P A P E R S

PUBLISHED IN THE

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

ON SOME RECENT ADVANCES IN THE THEORY OF ALGEBRAIC SURFACES

(Presidential Address.)

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[Read December 12th, 1912.]

MR. PRESIDENT, LADIES, AND GENTLEMEN,—

In accordance with a custom which I have no right to seek to evade, I have to-night to make some remarks to you on the occasion of my giving up the Presidential chair, which I have had the honour to occupy for two years. If you will believe me, it would in many ways be most agreeable to me merely to acknowledge the kindness I have received from the Council, and to emphasize the fact, which is obvious enough, that the success of a society such as this depends for the most part on the Secretaries and Treasurer ; and, having said this, to sit down. Tradition, however, requires that something more than this should be attempted. And I wish to recommend to your notice a department of mathematics which has already reached a stage in which the results, as it seems to me, are extremely remarkable, though our English mathematical literature bears as yet very little sign of its influence. I wish to speak to you of some modern developments of the theory of algebraical surfaces. As so often happens in the progress of science, these have come about by the union of two subjects, which, though their origin was largely identical, had drifted somewhat apart—namely, the general algebraical theory of surfaces, as we know it in books on Solid Geometry, and the theory of Abelian functions.

But before I come to my theme, which is, I fear, somewhat technical, I wish to make some remarks in which I may hope to have the sympathy

of all. I come before you to-night not in the character of discoverer, but in the character of dealer or importer. What I wish to speak about has been made for the most part outside these islands. And I want to ask the question: Why is it so often the case that the early history in England of a department of Pure Mathematics is a history of importation? We have teachers of Pure Mathematics in various places; our students are as quick and as ready to learn the subject as foreign students are; and it is not due to a reasoned feeling that Pure Mathematics is unworthy of study, for when a theory has acquired a certain age we require our students to learn at least the elements of it. Is the explanation to be found in our examination system, which rests like a network of red tape upon all our teaching, or is it to be found in a backward state of our civilisation? Pure Mathematics has the importance belonging to a subject which has interested the human mind for thousands of years. As a science it is both the basis and the model for all exact knowledge, and there can be no safer guide to how much of it is necessary for this purpose than the curiosity of those who know its relations. It is, however, as an art that the subject must appeal to those who practise it. And what an art it is, how vast, how delicate! With problems of every degree of difficulty, its humblest devotee may make something beautiful with his own hand; with conceptions of every degree of abstractness, it can allay the most restless imagination. If the student years are to leave an inspiring background for the daily task, if creative effort be any solace when life is beset with meanness, Pure Mathematics surely has peculiar claims in any scheme of intellectual education. Surely, in an advanced community there would be, in every University of the land, a school of students working at mathematics for mathematics' sake, in addition to working at something else, perhaps, for livelihood's sake. We, as a mathematical society, ought not to rest content with things as they are. We ought constantly to urge that our Science is of the same sort of importance for the intellectual health of the community as are, for instance, the arts of literature, and painting, and music for the emotional health of the community.

Further, I ought, perhaps, to make an apology to those members of the Society who have devoted themselves far more continuously than I have to the study of geometrical questions, for venturing to speak on a geometrical theme. It will indeed appear quite sufficiently that the matter is a branch of the theory of Abelian functions, with which the few publications for which I have been responsible have been mostly concerned. But perhaps I can make my apology best, at the risk of being personal, by explaining how I came to study the matter. The narrative, as it deals

with a familiar subject, may serve the purpose of eliciting your interest, in the same way as mine was secured. I was concerned, now five years ago, with the proof of the theorem, that any function of several variables, of certain characteristics, which is simultaneously periodic in regard to the variables, can be expressed in terms of proper combinations of theta functions. For this I was led to consider a corpus of algebraical functions of a certain number of independent variables—a corpus here meaning a set defined to include all functions obtainable from any functions of the set by one of the four fundamental algebraical processes, addition, multiplication, and so on. The problem was to prove the existence of certain elements in terms of which all functions of the corpus could be rationally expressed. In the case of a corpus of rational functions of one variable, it is known that all the functions are rationally expressible in terms of one function of the corpus. In the case of a corpus of rational functions of two independent variables, it is also true that two functions of the corpus can be chosen in terms of which all other functions of the corpus can be rationally expressed. A particular application of this is a theorem in regard to surfaces in three dimensions. Suppose that the equation to a surface can be reduced to an identity by substituting for each of the three coordinates a proper rational function of two independent variables or parameters. It may then happen that these parameters are incapable of being each expressed as rational functions of the coordinates. But when this is so, it is still possible to choose two other parameters, instead of these, for which this unique reversion, and rational expression, shall be possible. This seemed an obvious enough conclusion, and extremely probable. But I could not find a proof of it, and the enquiries I made were fruitless. Finally, after much searching, I came, by a sort of accident, across a memoir written in Italy, in which a proof was contained, written only in 1893. On examining this, I found that many of the ideas with which I had struggled had presented themselves already from a geometrical point of view. And I was thus led to study the matter. You may perhaps think that the particular instance to which I have referred is mainly of formal interest, and that the result might well have been assumed without proof. Indeed this remark was made to me. On the same general ground one would perhaps regard it as obvious that if a rational relation connecting four variables can be satisfied identically by replacing the variables by rational functions of three parameters, then these parameters can be taken so as to be rational functions of the four variables. This, however, appears not to be necessarily the case, though, in spite of various attempts, no proof of this statement was given until this year, 1912. Even so, the manner of the proof is instructive as to the difficulty of such

questions; it depends upon an elaborate and detailed study of the cubic complex of lines.

Passing from this particular problem, I proceed now to outline the general idea of what I am dealing with.

If we try to compare the theory of conic sections, as it would present itself to the mind of a Greek, and as it presents itself to us, we shall find, if my impression is just, that to us the theory is simpler because of the introduction of general principles, which enable us to collect under a few heads what were to him a vast number of disconnected propositions. And these general principles are, speaking roughly, consequences of only one—namely, of the principle of correspondence, whereby we consider two figures as identical when to any point of either corresponds a single point of the other. Geometrically, that is, we consider the older metrical geometry as a particular case of projective geometry. Analytically, this is one of the consequences of the enormously important step made by Descartes in the introduction of coordinates, and arises by the free use of linear transformation of coordinates. In brief, Projective Geometry is the study of the invariants of the group of linear transformations.

After the conic sections, and with the introduction of coordinates, other curves naturally come into consideration, and it is possible to fill many pages with the metrical and projective properties of special curves. But this alone would be of small interest to us to-day. The subject of all possible plane curves is so vast that we need a wider principle of identification, of correspondence, than is furnished by linear transformation. Any two conics, leaving aside straight lines, can be changed into one another by linear transformation; this is not the case for any two plane curves. But two curves which cannot be linearly transformed into one another, as, for instance, two curves of different orders, may nevertheless be capable of being transformed into one another by relations which ensure an unambiguous correspondence of point to point, the coordinates of the point of either curve being rational functions of the coordinates of the corresponding point of the other curve. And this is the leading idea of any general theory of plane curves—that two curves are to be regarded as identical when they are capable of such birational transformation into one another. And the properties of a curve that are considered are, thus, the invariants of birational transformation.

And I want particularly to emphasize some historical facts which ensured that this point of view, which has become the dominant one, should receive due recognition. The equation of a plane curve is merely an algebraical relation connecting two variables; it is therefore fundamental in a wide field of considerations. Euler and Legendre, in attempt-

ing to extend the possibilities of the integral calculus, were led to consider certain plane curves, and Cramer's *Introduction to the Analysis of Algebraic Curves* is as early as 1750. But it is to Abel's and Riemann's investigations in regard to Abelian integrals that we owe the due recognition of the modern point of view. Abel considered sums of algebraical integrals, and shewed that these are expressible by sums of a certain number P , of integrals, just as a sum of elliptic integrals is expressible in terms of only *one* integral, beside algebraic and logarithmic functions. This number P may be said, with certain conventions, to be what is now called the genus or deficiency of a curve, and a large part of Abel's great paper is concerned with shewing how this number P can be found from the equation of the curve. Riemann, both explicitly in words and implicitly by the number of results of his theory of Abelian integrals which can only be expressed in terms of the genus—as, for instance, by his remarkable result that the number of irreducible constants for a curve of genus p is $3p-3$, when $p > 1$ —further emphasized the same idea, and proposed to say that all curves capable of being birationally transformed into one another belong to the same *class*. From the point of view adopted by these writers, it appears at once that the genus of a curve is an invariant of birational transformation. It is, without doubt, the most important of such invariants.

Riemann's discoveries were made largely by help of a theorem of existence based on the theory of double integrals, and in connexion with his conception of a Riemann surface. But the geometrical results so obtained were expounded from a purely algebraical point of view later on, by Clebsch and Brill and Noether in Germany, and afterwards, with added refinement and generalisation, by a numerous school of Italian geometers. To-day none of us can think of a theory of plane curves without the apparatus of point sets, which gives to the theory a unity and simplicity which is very remarkable.

One of the obvious suggestions of such a theory of plane curves is that it ought to be capable of being extended to surfaces. This is a particularly welcome suggestion to anyone who knows by trial the difficulties of formulating a theory of functions of several variables in extension of the ordinary theory of functions of one variable. The idea presented itself to Clebsch. As the genus of a plane curve can be defined by the number of everywhere finite algebraical integrals attached thereto, so, if we consider everywhere finite integrals, double integrals however, attached to a surface, we obtain a number which is unaltered by birational transformation. Like many important steps in discovery, this idea was published in a brief note, at the end of 1868.

The suggestion involved in this remark found two different lines of development. On the one side, by Noether, in Germany, and afterwards by a powerful school in Italy, having its roots far down in the investigations which had been undertaken there for plane curves, a theory of the invariants of the birational transformation of surfaces was elaborated, working with an apparatus of sets of curves lying on surfaces, analogous to, but far more intricate than, the preceding theory of sets of points of curves, and the elaboration is as yet far from complete. No Englishman who attempts to acquaint himself with the exuberance, the power and the beauty of these enquiries, can fail to be struck with admiration, even with envy.

But this is not all. In the theory of plane curves it had happened, historically, that the theory of the rational functions was subsequent to a transcendental theory of the detailed properties of algebraic integrals. Possibly there might be an analogous circumstance in the case of surfaces. At any rate Picard, in France, set himself to imitate, on a surface, the theory of the various kinds of integrals known to exist for a plane curve, namely, those which are everywhere finite, those which possess algebraical infinities, and those which possess logarithmic infinities. The integrals he investigated in this regard which up to the present have led to the most definite results, were not the double integrals previously considered by Clebsch, though these also enter in important ways, but single integrals of complete differentials. To many, Picard's problem must have seemed an artificial and profitless one, a mere slavish imitation of Riemann's work in a field where analogous success was not to be looked for. It was, moreover, a problem of enormous labour, extending over many years, and sustained more by faith than by any striking definite result, as at least it would appear to some.

