$ is negative.

4. Returning to the case of $\eta>0$, other expressions for $\Omega^{-1}u$, not in general terminating, and consequently probably of little use, may be written down. For instance, since

$$\eta+O\Omega=\Omega O,$$
$$\eta O\Omega+(O\Omega)^2=\Omega O\,.\,O\Omega,$$
$$\eta(O\Omega)^2+(O\Omega)^3=\Omega O(O\Omega)^2,$$
$$\&c.,\qquad \&c.,$$

the operation being always on u, we get

$$\eta u=\Omega O\left\{1-\frac{1}{\eta}O\Omega+\frac{1}{\eta^2}(O\Omega)^2-\frac{1}{\eta^3}(O\Omega)^3+\dots\right\}u,$$

the remainder after the r^{th} term on the right being

$$(-1)^r\frac{1}{\eta^{r-1}}(O\Omega)^r.$$

Should u then be such that for any value of r this vanishes, we should have the expression for $\Omega^{-1}u$,

$$\frac{1}{\eta}\,O\left\{1-\frac{1}{\eta}\,O\Omega+\frac{1}{\eta^2}(O\Omega)^2-\ldots+(-1)^{r-1}\frac{1}{\eta^{r-1}}(O\Omega)^{r-1}\right\}.$$

5. The application of § 2 obtained, by putting

$$\begin{aligned}
a_n &= a_0x^n + nb_1x^{n-1} + \frac{n(n-1)}{1.2}b_2x^{n-2}+\ldots+b_n = v, \text{ say},\\
a_{n-1} &= a_0x^{n-1}+(n-1)\,b_1x^{n-2}+\frac{(n-1)(n-2)}{1.2}b_2x^{n-3}+\ldots+b_{n-1} = \frac{1}{n}\frac{dv}{dx},\\
a_{n-2} &= a_0x^{n-2}+(n-2)\,b_1x^{n-3}+\ldots+b_{n-2} = \frac{1}{n(n-1)}\frac{d^2v}{dx^2},\\
\ldots &\quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
a_1 &= a_0x+b_1 = \frac{1}{n(n-1)\ldots 2}\frac{d^{n-1}v}{dx^{n-1}},\\
a_0 &= a_0 = \frac{1}{n(n-1)\ldots 2.1}\frac{d^nv}{dx^n},
\end{aligned}$$

deserves to be more widely appreciated than is perhaps the case. With this substitution, we have

$$\Omega = \frac{d}{dx},$$

and

$$\Omega^{-1} = \int^x,$$

and our conclusion, involved in the Cayley-Sylvester theorem itself, and not brought to light only by the present method, is that, if u be any product, of degree w in x, of i functions chosen from among v, a quantic in x of degree n, and its successive differential coefficients, or any linear function of such products, then, provided $in-2w>0$, $\int u\,dx$ is a rational integral function of $v, \frac{dv}{dx}, \ldots \frac{d^nv}{dx^n}$, with the addition of an arbitrary constant. It is, of course, clear that $\int u\,dx$ can be expressed in terms of v and its derived functions, but it is a fact hard to prove from first principles that it can be expressed integrally in terms of them. We should expect a negative power of a_0 or $\frac{d^n v}{dx^n}$ to occur as a factor of the integral, and we have indirectly the conclusion that, were $(w+1\,;\ i,\ n) < (w\,;\ i,\ n)$, such must in fact be the case. A succession of integrations by parts of course gives such a form. Thus, returning to the Ω notation,

$$\begin{aligned}\Omega^{-1}u &= \frac{a_1}{a_0}u - \Omega^{-1}\left(\frac{a_1}{a_0}\Omega u\right)\\ &= \frac{a_1}{a_0}u - \frac{a_1^2}{2a_0^2}\Omega u + \Omega^{-1}\left(\frac{a_1^2}{2a_0^2}\Omega^2 u\right)\\ &= \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\\ &\quad\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\\ &= \frac{a_1}{a_0}u - \frac{a_1^2}{2a_0^2}\Omega u + \frac{a_1^3}{2.3a_0^3}\Omega^2 u - \frac{a_1^4}{2.3.4a_0^4}\Omega^3 u + \ldots,\end{aligned}$$

a series which must terminate.

6. Provided $in-2w > 0$, we have found an expression for $\Omega^{-1}u$ rational and integral in $a_0, a_1, a_2, \ldots a_n$, though it is not for a moment asserted that, knowing the fact that there is such an expression, there is no easier method for us to obtain the expression than by use of the formula. When $in-2w \not> 0$, we can use our knowledge to be sure that there is an expression for $\Omega^{-1}u$ or $\int u\,dx$ which has a_0^{-1} for a factor to a much lower power than that obtained by the integration-by-parts formula of the end of the last article, viz., to the r^{th} power, where r is 1 or the number next greater than $\frac{\eta'}{n}$, according as $in-2w = 0$ or $-\eta'$. For, on this supposition, we see that

$$a_0^r u$$

has n or $rn-\eta'$, in the one case or the other, for its η characteristic, which is accordingly positive. Whence $\Omega^{-1}(a_0^r u)$ is rational and integral ($= v$, say), and consequently

$$\Omega^{-1}u = \frac{1}{a_0^r}v.$$

[NOTE. *September 9th*, 1892.—Since communicating this paper I have noticed that the identity of § 2 may be given the more elegant form

$$\left\{1 - \frac{\Omega O}{1^2} + \frac{\Omega^2 O^2}{1^2.2^2} - \frac{\Omega^3 O^3}{1^2.2^2.3^2} + \ldots\right\} u = 0.]$$