To-day, however, the point of view is different. For it now appears that the results obtained by the Italian school, and the results obtained by Picard, are complementary to one another in a most wonderful way. By the union of these two diverse theories, of common ancestry, has arisen a remarkable series of consequences which really seem to bring the problem, of a classification of all possible surfaces, into manageable shape. To-day, as it appears to me, we are far advanced in such a general outlook as will make possible a general theory of surfaces, and of algebraic constructs of any number of dimensions.

This is the central proposition to which I wish to invite your consideration to-day.

So much for general exposition. I propose now to attempt to describe in more detail, but still roughly, some of the results which seem to me the

more important or interesting of those obtained. I must naturally give more attention to introductory notions than their final importance warrants, in order to try to be clear. You will understand also that what I say is necessarily incomplete, and in several important respects deliberately so. My object is not to make a scientific report, but to suggest the interest of the matter.

In the theory of a plane curve $f(y, x) = 0$, the genus may be defined as the number of integrals of the form

$$\int \frac{\phi(y, x)}{\partial f / \partial y} dx,$$

which are everywhere finite, $\phi(y, x)$ being an integral polynomial. If $f(y, x)$ is of order n , the polynomial ϕ must be of order $n-3$ at most, and must satisfy certain conditions at the multiple points of f in order to compensate the zeros of $\partial f / \partial y$, and certain conditions at infinity. These conditions we summarise by saying that ϕ must be an adjoint polynomial. It can be shewn that any everywhere finite integral must necessarily be of the form written down. It is an obvious fact that any everywhere finite integral is changed into such an integral by birational transformation. Thus the genus, the number of homogeneous coefficients left arbitrary in the polynomial ϕ , is an invariant of birational transformation.

Clebsch's proposal for a surface $f(z, x, y) = 0$ was similarly to consider everywhere finite integrals of the form

$$\iint R(z, x, y) dx dy,$$

in which R is rational in z, x, y . The number of such integrals is evidently unaltered by birational transformation. It can be shewn that such an integral must have the form

$$\iint \frac{\phi(z, x, y)}{\partial f / \partial z} dx dy,$$

in which ϕ is a polynomial, of order $N-4$, at most, if N be the order of f , subject to certain conditions at the multiple points and multiple curves of f , and at infinity, which we summarise by saying that ϕ is to be an adjoint polynomial; and conversely that, subject to these conditions, such an integral is everywhere finite. The number of such integrals is a first invariant for the birational transformation of surfaces. It is equal to the number of adjoint surfaces ϕ , of order $N-4$. It is interesting to-day, in the light of subsequent qualifications, to see the easy confidence which

was at first reposed in the finality of the step forward undoubtedly made by this remark.

The conditions of adjointness being stated, it becomes possible to calculate a formula for the number of surfaces ϕ . This formula applied to a rational surface is found to give zero, as it evidently should; for it is at once clear that if $F(\xi, \eta)$ be a rational function of two independent variables ξ, η , there is no integral of the form $\iint F(\xi, \eta) d\xi d\eta$ which is everywhere finite. But this same formula applied to a ruled surface gives a negative number; indeed, the negative of the genus of the plane section of the surface. This fact was remarked by Cayley in 1871. The formula is thus at fault in some cases, since there cannot be a negative number of everywhere finite integrals. We must hence conclude that for a surface of so low a comparative order as $N-4$, the conditions of adjointness that have been assumed in calculating it are not independent; and the circumstance is a warning always to revert to the geometrical conditions and not trust the general formula. We might be further tempted to conclude that, in the cases where the formula gives a negative number, there are no everywhere finite double integrals, and also that the general formula is of no use. Both these conclusions would be wrong. As regards the first, though it is true that for ruled surfaces there exist no adjoint polynomials of order $N-4$, yet this is not always so for the cases in which the general formula gives a negative number. For instance, the surface of which the coordinates are quadruply periodic functions of two variables, the so-called hyperelliptic surface, possesses one everywhere finite double integral, though the general formula leads to the value -1 . As regards the second conclusion, it was shewn by Zeuthen that, even though the general formula for the number of adjoint surfaces of order $N-4$ may have a negative value, the expression is, nevertheless, one which is unaltered by birational transformation of the fundamental surface, the proof being very much on the lines of the analytical proof that the expression for the genus of a plane curve is unaltered by birational transformation of the curve. In the case of a plane curve this expression always gives the number of linearly independent adjoint curves of order $n-3$, if n be the order of the fundamental curve. In the case of a surface, as we have been explaining, the facts shew that we have two invariants of birational transformation, which need not be equal. The first, the Clebsch invariant, now called the geometrical genus, and denoted by p_g , is the actual number of adjoint surfaces of order $N-4$. The second, the Zeuthen invariant, now called the arithmetical genus, and denoted by p_a (or p_n), is the number given by the general expression for the number of adjoint surfaces of

order m , when in this expression m is replaced by $N-4$. In all cases $p_g \geq p_a$.

It is extremely fortunate that neither of these genera was discarded in the earlier investigations; for subsequent work has shewn that it is their difference $p_g - p_a$ which is the most important invariant of the surface. In very many respects it is this difference which is the analogue of the genus of a plane curve, as we shall illustrate immediately. This difference is called the irregularity of the surface, and a surface for which it is zero is called regular. Familiar examples of regular surfaces are (1) rational surfaces, (2) surfaces the most general of their order, (3) surfaces whose only multiple line is a double curve which is a complete intersection of two surfaces. On the other hand, every ruled surface is irregular, the irregularity being the genus of the plane section of the surface; so also a surface representing the aggregate of the pairs of points of any two curves (not both rational) is irregular. I believe that, save for surfaces reducible to ruled surfaces, no surface of order less than 8 has yet been found which is irregular. Among the earliest English publications in this subject are Mr. Berry's investigations on elementary lines of the cases of the quartic and quintic surfaces (1899, *Camb. Phil. Trans.*, xviii, xix, xx).

Other invariants of birational transformation, after the genera, may now be mentioned. The notion of a linear system of curves on a plane is familiar. On any surface a linear system of curves is given by the intersection of the surface with a family of surfaces of the form

$$\lambda\phi + \lambda_1\phi_1 + \dots + \lambda_r\phi_r = 0,$$

in which $\lambda, \lambda_1, \dots, \lambda_r$ are arbitrary parameters. The surfaces $\phi = 0, \phi_1 = 0, \dots, \phi_r = 0$, which are supposed of the same order, may all pass through a common curve lying on the fundamental surface; this, which then belongs to every curve of the system independently of the values of the parameters $\lambda, \lambda_1, \dots, \lambda_r$, is not in general to be reckoned as part of the linear system considered. It may happen that the curves considered belong to a more general system,

$$\lambda\phi + \dots + \lambda_r\phi_r + \lambda_{r+1}\phi_{r+1} + \dots + \lambda_s\phi_s = 0,$$

of curves of the same order, depending, however, on $r+s$, instead of r , parameters. When this is the case the original system is called incomplete. In what we are to say we speak only of complete systems unless the contrary is asserted. The distinction is of the most fundamental importance. An analogous distinction arises for linear series of point sets upon a curve. A linear series of sets of points is then called complete when it is not contained in a series of sets, each of the same

number of points as the original sets, given by a variable curve containing linearly a larger number of parameters than in the original series. Consider, for instance, a rational quartic curve in space, defined by the partial intersection of a cubic and a quadric surface whose remaining intersection consists of two skew lines. We have, then, upon the quartic curve a linear series of sets of four points, obtained by the intersections of the curve with variable planes. This, however, is not complete, being contained in the series obtained by variable quadric surfaces passing through four fixed points of the curve. The former series depends on only three parameters, the latter on four.

It is clear that a linear system of curves on a surface changes, by birational transformation, into a linear system of curves on the transformed surface, depending on the same number of linear parameters. If this number, entering homogeneously, be $r+1$, the system is said to be of *freedom*, or multiplicity, or dimension, r ; and a curve of the system is identified by assigning r arbitrary points of it. Further, the number of intersections of any two curves of the system, called the *grade* of the system, is also evidently unaltered by the transformation. Further, as two curves which are in birational relation with one another are of the same genus, this genus is a third invariant of the birational transformation.

In making these remarks about linear systems we have avoided reference to the possibility that all the curves of the system may have certain fixed common points, which may be multiple points of the same multiplicity for all the curves of the system. It is evident that a system of which all the curves have certain base points prescribed can be regarded as included in a system of greater freedom, having no prescribed base points. Whether a system is complete or not depends, therefore, on what base points are supposed to be prescribed. The definition of the grade and genus of the system is also in general modified. Further, we have taken no account of the fact, which is a capital source of difficulty in the theory of the birational transformation of surfaces, that a curve may transform to a point and a point transform to a curve. It appears undesirable in a brief survey to attempt to deal with this matter.

Beside the numbers associated with a linear system which are invariant in a birational transformation, there are certain linear series of sets of points upon any curve of the system which are also invariant. Chief among these is the so-called canonical series existing upon any curve, having, when the curve is of genus π , a freedom $\pi-1$ and, in each set, $2\pi-2$ points, these being the zeros of the differential of an everywhere finite integral associated with the curve; that is, of an adjoint polynomial when the curve is plane. Another series, in the case of a curve belonging

to a linear system of freedom r , is the series of freedom $r-1$ cut upon that curve by all the other curves of the same system, of which each set contains a number, n , of points equal to the grade of the linear system. This is called the characteristic series.

In the case of a plane curve of order m , the canonical series is given by the intersections of the curve with a variable adjoint curve of order $m-3$. The question arises whether it may not be possible, in the case of a linear system, $|C|$, of curves upon a surface, to define in connexion therewith, another system of curves whose intersections with any curve C of $|C|$ shall constitute, or be contained in, the canonical series of $|C|$. The answer is that this is generally possible. For instance, when $|C|$ consists of all possible plane sections of the surface, the new system of curves which is desired is given by the system of adjoint surfaces of order $N-3$, if N be the order of the fundamental surface. This system of curves is called the system adjoint to the original system, and denoted by $|C'|$. It is not, however, generally the case, though $|C'|$ gives upon any curve C sets of the canonical series, that all sets of the canonical series upon C are obtainable in this way. The series obtained upon C by means of the curves of $|C'|$ is not the complete canonical series of freedom $\pi-1$, but a series of only freedom $\pi-1-p$ contained therein, p being the irregularity of the surface. This fact has only recently been obtained. The original result was that the deficiency in freedom of the series cut upon C by $|C'|$ was never greater than p . The revised result was first proved by Picard, in a transcendental way, as an *a posteriori* result of a fundamental theorem, in regard to the number of integrals existing upon a surface, obtained in Italy in 1905, and the geometrical proof was then given in 1908 by Severi, upon the basis of a theorem due to Enriques (1904-5). This result, in regard to the deficiency of the series cut by $|C'|$ on C , may be looked upon as furnishing a definition of the irregularity of a surface. Considering the linear system of curves on the surface constituted by all plane sections, the adjoint surfaces of order $N-3$, where N is the order of the surface, give upon an arbitrary plane, not the complete system of curves of order $N-3$ adjoint to the plane sections of order N and genus π , which is of freedom $\pi-1$, but give a system of plane curves of freedom $\pi-1-p$. This defines p and shews that it is unaltered by birational transformation. As $p = p_g - p_a$, and p_g is evidently unaltered by birational transformation, we can thus infer that so also is p_a . The original proof of this fact, given by Zeuthen, depending upon a consideration of the possible multiplicities of the surface, may thus be dispensed with, save for the necessary rearrangement in the order of development of the theory.

Another result in which the irregularity of the surface enters relates to the characteristic series. Any curve of a linear system of freedom r cuts upon an individual curve of the system a series of freedom $r-1$. But it does not follow from the fact that the linear system of curves is complete that this linear series of sets of points is also complete. We will give an example in a moment in which this is not so. The proposition to which we are referring is that if the complete series on the individual curve of the system, which contains the sets cut thereon by the other curves of the system, be of freedom $r-1+k$, then k is not greater than the irregularity p of the surface. And to this there is the complementary proposition that upon any surface there exist linear systems for which k is exactly p .

To give an example of the theorem that the characteristic series of a complete linear system is not necessarily complete, we may consider a particular ruled surface. Consider two plane cubic curves of the same modulus, between which a $(1, 1)$ correspondence has been defined; let them be placed in space in two different planes, so that two corresponding points coincide. If, then, pairs of corresponding points, one on each curve, be joined by straight lines, these lines will give rise to a ruled surface of order 5, having a double curve of order 5 and genus 1. On this surface there is a linear system, of freedom 3, of curves defined by all the plane sections of the surface. This system is not complete, however. For if we take 5 arbitrarily chosen fixed generators of the surface, and construct quartic surfaces passing through the double curve of the surface, and also through these 5 generators, we shall obtain 5 linearly independent quartic surfaces, whose variable intersections with the ruled surface constitute a linear system, of freedom 4, of curves of order 5 and genus 1. This linear system contains that of the plane sections of the surface; for when the 5 fixed generators are properly chosen, as is possible with arbitrary positions for 4 of them and a suitable position for the remaining one, a cubic surface can be drawn through the double curve of the surface and these 5 generators; and then the quartic surfaces spoken of contain as part of themselves the product of this cubic surface and an arbitrary plane. Considering now this complete system, of freedom 4, of curves of order 5, obtained by the variable quartic surfaces, any two curves of the system intersect in 5 points, and there is upon any separate curve of the system a characteristic series of freedom 3 of sets of 5 points. As the separate curve is of genus 1, the complete series of sets of 5 points has freedom 4, not 3, and is not determined upon this curve by the other curves of the system. There is thus a deficiency of freedom in the characteristic series equal to 1. This is, in fact, equal to the irregularity of the ruled surface. For

there are no adjoint surfaces of order $N-4$, the double curve not being plane; so that $p_g = 0$. The number of conditions that a surface of order m , sufficiently high, should contain the double curve, of genus 1, is $5m$; such a surface will thus contain

$$\frac{1}{6}(m+1)(m+2)(m+3) - 5m,$$

homogeneously entering arbitrary parameters. When $m = 1$, this gives

$$p_a = \frac{1}{6} 2 \cdot 3 \cdot 4 - 5 = -1,$$

which is equal, as previously remarked, to the negative of the genus of the plane section of the ruled surface. And the irregularity

$$p = p_g - p_a = 1.$$

Any ruled surface might similarly have been taken to furnish an example of a complete linear system with an incomplete characteristic series, as follows from the researches of Segre. Another example is furnished by the surface investigated by Picard and Humbert, of which the coordinates are quadruply periodic functions, of a certain kind, of two independent variables. These examples are referred to as having been among the earliest to suggest themselves; on any surface, however, which is not regular an example is obtained by considering the system of curves adjoint to an arbitrary linear system (without base points).

The theorem of the possible incompleteness of the characteristic series of a linear system of curves undergoes a modification when we consider systems of curves depending on variable parameters which enter algebraically but not linearly, in the equations of the surfaces determining the curves of the system. A curve in space can be determined completely as the common part of the intersection of at most four surfaces. Thus a curve on a given surface can be completely specified by at most three other surfaces, or, if we are considering a family of curves and neglect isolated points, can be determined by at most two other surfaces. If the equations of these surfaces depend algebraically upon a certain number of parameters, we obtain an algebraic system of curves. For instance, if we wished, on a given surface, to obtain all the curves of a specified order, we might begin with a variable intersecting surface, and should be led, in specifying the order, to a certain number of algebraic conditions for its coefficients. It is known that any system of algebraic relations breaks up into a finite number of irreducible systems, each representable by the points of an algebraic construct such as $f(z, x_1, \dots, x_R) = 0$, where f is a polynomial in $z, x_1, \dots, x_R$. We may thus say that an algebraic system of curves consists of a finite number of irreducible systems, each curve of such an

irreducible system corresponding without ambiguity to one point of such an algebraic construct as is represented by an equation $f(z, x_1, \dots, x_R) = 0$. A linear system of curves is the particular case when the representative construct may be taken to be all the points of a linear space. The number R is called the freedom, or dimension, or multiplicity of the system. A complete algebraic system contains all the curves of the same order represented by all the points of the construct, and, in particular, contains the curves of the linear systems to which individual curves of the algebraic system belong. If we consider an algebraic system of one dimension, represented therefore by the points of an algebraic curve, the intersections of any curve of the system with a consecutive curve of the system have limiting positions as the consecutive curve tends to coincidence with the original. These limiting positions constitute what is meant by a characteristic set in this case. If we consider an algebraic system of R dimensions, the intersections of a particular curve of the system, with all possible consecutive curves, give rise, by their limiting positions, to a linear series of characteristic sets upon the original curve, represented, on the representative construct, by the pencil of directions which exist on the construct at any point of it, a series of freedom $R-1$. Any set of this characteristic series determines on the particular curve chosen, a complete series. Contrary now to what may happen in the case of a *linear* system of curves, this complete series coincides with the series determined by the consecutive curves of the algebraic system, being of freedom $R-1$; namely, every set of the complete characteristic series is given by the limiting position of a consecutive curve of the algebraic system. This theorem is due to Enriques and is of the most fundamental importance in the theory. In order to understand it, we may recur to the example of the ruled quintic surface previously discussed. We considered a linear system of quintic curves, of freedom 4, determined by variable quartic surfaces passing through the double curve and 5 fixed generators. Take an arbitrary plane section of the ruled surface, which is a quintic curve with 5 double points, and hence of genus 1. Any of the quartic surfaces will meet this plane quintic curve in points lying upon a quartic curve, which passes through the double points and through 5 fixed points of the plane quintic curve; there will thus be 5 residual intersections, which vary, when the quartic surface varies, in a linear series. The sum of the elliptic arguments at the 5 points of a set of this series is the same whatever be the quartic surface, and is a constant belonging to the linear system of curves, on the ruled surface, determined by taking all the possible quartic surfaces passing through the same set of 5 fixed generators. By taking a different set of 5 fixed generators through which to pass the

adjoint quartic surfaces we obtain another linear system of curves on the ruled surface, in general associated with another value for the sum of the 5 elliptic arguments of the points where a curve of the system cuts a plane section. The various possible linear systems of quartic surfaces thus obtained are therefore all included in an algebraic system of quartic surfaces, involving, beside the original 4 parameters which enter linearly (not homogeneously), a further elliptic parameter. The equation of a general quartic surface of this algebraic system will contain, beside the 4 linear parameters, rational functions of two independent elliptic functions of this elliptic argument [say $\wp(u)$ and $\wp'(u)$]. Thus the algebraic system is of dimension 5, and the characteristic series it determines on any one of its curves is of freedom 4, and is complete, the series consisting of sets of each 5 points on a curve of genus 1. The circumstance we have illustrated by this case is general. Namely, when in a linear system of curves of freedom r , the characteristic series on any curve, which is of freedom $r-1$, belongs to a complete series of freedom $r-1+k$, it is because the linear system of curves belongs to an algebraic system of freedom $r+k$, of which the general curve, beside depending linearly on r parameters, depends further on parameters, effectively k in number, which are rational functions of Abelian functions of p independent arguments, where p is the irregularity of the surface. This result, developed by Castelnuovo from previous theorems of Picard and Enriques, shows how intimately connected are the problems of the theory of surfaces with the theory of Abelian functions.

A particular corollary from this result may be stated, as it is an example of the fact which has been remarked, that the irregularity of a surface is analogous in many respects with the genus of a curve. As we may have on a surface an algebraic system of curves, so we may have on a curve an algebraic series of sets of points, each set of the series being then represented uniquely by a point of another representative construct, which, when it is of dimension 1, is also a curve. Suppose now the fundamental curve is of genus zero. We may then suppose it to be a straight line. Upon this an algebraic series will be given by an equation of the form

$$x^n + u_1 x^{n-1} + \dots + u_n = 0,$$

wherein $u_1, \dots, u_n$ are rational functions of the coordinates of a point of the representative construct. These functions $u_1, \dots, u_n$ must be such that when their values are given it is possible, with the help of the equation of the construct, to express rationally the coordinates, say $\xi_1, \dots, \xi_r, \eta$, of the representative point which corresponds to the set of points of the straight line given by this equation. The series on the straight line

is therefore evidently contained in the linear series represented by

$$\lambda x^n + \lambda_1 x^{n-1} + \dots + \lambda_n = 0,$$

wherein $\lambda, \lambda_1, \dots, \lambda_n$ are arbitrary parameters. In other words, on a rational curve any algebraic series of sets of points is contained in a linear series. What we are pointing out is that this theorem for rational curves is analogous to a theorem for regular surfaces. We have been remarking that the variable parameters in an algebraic system of curves on a surface consist of a certain number entering linearly and a certain number, depending on the irregularity of the surface, entering as Abelian functions of variable parameters. When the surface is regular, parameters of this kind cannot enter. Wherefore on a regular surface every algebraical system of curves consists wholly of curves belonging to a certain linear system. This is analogous to the theorem established above for rational curves.

The results so far stated have dealt for the most part with a single linear system of curves on a surface. Consider now *two* linear systems, say $|A|$ and $|B|$, both existing on the surface. There is then a linear system, denoted by $|A+B|$, containing every composite curve obtained by combining a curve of $|A|$ with a curve of $|B|$, which, completed, will in general contain also other curves which are not reducible. This system $|A+B|$ may then be defined as the complete linear system containing the curve A_1+B_1 , where A_1 is any chosen curve of $|A|$, and B_1 any chosen curve of $|B|$, it being understood that if $|A|$ and $|B|$ have assigned base points, such a point, i -ple for curves of $|A|$, and j -ple for curves of $|B|$, is $(i+j)$ -ple for curves of $|A+B|$. In order to understand this completely it is necessary to be clear about the fact that any curve $|A|$ determines a definite complete linear system to which it belongs. For take any definite curve A of the surface. Either there is no linear system of curves of which A is a member; this is possible; then A is called an isolated curve, and may be said to define a linear system of freedom zero. Or there is such a linear system with assigned base points. In this case the system is of limited freedom r ; for if n be the number of free intersections of any two curves of the linear system it is evident that $r-1$, the freedom of the characteristic series, cannot be greater than n . Take now r as great as possible, so that A belongs to a complete system of freedom r . The only doubt then remaining is whether A can belong to two complete systems of the same assigned fixed points. If this is possible, suppose it belongs to both the systems $|A|$ and $|B|$ given, suppose, by

$$\lambda\phi + \lambda_1\phi_1 + \dots + \lambda_r\phi_r = 0, \quad \mu\psi + \mu_1\psi_1 + \dots + \mu_s\psi_s = 0,$$

being given by the part of $\phi = 0$ which is not common to all the curves

$\phi = 0, \phi_1 = 0, \dots, \phi_r = 0$, and also given by the part of $\psi = 0$ which is not common to all the curves $\psi = 0, \psi_1 = 0, \psi_s = 0$. Then consider the linear system given by

$$\phi(\mu\psi + \mu_1\psi_1 + \dots + \mu_s\psi_s) + \psi(\lambda\phi + \lambda_1\phi_1 + \dots + \lambda_r\phi_r) = 0.$$

The fixed intersection, independent of the parameters, of the surfaces here represented consists of the fixed common intersection of all the surfaces of both the two original surfaces, every multiple point assigned to be i -ple for all the curves of the first system and j -ple for all the curves of the second system, being $(i+j)$ -ple for the curves of the new system; evidently both of i and j must be at most equal to an actually occurring multiplicity for the original curve given, A . Another curve common to all the surfaces of the new system is the curve A itself, as satisfying both $\phi = 0$ and $\psi = 0$. These fixed intersections being taken account of, the new system is a linear system whose variable intersections contain all the curves of $|A|$ and also all the curves of $|B|$. Moreover, the fixed points, simple or multiple, supposed to be assigned for $|A|$ and for $|B|$ being taken the same, part of those actually belonging to the curve A , will also be common to the new system and of the same multiplicities. But the new system is of greater freedom than either of the systems $|A|, |B|$. And this is impossible, since each of these is supposed complete with the assigned fixed points. Wherefore the curve A can only belong to one complete system of assigned fixed points, and determines this system.

Thus we are justified in defining the system $|A+B|$ as that complete system determined by one composite curve A_1+B_1 of assigned fixed points.

Conversely, it may happen, if $|P|$ and $|Q|$ be two linear systems, that the general surface defining the curves of $|P|$ can, by proper limitation of its parameters, be made to contain a curve, Q_1 of $|Q|$, and still retain linearly arbitrary parameters. The residual intersection will then be a linear system of curves defining a complete linear system, which we may denote by $|P-Q_1|$. For definiteness suppose that neither $|P|$ nor $|Q|$ has any assigned fixed points. If Q_2 be another curve of $|Q|$ we can, in that case, similarly obtain a complete linear system $|P-Q_2|$. These two systems $|P-Q_1|, |P-Q_2|$ must, however, coincide. For if R_1 be a particular curve of $|P-Q_1|$, the composite curve $|R_1+Q_1|$ is a curve of $|P|$, and the complete system $|P-R_1|$, which thus contains Q_1 , must coincide with $|Q|$ by what has been proved, and must hence contain Q_2 . Thus there is a curve of $|P|$ formed by the composite curve R_1+Q_2 , and so the complete system $|P-Q_2|$ must contain R_1 . As $|P-Q_1|, |P-Q_2|$

thus contain a common curve R_1 they must coincide. For this reason we denote the system so obtained by $|P-Q|$.

One of the most important applications of these ideas may now be referred to. If $|C|$ be any linear system, and $|C'|$ its adjoint system, it may happen that the latter system contains the former, so that we can form a system $|C'-C|$. It can be proved that when this is the case for one linear system $|C|$, it is equally so for any other linear system $|D|$, and that the curves $|D'-D|$ are the same as the curves $|C'-C|$. Some qualifications of this statement are noticed below. For instance, if $|C|$ be the system of the plane sections of the surface of order N , then $|C'|$ is determined by adjoint surfaces of order $N-3$. Thence $|C'-C|$ is determined by adjoint surfaces of order $N-4$. These are, in fact, the surfaces which arise in the Clebsch everywhere finite double integrals whose number is p_g . The curves on the surface determined by them are called the canonical curves.

We have indicated that some qualification may be necessary in the above definition of the canonical system of curves. This arises owing to two circumstances: (a) the linear system $|C|$ may have multiple points assigned as fixed base points; (b) the system $|C'-C|$ may have fixed components. As regards the former, if $|C|$ have a certain point of the surface as a fixed point of multiplicity i , it is contained in a more ample system $|C_1|$, having this point only as an $(i-1)$ -ple point; if $|\Gamma|$ be the system formed from $|C|$ in this way, by reducing the multiplicity of every assigned fixed base point by unity, it is the system $|C'-\Gamma|$ which gives the canonical system. As regards (b), there may be upon the surface curves which are capable, by a properly taken birational transformation, of being transformed to simple points. Such curves are called exceptional curves, and they arise as fixed components in the curves of the canonical system. Moreover, in what we have said throughout, we have made no allusion to the question of multiple points belonging to the fundamental surface itself. In a general survey such as the present it is impossible to enter into the consideration of such qualifications with any clearness without unduly increasing the length of what is said. But it may be stated that the advance of the theory shews how to deal with the restrictions referred to.

From the statement which has been made, it follows, when the surface possesses a canonical system of curves, that the series cut upon any curve of a linear system by the curves of the adjoint system is the sum of the characteristic series of the curve and the series cut upon the curve by the canonical system of the surface.

Hence, when there exists a canonical system of curves on the surface, the characteristic series of any linear system is special, namely, contained

in the canonical series of the curve of the system upon which it is considered. Now, upon a curve of genus π , a special series of sets of m points is of freedom $r = m - s$, where s is the strength of a set of the series as conditions for a canonical set. And, if j be the number of linearly independent canonical sets which contain a set of the given series, the so-called index of specialty of the series, and r' the freedom of the complementary series, obtained by subtracting the given series from the canonical series of the curve, we have

$$j = \pi - s, \quad r' + 1 = \pi - s;$$

and

$$r = m - \pi + j = m - \pi + r' + 1.$$

We apply this to the case of the characteristic series of a linear system of curves upon the surface, of freedom r . We have already stated that the characteristic series is of freedom

$$r - 1 + p_g - p_a - h,$$

where $0 \leq h \leq p_g - p_a$; if i be the number of linearly independent canonical curves of the surface which contain any curve of the linear system, so that $p_g - i$ is the number of linearly independent sets cut by the canonical system upon the curve of the system under consideration, and if $p_g - i + \delta - 1$ be the freedom of the complete series to which these sets belong, we obtain

$$r - 1 + p_g - p_a - h = n - \pi + p_g - i + \delta,$$

where n , π are the grade and genus of the linear system considered. Thus we have

$$r = n - \pi + 1 + p_a + s - i,$$

where i , defined above, is called the index of specialty of the linear system of curves, and s , which is called the superabundance of the linear system, is the sum of the "algebraic deficiency" of the characteristic series and of the deficiency of the series cut upon a curve of the given linear system by the canonical curves of the surface. Neither of the numbers s , i can be negative. When they are both zero, the linear system is said to be regular. It can be shewn that on any surface there exist linear systems of curves which are regular—and, indeed, by a recent theorem, the adjoint system of any linear system of curves is a regular system. If the surface possess no singularities beside an ordinary double curve and triple points

triple for the curve, all adjoint surfaces of sufficiently high order give rise to regular systems.

The above formula, known as the Riemann-Roch theorem for surfaces, is obtained on the assumption that the adjoint system of a linear system of curves is the sum of the system and the canonical system of the surface. This statement requires modification when the original system of curves possesses multiple base points. Then the number s in the formula must be increased by a number μ , where μ is the number of conditions that a curve of the adjoint system passing through the points of a characteristic set, should pass through each multiple base point, or μ is the index of speciality of the characteristic series less the index of speciality of the series cut upon a curve of the given linear system by curves differing therefrom only in the fact of having an $(i-1)$ -ple base point at each i -ple base point of the original system.

A regular system of curves has the properties (1) of not being special, (2) of having its completed characteristic series "algebraically complete," that is, of freedom $r-1+p_g-p_a$; (3) of having for the index of speciality of its characteristic series the number p_g .

The Riemann-Roch theorem, and a certain converse of it, can be extended to reducible curves, provided we define the genus and grade of a curve consisting of several components. If $|A|$, $|B|$ be two linear systems, without base points, the general curve of the sum system $|A+B|$ will, most often, not be reducible. If n, π be the grade and genus for $|A|$, n' and π' those for $|B|$, and N, Π those for $|A+B|$, and i the number of common points of a curve of $|A|$ with a curve of $|B|$, it can be shewn that

$$N = n + n' + 2i, \quad \Pi = \pi + \pi' + i - 1.$$

If now we have any single curve whatever, composite or not, say B , it is possible to take a system $|A|$, of irreducible curves, of order so great that the sum system $|A+B|$ consists of irreducible curves. Let n, π be the grade and genus for $|A|$, and N, Π those for $|A+B|$. Then the grade and genus for the curve B are defined by the relations above given. And this is legitimate, provided it be shewn that the result so obtained is the same whatever system $|A|$ be taken. This is easily established by choosing another system $|A_1|$ and considering the sum system $|A+A_1+B|$, first as the sum of $|A_1|$ and $|A+B|$, and then as the sum of $|A|$ and $|A_1+B|$.

It is at once clear that the equations above give rise to

$$2\Pi - 2 - N = (2\pi - 2 - n) + (2\pi' - 2 - n').$$

The number $2\pi - 2 - n$, for the system $|A|$, may be called the canonical

number; when the surface possesses a canonical system of curves, it is equal to the number of intersections of any curve of $|A|$ with a canonical curve. The canonical number of the sum of any number of linear systems, without base points, is the sum of their canonical numbers. Denoting by n, π, γ , respectively, the grade, genus, and canonical number for any linear system $|C|$, and by n', π', γ' , the corresponding numbers for the adjoint system $|C'|$, the canonical number for $|C' + C|$ is $\gamma' + \gamma$. This, however, when a canonical system of curves exists, is equal to the number of intersections of $|C' - C|$ with $|C' + C|$, which is $n' - n$. Equating these, we find

$$n' = \pi' + \pi - 2,$$

which can be put also into the form

$$\pi' - \pi = \gamma',$$

and is easily established independently of the hypothesis of the existence of a canonical system.

Since the system $|C' - C|$ is the same whatever system $|C|$ may be, so also is $\gamma' - \gamma$. This number is denoted by $\omega - 1$, and is thus an invariant of the surface. Various forms for it are

$$\omega - 1 (= \gamma' - \gamma) = \pi' - \pi - (2\pi - 2 - n), \quad \text{or} \quad \pi' - 1 + n - 3(\pi - 1),$$

$$(&= \gamma'' - \gamma') = \pi'' - \pi' - (\pi' - \pi), \quad \text{or} \quad \pi'' - 2\pi' + \pi,$$

where π'' and γ'' refer to the adjoint system, $|C''|$, of $|C'|$.

As a simple illustration of the formula

$$\pi - 1 = n' - \pi' + 1,$$

we may make the following remark. We have previously stated that $|C'|$ cuts on $|C|$ a series of freedom $\pi - 1 - (p_g - p_a)$. There are p_g curves $|C'|$ which contain C ; hence, if r' be the freedom of $|C'|$, we have

$$r' - p_g = \pi - 1 - (p_g - p_a),$$

or

$$r' = \pi - 1 + p_a.$$

By the formula in question this is the same as

$$r' = n' - \pi' + 1 + p_a,$$

which is in accordance with a statement above made that $|C'|$ is regular.

We turn aside now from these geometrical investigations, for the most

part elaborated in Italy, to the question of the integrals of the form

$$\int (A dx + B dy),$$

that can exist on the surface, A and B being rational functions of the coordinates x, y, z . Such integrals may be of three kinds. First, integrals which are everywhere finite. Second, integrals which become algebraically infinite along curves, so that on an arbitrary curve, say a plane section of the surface, they give rise to integrals having points of algebraic infinity or poles. Third, integrals which are logarithmically infinite along curves, so that, on a plane section say, they give rise to Abelian integrals with points of logarithmic infinity. It is now known that the number of linearly independent everywhere finite integrals is p , or $p_g - p_a$, the irregularity of the surface, and the number of linearly independent integrals of the second kind, independent of these and of rational functions, is also p , the number of periods for any such integral of the first or second kind, being $2p$. But this result was not obtained until many partial results had been first obtained by different workers. Humbert proved, quite early, in 1894, in a paper which contains many ideas since utilised, that on a surface not possessing everywhere finite integrals of perfect differentials, every algebraic system of curves must consist of curves all belonging to the same linear system, so that a surface containing an algebraic system of curves not contained in a linear system must possess integrals of the first kind. Severi proved that any integral of the second kind can be reduced by subtraction of rational functions, to an integral with only a single curve of infinity of the first order, and then, that this curve, if belonging to a linear system of curves, must have an incomplete characteristic series. As, by Castelnuovo's theorem, the deficiency of the characteristic series of a linear system is not greater than p , it follows hence that a surface with integrals of the second kind must have a value of p greater than zero. Denoting the number of linearly independent integrals of the first kind by q , and the number of linearly independent integrals of the second kind, including those of the first kind, but excluding rational functions, by r , Severi deduced that $r - q \leq p$. This was in 1904, about eight years ago. Picard had already proved (in 1885), by consideration of the group of the linear differential equation whose solutions are the periods of an integral of the second kind, that such an integral could be constructed with arbitrary periods. From this it can be inferred that $r \geq 2q$. Finally, on the basis of a theorem due to Enriques, which we have quoted, that the characteristic series of a complete algebraic system of curves is necessarily complete, Castelnuovo and

Severi, early in 1905, established that $q \gg p$. These inequalities necessitate

$$r = 2q = 2p.$$

This enumeration is not intended to be complete. It makes no reference, for instance, to Picard's paper in the *Comptes Rendus* early in 1905.

The various investigations that have been referred to are all of permanent interest even apart from the fact that together they established the fundamental theorem $q = p$.

Picard's use of the linear differential equation satisfied by the periods of an integral is extremely remarkable, and his investigation, in connexion therewith, of the Riemann surface of four dimensions which represents the original algebraic surface, is striking especially from the disproportion between the apparent meagreness of the method and the evident difficulty of the problems considered. Perhaps it is not fanciful also to note that Riemann's development, for the case of complex variables, of Gauss's consideration of the linear differential equation satisfied by the hypergeometric series, was subsequent to his preoccupation with complex variables for his theory of Abelian functions; and that it was Fuch's application of Riemann's ideas to linear differential equations in general which led to Poincaré's and Picard's investigations for the group of such an equation.

Severi's investigation of the formula $r - q \leq p$ is not only remarkable as giving incidentally a practical method of finding the algebraical part of an integral $\int A dx + B dy$, but also by the great light it throws on the characteristic series of a curve. In association with an integral of the second kind having a given curve for its sole polar curve, of the first order, he builds a rational function which, in a sense, serves to identify the integral. Finding the poles and zeros of this rational function upon the polar curve, he obtains the result that its poles are the same for every integral of the second kind having this curve as sole polar curve of the first order, while its zeros are partly the same. The remaining zeros are the points of the curve where the integral is finite, and are different for different integrals with this same polar curve. Severi show that these variable zeros are a set of the characteristic series of the curve; and proves that, if the polar curve be a curve of a linear system, these points cannot be a set of the characteristic series determined on the polar curve thereby, unless the integral is reducible to a rational function. This explains then why the characteristic series of a linear system may be incomplete. Moreover, the matter is thus in a certain sense analogous to what occurs in the theory of Abelian integrals on a curve, where the rational functions are found to be members

of a larger class of functions, namely, of the integrals of the second kind. Another striking part of the same investigation is the reduction, already incidentally referred to under another point of view, of an integral of the second kind with any set of polar curves, to an integral with a single polar curve of the first order for the integral.

Severi's proof that the number g , of everywhere finite integrals is as great as the irregularity p , of the surface, is associated with a theorem which is an extension of the converse of Abel's theorem for a curve. If we have two sets, of the same number of points, upon a curve, the converse of Abel's theorem gives, as a sufficient and necessary condition that the two sets should belong to the same linear series, or, in other words, as a sufficient and necessary condition that the two sets should be the poles and zeros, respectively, of a rational function on the curve, the condition that the sums of the values of the everywhere finite integrals at the points of the first set should be equal, save for periods, to the corresponding sums at the points of the second set. Severi sets out to obtain conditions, expressed in terms of the everywhere finite Picard integrals of the surface, in order that an algebraic system of curves on the surface should consist of curves all belonging to the same linear system. Of the various forms of the result which he obtains, the first may be quoted: "If we have an algebraic system of curves on a surface, a sufficient and necessary condition that its curves should belong to the same linear system is that the sums of the everywhere finite integrals of the surface should be the same at the points of intersection of every two curves of the system." Another interesting result, contained as a lemma in this investigation, has subsequently led to an important theorem, proved by Castelnuovo, for algebraic series of points upon a curve. If we have an algebraic series, of freedom unity, of sets of m points upon a curve of genus π , with the property that, containing an arbitrary point of the curve, there are m' of its sets, then a sufficient and necessary condition that the sets should all belong to the same linear series is that the number of united points, at which two points of a set coincide, should be $m'(2m+2\pi-2)$; when the sets do not belong to the same linear series the number of united points is less than this. To read the descriptive proof of this given by Castelnuovo, and to follow out the proof given by Segre, of a formula, originally due to Schubert, upon which it depends, is to realize how deep are the roots of the present brilliant school of geometry in Italy. Further, I would remark, in this connexion, the following intimately connected problem. If we have on a surface of irregularity p a curve of genus π , where $\pi > p$, the p integrals of the surface which are everywhere finite will give upon this curve, generally, just p integrals of the first kind forming a defective system; and

there will be, in addition, upon this curve, $\pi-p$ integrals of the first kind, forming the complementary defective system. If now we consider a set of m points upon the curve, what relation exists between the sums of the p integrals of the first defective system at these m points and the sums of the $\pi-p$ integrals of the second defective system at the same points?

The fundamental theorem upon which we have been dwelling, that the number of everywhere finite Picard integrals upon a surface is equal to the irregularity of the surface, may be regarded as being to a large extent a corollary from Picard's theory of the integrals (of total differentials) which possess curves of algebraic infinity. The theory elaborated by him of the integrals which possess curves of logarithmic infinity has led to another geometrical theory of a singularly interesting kind. We can illustrate the character of the results by referring first of all to the cases of the quadric and cubic surfaces, though the theory referred to explicitly leaves aside all rational surfaces. Two curves, A and B , upon a surface are coresidual when there exists a third curve C , such that C and A together form a complete intersection, of the fundamental surface with another surface ϕ , and also C and B form a complete intersection, of the fundamental surface with another surface ψ , of the same order as ϕ . In such case we may also speak of A and B as equivalent, and express this result by writing $A \equiv B$. In this sense, it is a statement of a familiar theorem to say that upon a quadric surface every curve is equivalent with a curve expressible in terms of multiples of two arbitrarily chosen intersecting generators u, v —every curve C can be written in the form

$$C \equiv mu + nv,$$

where m, n are integers. In the same way upon a cubic surface every curve C can be written in the form

$$C \equiv mu + m_1 a_1 + \dots + m_6 a_6,$$

where $a_1, a_2, \dots, a_6$ are six lines of the surface forming the upper, or the lower, row of a double six of lines upon the surface, and u is a certain associated twisted cubic curve of the surface, while $m, m_1, \dots, m_6$ are integers appropriate to the particular curve C . In a similar, but much more general way, upon any irregular surface, any curve can be expressed linearly in terms of a certain finite number of curves, with integral coefficients, in such a form as

$$C \equiv \lambda v + \lambda_1 v_1 + \dots + \lambda_k v_k,$$

where $v, v_1, \dots, v_k$ are the base curves, $\lambda, \lambda_1, \dots, \lambda_k$ are integers appropriate to the curve C , but the sign $\equiv$, instead of meaning that the curves on

the two sides of the sign belong to the same linear system, means that they belong to the same algebraic system. For instance, it can be shewn that on a Kummer surface, every curve of the surface can be expressed in terms of 17 elements. We explained at the commencement that we are dealing with a theory of birational transformation, and it is sufficiently clear, though for simplicity's sake we have avoided discussion of this, that a point may transform into a curve or conversely. Thus in the expression of an arbitrary curve in terms of elements, it may be convenient to take, upon a particular representative of a surface, points, instead of actual curves, as elements. And indeed upon the Kummer surface, 16 of the 17 elements may be taken to be the double points of the surface.

It has been said that this theory is in intimate connexion with the theory of Picard's integrals of the third kind, having curves of logarithmic infinity. To explain this, we must premise that curves $C_1, C_2, \dots, C_k$ upon a surface are said to be algebraically connected when there exist integers $\lambda_1, \dots, \lambda_k$, such that the two curves

$$\lambda_1 C_1 + \dots + \lambda_m C_m, \quad \lambda_{m+1} C_{m+1} + \dots + \lambda_k C_k$$

belong to the same algebraic system, for some value of m , where $1 \leq m < k$. This being clear, there is the theorem that, for any integral of a total differential which becomes logarithmically infinite along curves of the surface, these curves of logarithmic infinity are algebraically connected; and conversely that, given any set of curves which are algebraically connected, an integral of the third kind exists having these for its curves of logarithmic infinity, with, in fact, logarithmic coefficients (or periods) directly deducible from the coefficients occurring in the expression of the algebraic connexion of the curves. Further, a necessary and sufficient condition for the curves $C_1, \dots, C_k$ to be algebraically connected, is that every determinant of the matrix, of $(k+1)$ rows and k columns,

$$\begin{vmatrix} n_{11} & n_{12} & \dots & n_{1k} \\ n_{21} & n_{22} & \dots & n_{2k} \\ \dots & \dots & \dots & \dots \\ n_{k1} & n_{k2} & \dots & n_{kk} \\ m_1 & m_2 & \dots & m_k \end{vmatrix}$$

should vanish, where $m_1, \dots, m_k$ are the orders of these curves, n_{ij} is the number of intersections of C_i and C_j and n_{ii} is the grade of C_i .

We have remarked, as an example of the analogy between the genus of a curve and the irregularity of a surface, that on a regular surface every

algebraic system of curves consists of curves belonging to the same linear system. The curves of logarithmic infinity of a Picard integral of the third kind, being in general algebraically connected, are thus, on a regular surface, linearly connected. This enables us to show that, on a regular surface, every Picard integral reduces to rational functions and logarithms of rational functions. And the converse statement, that this only occurs on a regular surface, is deducible from Enriques' theorem in regard to the characteristic series of an algebraic system of curves. Surely this is an extremely interesting, if only a particular, result.

We have now attempted to give in rough outline some account of two of the main divisions of the new theory. First of the method of defining invariants of birational transformation by means of systems of curves lying upon the surface, due largely to the Italian geometers; second of the algebraic integrals, due to Picard, and of the influence which either of these has exerted upon the other. One of the practical outcomes of the combined theory is a set of theorems, still, of course, far from complete, enabling us to classify surfaces according to the values of their invariants. Some of these results may now be referred to.

One striking theorem may be mentioned at once, because it can be stated very simply. We see very soon that ruled surfaces, which may be looked upon as aggregates of straight lines, differ essentially, from the point of view we are considering, from surfaces in general. They possess no adjoint surfaces of order $N-4$, where N is their order, and their plane sections possess no adjoint system of curves. The geometrical genus p_g is in all cases zero, and the numerical genus p_a equal to the negative of the genus of a plane section. Thus the irregularity is equal to this genus, and the everywhere finite integrals of the surface are the integrals associated with the curve of any plane section, all of which are functions of one of them. It is thus of interest to have a criterion for the birational transformation of any surface into a ruled surface. It can be shown that any surface possessing a linear system of curves, of grade n and genus π , such that the canonical number $\gamma, = 2\pi-2-n$, is negative, can be transformed birationally into a ruled surface. The proof of this which has been given is certainly of a very delicate character; it requires a careful examination of the fact, to which we have adverted, which makes so much of the difficulty of the whole theory, that curves can exist on a surface, called exceptional curves, which can be transformed birationally into simple points of a new surface; but beside this, a proof was impossible until a sufficient number of special cases of transformation into ruled or rational surfaces, had been examined in detail. One of the first of such detailed examinations was Noether's proof that a surface possessing a

linear system of rational curves of freedom 1 is a rational surface. A consequence of the general theory is that any surface which is not birationally transformable into a ruled surface possesses only a finite number of exceptional curves, which do not intersect, and can be transformed birationally into a surface with no exceptional curves. A particular theorem obtained is that a surface possessing a linear system of elliptic curves with one common point at least is rational or transformable into a ruled surface; and the same may be said of a surface of which all the plane sections are hyperelliptic curves of any genus. Or, again, a surface of order greater than 4 which has plane sections of genus 3 is transformable into a ruled surface (or is rational).

Another class of surfaces which, though very special from the present point of view, are interesting from their comparative frequency, are the rational surfaces. We have already referred to them at the outset. Evidently they must have $p_g = 0$, $p_a = 0$. But these two conditions are not sufficient to ensure that a surface be rational. For instance, we may consider (after Enriques) a sextic surface having six edges of a finite tetrahedron as double lines. Such a surface is represented by an equation

$$\phi(x, y, z, t) + xyz t \psi\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}, \frac{1}{t}\right) = 0,$$

where ϕ, ψ are both homogeneous quadratic functions of their four arguments. For this surface evidently $p_g = 0$, there being no adjoint surfaces of order $N-4$, that is, quadric surfaces, passing through the six edges of the tetrahedron. Moreover also $p_a = 0$; for a surface of order m containing the 6 edges of the tetrahedron would have

$$\frac{1}{6}(m+1)(m+2)(m+3) - 4 - 6(m-1)$$

homogeneous coefficients, which when $m = 2$ is $10-10$, or zero. Nevertheless, this surface is not rational. The proof of this, from general principles, we may give some account of, because it offers an opportunity of explaining an invariant of birational transformation to which we have not as yet referred. The property $p_g = 0$ of a surface is an expression of the fact that the adjoint system $|C'|$ of a linear system $|C|$ of curves upon the surface does not contain the system $|C|$. It may, however, happen, even when this is so, that the system $|2C'|$ contains the system $|2C|$. Sufficient conditions for a surface to be rational are (1) that the arithmetical genus p_a be zero; (2) that for some linear system $|C|$ of curves upon the surface the system $|2C'|$ does not contain the system $|2C|$; and these conditions are necessary. The possibility of curves $|2C' - 2C|$ for one system $|C|$ involves the possibility for every other system, and the impossibility of

curves $|2C' - 2C|$ involves the impossibility of curves $|C' - C|$, or $p_g = 0$. Hence, in order to show that the sextic surface under consideration is not rational, it is sufficient to find a linear system $|C|$ for which the system $|2C' - 2C|$ is possible, and any linear system $|C|$ may be taken. Taking, then, the plane sections, the adjoint system is given by cubic surfaces passing through the edges of the tetrahedron. Such a surface cannot contain a plane section, which is a curve of order 6, without breaking up into the plane of this section and a quadric surface. As there exists no quadric surface passing through the edges of the tetrahedron, we see that the system $|C' - C|$ is impossible. The system $|2C|$ is given by the intersection of the sextic surface with all possible quadric surfaces. The system $|2C'|$ is given by the intersections of the sextic surface with all possible sextic surfaces having the edges of the tetrahedron as double lines; for evidently such a surface, when it breaks up into two cubic surfaces passing each through the edges of the tetrahedron, has for its free intersection with the fundamental sextic surface the sum of two curves of $|C'|$. In order to shew that the system $|2C' - 2C|$ is possible we have then merely to shew that a sextic surface can be given, one at least, which passes through the intersection of the fundamental sextic surface with an arbitrary quadric surface, and has, moreover, the six edges of the tetrahedron as double lines. Such a sextic surface is evidently given by the product of the quadric and the four faces of the tetrahedron. Thus the system $|2C' - 2C|$ exists, though it consists of a curve of order zero. Another example may be given in which there is an infinity of curves $|2C' - 2C|$, though both $p_g = 0$ and $p_a = 0$. This second example, due to Castelnuovo, was suggested by the former, which we have thought should be mentioned for its historical importance as well as because it will be referred to below in another connexion. Consider a surface of the seventh order having a double conic, a triple straight line not intersecting the conic, and three tacnodes whose tangent planes intersect in the triple line. That such a surface exists is evident thus. From a line lying on a cubic surface C five tangent planes can be drawn to touch the surface at points not on the line, and the surface possesses conics not intersecting the line. Let α, β, γ be three of the five tangent planes, P an arbitrary plane through their line of intersection; and let Q be a quartic surface having one of the conics of the cubic surface which do not intersect the line, as a double conic, and, further, touching the planes α, β, γ at their points of contact with C . Then the equation

$$C^2P + Q\alpha\beta\gamma = 0,$$

represents such a septic as described. The curves adjoint to the plane

sections of this surface will be obtained by quartic surfaces, having the triple line of the surface as a double line, passing through the double conic, and passing through the tacnodes. A quartic surface cannot contain a plane section of order 7, unless it break up into the plane and a cubic surface, which will then have the triple line as a double line, pass through the double conic, and also pass through the tacnodes. Such a cubic surface, being intersected in four points by every line meeting the triple line and lying in the plane of the conic, would contain this plane. The residual would be two planes through the triple line; these could not contain the three tacnodes. Hence it is clear that if $|D|$ be the system of the plane sections of the septic, the system $|D'-D|$ is impossible, or $p_g = 0$. But also $p_a = 0$; for the conditions that a surface of order m should have a given line as double line are $3m+1$ in number, and so a surface of order m having the proper adjoint behaviour would in this case contain, homogeneously,

$$\frac{1}{6}(m+1)(m+2)(m+3) - (3m+1) - (2m+1) - 3$$

terms, which, when $m = N-4 = 3$, gives

$$20 - 10 - 7 - 3 = 0.$$

Consider now the formation of a system $|2D'-2D|$. The system $|2D|$ is given by arbitrary quadric surfaces. The system $|2D'|$ will be given by octavic surfaces, having the conic as a double conic, passing through the triple line so as to give on an arbitrary plane section twice the number of intersections with the section of the fundamental septic which are given by a surface of the system $|D'|$, and passing through the tacnodes in such a way as would be given by two surfaces $|D'|$. The octavic must then give twelve intersections at an arbitrary point of the triple line, and, as we shall assume, must have at the tacnodes the planes α, β, γ as tangent planes. These conditions are satisfied by a surface of the form

$$H\alpha\beta\gamma k^3\phi,$$

wherein H is an arbitrary plane through the triple line, k is the plane of the double conic, and ϕ is an arbitrary quadric surface. Thus we see that the system $|2D'-2D|$ is possible, and contains, beside fixed intersections, the intersection of the septic with any arbitrary plane through the triple line. It can be shewn that this is its most general form. The number of linearly independent curves $|2D'-2D|$ is, in general, for any fundamental surface, denoted by P_2 , or indeed of linearly independent curves $|sD'-sD|$ is denoted by P_s . If $P_s = 0$, then $P_1 = 0$, that is $p_g = 0$, since else we could obtain curves $|sD'-sD|$ by repeating, s times,

curves $|D' - D|$. But conversely when $p_g = 0$, we may have $P_2 \geq 0$, or, in general, $P_s \geq 0$. In the two examples we have considered, for both of which $p_g = 0$, $p_a = 0$, we have respectively $P_2 = 1$, $P_2 = 2$. For a surface to be rational, it is necessary and sufficient that $p_a = 0$ and $P_2 = 0$, these involving $p_g = 0$. If the surface have no multiple curve other than a double curve with triple points which are triple also for the surface, these conditions, for a surface of order N , with a double curve of order m , of genus π , with t triple points, can be stated as follows

$$(1) \frac{1}{6}(N-3)(N-2)(N-1) - m(N-4) + 2t + \pi - 1 = 0.$$

(2) No surface, other than those containing the original as part of themselves exists, of order $2(N-4)$, having the double curve as double curve for it.

In regard to the genera P_s , another remark may be made in passing. Enriques has found that a necessary and sufficient condition that a surface should be reducible to a ruled surface by birational transformation, is that $P_4 = 0$, $P_6 = 0$. And in connexion with this and results previously given, we may also state that not only is it true that for every ruled surface is $p_a \leq 0$, but conversely, every surface for which $p_a < -1$ can be reduced to a ruled surface. The cases $p_a = 0$, $p_a = -1$ have been examined.

Another class of surfaces to which reference may be made is that of which the points represent the couples of points of one or two curves. Taking first two curves, if we associate every point of the first curve with every point of the second curve, we obtain an ∞^2 of couples. If X, Y, Z be three functions, each rational in the coordinates of each of the two associated points, these will be connected by a rational equation, which we may interpret as the equation of a surface. If these functions X, Y, Z can be chosen, so that the coordinates of each of the associated points, one on each curve, are determinable uniquely, and therefore rationally, in terms of X, Y, Z , we shall have such a surface as is under consideration. Similar remarks apply when we begin with a single curve and consider all possible pairs of points upon this. Here, if X, Y, Z are not symmetrical in regard to the coordinates of the two associated points, we merely have a particular case of the former state of affairs, in which the two curves are repetitions of one another. When X, Y, Z are each symmetrical functions of the coordinates of the two associated points, the case is practically distinguishable from the former, in that, when X, Y, Z are given, we cannot evidently obtain the two points from which these arose, but only the symmetrical functions of them. Consider these cases more in detail, and first when we start from two curves, say of genera π and π' . Then it is easy to see that each of the everywhere finite integrals of the two curves gives

rise to an everywhere finite integral on the surface, and conversely, that every such integral arises in this way. Thus, if p_g, p_a be the geometrical and arithmetical genera of the surface, we have

$$p_g - p_a = \pi + \pi'.$$

Further, there are everywhere finite double integrals on the surface, of the form

$$\iint du dv,$$

where u is an everywhere finite integral of the first curve and v of the second, and it can be shewn that all everywhere finite double integrals of the surface arise in this way. Hence we also have

$$p_g = \pi\pi'.$$

By associating a definite point of one curve C with all the points of the other curve C' in turn, we evidently obtain a curve on the surface birationally equivalent with C' . There are thus two systems of curves on the surface, each of freedom 1, those of one system being birationally equivalent each with the curve C , and those of the other system each birationally equivalent with C' . Through an arbitrary point of the surface there clearly passes one curve of each system. The composite curve on the surface consisting of $2\pi - 2$ curves of the system $\{C'\}$, each birationally equivalent with the curve C' , of genus π' , taken with $2\pi' - 2$ curves of the system $\{C\}$, each birationally equivalent with C , defines a linear complete system of curves on the surface, which is seen without much difficulty to constitute the canonical system of curves on the surface. This may be symbolically denoted by $(2\pi - 2)C' + (2\pi' - 2)C$. As two curves C do not intersect, and two curves C' do not intersect, it is thus clear that the grade of the canonical system is $2(2\pi - 2)(2\pi' - 2)$ or $8(\pi - 1)(\pi' - 1)$. The surface does not in general possess any rational curve, because through any point of such a curve would pass a single curve of each of the systems $\{C\}$ and $\{C'\}$, and the intersections of these with the rational curve would give on this curve involutions of respective genera π' and π , whereas such cannot exist on a rational curve, unless both π and π' be zero. Wherefore the surface does not possess any exceptional curves, and the invariant $\omega - 1$, previously explained in this sketch, is thence easily seen to be equal to the grade of the canonical system, or

$$\omega - 1 = 8(\pi - 1)(\pi' - 1).$$

For a surface with e exceptional curves, as we may remark, unless it is

transformable to a ruled, or rational, surface, the grade of the canonical system is $\omega-1+e$.

In this account of the surface we have utilised the fact that the number of everywhere finite integrals is p_g-p_a , and our formulæ then enable us to calculate both p_g and p_a . But, in fact, these values were obtained for the surface under consideration before it was definitely known that the number of everywhere finite integrals is in general equal to p_g-p_a , thus forming some presumptive evidence for the general result. It is interesting to see how this was done, especially as this furnishes an excuse for explaining an invariant of birational transformation to which no reference has so far been made. For this, it should be premised that there may exist upon a surface a system of curves of freedom 1, of which a number i , greater than 1, pass through an arbitrary point of the surface. When $i=1$, and only one curve of the system passes through an arbitrary point, it is very convenient to have a name for such a system. We may call it a *pencil* of curves, though the adoption of the Italian word *fascio* would seem a preferable course. Such a pencil may either be linear, or it may be algebraical. In the latter case, the curves of the system can be associated with the points of an algebraic curve of a certain genus. The linear system arises when this curve is rational. Now first suppose we have upon the fundamental surface a linear pencil of curves, each of genus π , having n points common to all; no two curves can intersect except in points common to all the curves of the pencil. Let δ be the number of curves of the pencil which have a double point, the case of several double points upon one curve, or of higher singularities, being neglected for our explanatory purposes. For instance, the curves may be the intersections of the surface with planes passing through a fixed axis; then π will be the genus of a plane section of the surface, n the order of the surface, and δ the number of tangent planes of the surface passing through the axis. With these definitions it can be shewn that the number

$$I = \delta - n - 4\pi$$

is the same as for any other linear pencil of curves upon the surface, and, moreover, if e be the number of exceptional curves upon the surface, which is, in fact, finite, unless the surface can be transformed into a ruled (or rational) surface, then $I-e$ is unaltered by birational transformation. The number I is called, from its origin, the Zeuthen-Segre invariant. The same number can be defined by an irrational pencil of curves on the surface. Let, as before, π be the genus of each curve of the pencil, π' the genus of the algebraic curve which represents the pencil, and δ the

number of curves of the pencil which have a double point. Then

$$I = \delta + 4(\pi - 1)(\pi' - 1) - 4.$$

And it is remarkable, the result being ultimately due to Noether, that the invariants ω and I are connected by the relation

$$I + \omega = 12p_a + 9.$$

Returning now to the case of the surface which represents the pairs of points of two curves of genera π and π' , recalling that it possesses a pencil of curves, each of genus π , with genus π' (as well as another such pencil), we have for this surface

$$I = 4(\pi - 1)(\pi' - 1) - 4.$$

Further, we shewed how to prove that

$$\omega - 1 = 8(\pi - 1)(\pi' - 1).$$

Hence, from the formula above,

$$p_a = \frac{1}{12}[12(\pi - 1)(\pi' - 1) - 3 - 9] = \pi\pi' - \pi - \pi'.$$

We saw, however, that $p_g = \pi\pi'$. From these we derive

$$p_g - p_a = \pi + \pi',$$

shewing that the irregularity of the surface is equal to the number of everywhere finite integrals.

Consider now briefly the case of the surface representing pairs of points upon the same curve C of genus π , without regard to the order of these. For this surface we find, as before, the equations

$$p_g - p_a = \pi, \quad p_g = \frac{1}{2}\pi(\pi - 1).$$

Combining any point of the curve C with all other points of this curve, we have upon the surface, corresponding to the couples so obtained, a curve which is birationally equivalent with C . These curves form a system of freedom 1 upon the surface, with the property, however, that through an arbitrary point of the surface 2 of these curves pass. This is clear because any point of the surface corresponds to two points of the curve C , and either of these may be kept fixed, while the other describes the curve. The surface is thus essentially different from that before considered, being in a correspondence (1, 2) with it. We do not enter further into its properties. Evidently the hyperelliptic surface, of which the co-

ordinates are quadruply periodic functions of two arguments, of a certain kind, is a particular case of it, in which $\pi = 2$.

Other kinds of surfaces which it may be interesting to refer to, are those known as double planes, of which the equation can be written in the form

$$t^2 = \phi(x, y),$$

wherein $\phi(x, y)$ is a polynomial in x and y . Of these the cases which are rational surfaces have been ascertained. By a birational transformation, in x and y only, all the rational surfaces of this form may be supposed to arise from the three cases in which $\phi(x, y) = 0$ represents (1) an ordinary plane quartic curve, (2) a sextic curve with a triple point of which the three tangents coincide, (3) a curve of order $2m$ with a $(2m-2)$ -fold triple point.

The verification that these represent rational surfaces may be given as follows. In case (1),

$$t^2 = y^4 + y^3a + y^2b + yc + d,$$

where a, b, c, d are integral polynomials in x , of respective dimensions 1, 2, 3, 4, if we replace y by $y_0 = Ax + B$, and determine the two constants A, B , so that the right side is the square of a quadratic polynomial $t_0 = Px^2 + Qx + R$, and then, in the original equation, write

$$t = t_0 + \tau(y - y_0),$$

we shall obtain a relation connecting x, y and τ representing a cubic surface. This is known to be rational. In case (2),

$$t^2 = y^3 + y^2a + yb + c,$$

where a, b, c are integral polynomials in x , of respective dimensions 2, 4, 6, if we replace y by $y_0 = Ax^2 + Bx + C$, and determine the constants A, B, C , so that the right side is the square of a cubic polynomial

$$t_0 = Px^3 + Qx^2 + Rx + S,$$

and then, in the original equation, write

$$t = t_0 + \tau(y - y_0),$$

we obtain the square of $y + \frac{1}{2}(y_0 + a - \tau^2)$ expressed as a quartic polynomial in x and τ , whereby this case is reduced to the former. The algebraic problem which is to be solved in each of these cases is a particular case of the standard problem of finding, for a plane curve of order n , the

adjoint polynomials of order $n-3$ whose free intersections are all contacts, and arises in the theory of the vanishing of the theta functions. The number of solutions is known. In case (3),

$$t^2 = y^2a + yb + c,$$

where a, b, c are polynomials in x of respective orders $2n-2, 2n-1, 2n$, we may similarly determine

$$y_0 = u/w, \quad t_0 = v/w,$$

where u, v, w are polynomials in x of respective dimensions $n-1, 2n-2, n-2$, and then, putting

$$t = t_0 + \tau(y - y_0),$$

we find y rationally in terms of x and τ .*

And, as double planes which are rational are interesting to consider, so double planes of genera other than zero are also worthy of investigation. A very general and interesting result may be mentioned. Every double plane whose branch curve ($\phi = 0$) is irreducible is rational or regular, having $p_g = p_a$. More precisely, if the surface $z^2 = \phi(x, y)$ possesses q integrals of the first kind, the curve $\phi(x, y) = 0$ is formed by $2q+2$ or $2q+1$ separate curves belonging to the same pencil. Every curve of this pencil is obtainable by projection from two curves of the surface which vary in a hyperelliptic pencil of genus q . We recall, in passing, the comparison with the case of a Kummer surface, which possesses no integrals of the first kind, but is in (1, 2) correspondence with a surface which possesses two such integrals. It is known that the Kummer surface is reducible to the form $z^2 = P_1P_2 \dots P_5$, where $P_1, \dots, P_5$ are five

* The most general surface of order n having the line at infinity of $x = 0$ for an $(n-2)$ -fold line can similarly be satisfied by writing for the other coordinates y, t respectively $y_0 = u/w, t_0 = v/w$, where u, v, w are polynomials in x of respective orders $n-2, n-2, n-3$. Then the substitution $t = t_0 + \tau(y - y_0)$ enables us to express y rationally in terms of τ and x . The cones $y = y_0, t = t_0$ intersect in a (rational) curve of order $n-2$, meeting the multiple line in $n-3$ points. Such a curve of the surface may also be obtained by considering the $3n-4$ planes through the multiple line whose remaining intersection is a pair of lines a_r, b_r . Through the $3n-6$ lines $(a_1, \dots, a_{3n-7})$ and a_{3n-6} , can be drawn a surface of order $2n-4$ having the multiple line of the given surface as $(2n-5)$ -ple line. Another such surface can be drawn through $(a_1, \dots, a_{3n-7})$ and a_{3n-5} . The intersection of these surfaces determines such a rational curve of order $n-2$ as has been described; and there is only one such curve meeting the $3n-5$ lines $a_1, \dots, a_{3n-7}, a_{3n-6}, a_{3n-5}$.

straight lines. The general result in regard to irregular double planes has been obtained in connexion with the following theorems:—(a) The necessary and sufficient condition in order that a surface should possess an irrational fascio of curves is the existence of two Picard integrals which are functions one of the other; (b) if $p_g > 2p_a + 3$ the surface possesses an irrational fascio of curves which is of genus 2 at least. There may, however, be an irrational fascio when this condition is not satisfied, as is shown by the case of surfaces for which $p_g = 0$, $p_a = -1$. Such surfaces arise by considering the surfaces representing the pairs of points of an elliptic curve and a rational curve. In general, when a surface possesses an irrational fascio of curves, the genus of the fascio can never exceed the irregularity of the surface.

Next we refer in brief terms to another general question of the theory of surfaces—that, namely, of surfaces which allow birational transformations into themselves.

Such transformations may be possible which are expressible by formulæ wherein an arbitrary parameter enters, so that, by taking different values of this parameter, the same point may be capable of being transformed into an infinite number of other points of the surface. When this is the case the combination of two such transformations, with the same general formula and different values of the parameter, may lead to a transformation expressible by the same general formula with a parameter depending on the parameters of the component transformations. More generally the surface may be transformable into itself by the formulæ of a continuous group of birational transformations depending upon a finite number of parameters. Or, while depending upon variable parameters, the formulæ may not be included in those of a continuous group with a finite number of parameters. But a surface may be capable of birational transformations in which no variable parameter enters, these being incapable of being regarded as particular transformations of a continuous group. And, when this is the case, the number of transformations may be only finite or it may be infinite. It is probably unnecessary to remark that for a surface a transformation may be uniquely reversible without being birational, a circumstance which does not arise for the transformation of a curve into itself.

A surface allowing continuous transformations into itself (that is, birational transformations containing arbitrary parameters) which do not form a continuous group of a finite number of parameters is reducible to a ruled (or rational) surface.

Surfaces allowing the birational transformations of a continuous group of finite order are either (1) surfaces with a rational group depending on

one parameter (these are reducible to ruled surfaces) or (2) what are called elliptic surfaces, allowing an elliptic group with one parameter. The surfaces contain two fasci of curves—first, a fascio of elliptic curves, trajectories of the group, not necessarily a linear fascio; second, an elliptic fascio of curves of any genus; or (3) the surfaces may be hyperelliptic surfaces, of which the coordinates are quadruply periodic functions of two variables. The necessary and sufficient condition for the existence of a birational group of finite order of self-transformations of a surface is $p_a < 0$. When $p_a < -1$ the surface is reducible to a ruled surface. When $p_a = -1$ the surfaces are elliptic, unless $p_g = 1$, the fascio of elliptic trajectories of the group being of genus p_g . Even when $p_g = 1$, if beside $p_a = -1$, we have $P_4 > 1$, the surfaces are still elliptic. But when $p_a = -1$, $p_g = 1$, $P_4 = 1$ the surfaces are hyperelliptic, with a continuous group of two parameters, which may have two elliptic subgroups.

As regards transformations in which no variable parameter enters, these may be of finite period, and, in that case, may belong to a finite discontinuous group. Any quartic surface with a double point evidently possesses such a periodic self-transformation, and the case of a Weddle or Kummer surface is a familiar example of a finite group of such transformations; while, since the Cremona transformations of a plane do not form a finite group, any rational surface furnishes an example of an infinite group of discontinuous birational transformations. A surface possessing a continuous group may also possess discontinuous transformations which are not of finite period, as has been shewn by Humbert and Painlevé for particular classes of hyperelliptic or elliptic surfaces. Two other cases of surfaces possessing discontinuous transformations of infinite period, which have been remarked more recently, may also be referred to. First, the sextic surface, previously considered, having each of the edges of a finite tetrahedron as a double line; second, the quartic surface of the most general kind described through a sextic curve in space of genus 2. This is not a completely general quartic surface, since the curve in question is not a complete intersection, being the partial intersection of two cubic surfaces whose remaining intersection is a conic and a straight line not intersecting this. This line cuts the sextic in four points. In general a surface which allows a birational transformation of infinite period, and does not allow a continuous group of finite order of birational self-transformations, must have $p_a \geq 0$ as well as $P_2 > 0$ and $p^{(1)} = 1$. Here $p^{(1)}$ is the invariant equal to the previously defined ω , when no exceptional curves exist, but otherwise equal to $\omega + e$, where e is the number of exceptional curves, which, under the hypotheses $p_a \geq 0$, $P_2 > 0$, must be finite. The surface must in general have a fascio of

elliptic curves. The exceptional case is when $p_a = P_2 = 1$; these are satisfied in the case of the quartic surface just instanced, which is therefore a particularly interesting example. Conversely, a surface with $p_a \geq 0$, $P_2 > 0$, $p^{(1)} = 1$ always has a rational or irrational fascio of elliptic curves except when $p_a = P_2 = 1$. Whether the existence of this fascio is in general sufficient to prove the existence of such a transformation seems as yet not quite clearly decided.

As a particular result, if a surface have a non-periodic projective transformation, it allows a continuous group of projective transformations and is ruled or rational.

Notwithstanding the length to which these remarks have extended, we have by no means referred to all the topics which seem to be of capital importance for the theory. One more matter at least, which well illustrates the intimate relation between the geometrical and analytical aspects of the theory, must be mentioned in brief terms, though a competent description would need to be of some length. Our references to the algebraical integrals existing upon the surface, though beginning with Clebsch's everywhere finite double integral, have been, for the most part, to integrals of total differentials. But Picard has also investigated a theory of double integrals analogous in many ways to that of the Abelian integrals upon a curve which are called of the second kind, possessing algebraical infinities, calling them double integrals of the second kind. In rough terms we may say that on the Riemann surface of four dimensions which represents the fundamental surface, they are related to the two-dimensional barriers, which arise in considering the connectivity of that surface, very much as are the Abelian integrals of the second kind, for a Riemann surface representing a curve, related to the curve barriers or period loops described thereon. It was natural then to seek the number of such double integrals of the second kind which are independent of rational functions. Picard has found a formula for this number, working through the connexion of single and double integrals to the linear differential equation whose group is fundamental in his theory of the connectivity of the Riemann surface. Such a formula is evidently an invariant of birational transformation. Now it is particularly interesting to see that he is thus led to consider an expression, involving the invariants previously considered in this sketch, which had already offered itself in the early algebraic investigations of Zeuthen and Noether. This is surely very remarkable, and offers a fresh encouragement to develop the theory of the connectivity of the four-dimensioned Riemann surface in further detail.

With this cursory reference to a large and fascinating theory, I will

conclude these remarks. They will have served a purpose if they should help towards a wider study of the subject.*

* The address is limited to developments of which I have myself been able to study the proofs given by their several authors. Picard's investigations are collected in the volumes entitled *Théorie des Fonctions Algébriques de deux variables indépendantes*, t. I, 1897 ; t. II, 1900 and 1906. On pp. 485–522 of the second volume is an exhaustive account of the Italian work to 1906, by MM. Castelnuovo et Enriques, under the title *Sur quelques résultats nouveaux dans la théorie des surfaces algébriques*. An earlier paper by the same authors, with a similar title, is on pp. 241–316 of the *Math. Ann.*, Bd. XLVIII, 1897. These will furnish a guide to the demonstrations of the results, which cannot, however, in many cases, be at present developed in the order, or with the simplicity, with which they can be explained.