seem to be something very different from definitions of words. Kant considers $7 + 5 = 12$ to be a synthetical judgment (Proleg., pp. 22, 23).

Metaphysics and mathematics are consorts in the East as well as the West. Bhascara says that the analytical art is merely sagacity exercised, and is independent of symbols, which do not constitute the art.¹ If De Morgan² be right in placing Diophantus as late as the beginning of the seventh century, Aryabhata was earlier, by two centuries, than Diophantus. The name, certainly, seems to have been a very common one. Josephus³ relates that Alexander (a son of Herod the Great) said that Diophantus the scribe had imitated his hand. But Mr. Heath's work⁴ renders it scarcely possible to sustain De Morgan's contention.

On Raabe's Bernoullians. By J. D. HAMILTON DICKSON, M.A.

[Read Nov. 8th, 1888.]

Raabe has given the name Bernoullian to the function

$$\phi(z, m) = z^m + m_1 B_0 z^{m-1} + m_2 B_1 z^{m-2} + m_3 B_2 z^{m-3} + \dots,$$

where there is no term without z ; $m_1, m_2, \dots$ being the successive Binomial coefficients after the first in the expansion of $(1+t)^m$, and $L_0, B_1, \dots$ being Bernoulli's numbers, viz.,

$$B_0 = -\frac{1}{2}, \quad B_2 = B_4 = B_6 = \dots = B_{2n} = \dots = 0,$$

$$B_1 = \frac{1}{6}, \quad B_3 = -\frac{1}{36}, \quad B_5 = \frac{1}{42}, \quad B_7 = -\frac{1}{36}, \quad B_9 = \frac{5}{648}, \text{ \&c.}$$

¹ Colebrooke, "Algebra," etc. (London, 1817), p. xix.

² De Morgan, "Arithmetical Books" (London, 1847), p. 47.

³ Josephus, "Antiquities of the Jews" (Burder's Translation, Vol. i., pp. 616, 617). Burder's Preface is dated London, October 1, 1811.

⁴ T. L. Heath, "Diophantos of Alexandria: a Study in the History of Greek Algebra" (Cambridge University Press, 1885).

For instance, the first Bernoullians are

$$\begin{aligned}\phi(z, 1) &= z, \\ \phi(z, 2) &= z^2 - z, &= u \text{ (say)}, \\ \phi(z, 3) &= z^3 - \frac{3}{2}z^2 + \frac{1}{2}z, \\ \phi(z, 4) &= z^4 - 2z^3 + z^2, \\ \phi(z, 5) &= z^5 - \frac{5}{2}z^4 + \frac{5}{3}z^3 - \frac{1}{6}z, \\ \phi(z, 6) &= z^6 - 3z^5 + \frac{5}{2}z^4 - \frac{1}{2}z^3, \\ &\quad \&c. \qquad \&c.\end{aligned}$$

Jacobi has shown [*Crelle*, Bd. xii. (1834), p. 271], and it is already well known, that

$$\begin{aligned}\phi(z, 2n-1) &= \phi(z, 3) \cdot F(u), \\ \phi(z, 2n) &= \phi(z, 4) \cdot G(u),\end{aligned}$$

where $F(u)$, $G(u)$ are functions of u of degree $n-2$. It might, therefore, be inferred that $F(u)$ and $G(u)$ might be expressed in terms of Bernoullians not higher than the $(2n-4)$ th. The object of the present paper is to prove this and find the development.

Writing

$$\phi(z, 2n-1) = \phi(z, 3) \{a_0 z^{2n-4} + a_1 z^{2n-5} + \dots + a_r z^{2n-r-4} + \dots + a_{2n-4}\} \dots\dots\dots(1),$$

and noting that

$$\begin{aligned}\phi(z, 3) &= z^3 - \frac{3}{2}z^2 + \frac{1}{2}z, \\ \phi(z, 2n-1) &= z^{2n-1} + (2n-1) B_0 z^{2n-2} + (2n-1)_2 B_1 z^{2n-3} \\ &\quad + (2n-1)_3 B_2 z^{2n-4} + \dots\end{aligned}$$

on expanding the right-hand side of (1), and equating coefficients of powers of z , we get the equations

$$\begin{aligned}a_r - \frac{3}{2}a_{r-1} + \frac{1}{2}a_{r-2} &= (2n-1)_r B_{r-1}, \\ a_{r-1} - \frac{3}{2}a_{r-2} + \frac{1}{2}a_{r-3} &= (2n-1)_{r-1} B_{r-2}, \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ a_2 - \frac{3}{2}a_1 + \frac{1}{2}a_0 &= (2n-1)_3 B_1, \\ a_1 - \frac{3}{2}a_0 &= (2n-1) B_0, \\ a_0 &= 1.\end{aligned}$$

Hence, the value of a_r may be expressed as a determinant of the $(r+1)^{\text{th}}$ order, viz., after a slight simplification,

$$2^r a_r = \begin{vmatrix} 3, 1, ., ., & \dots, & \pm(2n-1)_r B_{r-1} \\ 2, 3, 1, ., & \dots, & \mp(2n-1)_{r-1} B_{r-2} \\ ., 2, 3, 1 & \dots, & \pm(2n-1)_{r-2} B_{r-3} \\ \dots & & \dots \\ \dots & & \dots \\ \dots & 2, 3, 1, ., & -(2n-1)_3 B_3 \\ \dots & ., 2, 3, 1, & (2n-1)_2 B_1 \\ \dots & ., ., 2, 3, & -(2n-1) B_0 \\ \dots & ., ., ., 2, & 1 \end{vmatrix} \quad (r+1)^{\text{th}} \text{ order},$$

the signs of the terms in the last column being alternately + and -, beginning with the lowest term as +. This is easily evaluated, and gives

$$\begin{aligned} 2^r a_r &= (2^{r+1}-1) + 2(2^r-1)(2n-1) B_0 + 2^3(2^{r-1}-1)(2n-1)_3 B_1 \\ &\quad + 2^5(2^{r-2}-1)(2n-1)_5 B_2 + \dots \\ &\quad \dots + 2^{r-1}(2^3-1)(2n-1)_{r-1} B_{r-2} + 2^r(2n-1)_r B_{r-1} \dots (2). \end{aligned}$$

Similarly, writing

$$\phi(z, 2n) = \phi(z, 4) \{ b_0 z^{2n-4} + b_1 z^{2n-6} + \dots + b_r z^{2n-r-4} + \dots + b_{2n-4} \} \dots (3);$$

and noting that

$$\phi(z, 4) = z^4 - 2z^3 + z^2,$$

$$\phi(z, 2n) = z^{2n} + 2n B_0 z^{2n-1} + (2n)_3 B_1 z^{2n-2} + (2n)_5 B_2 z^{2n-3} + \dots;$$

on expanding the right-hand side of (3), and equating coefficients of powers of z , we get the equations

$$b_r - 2b_{r-1} + b_{r-2} = (2n)_r B_{r-1},$$

$$\vdots \qquad \qquad \qquad \vdots$$

$$b_2 - 2b_1 + b_0 = (2n)_3 B_1,$$

$$b_1 - 2b_0 = (2n)_5 B_2,$$

$$b_0 = 1,$$

whence, as before,

$$b_r = \left| \begin{array}{l} 2, 1, \dots, \dots, \pm(2n)_r B_{r-1} \\ 1, 2, 1, \dots, \dots, \mp(2n)_{r-1} B_{r-2} \\ \dots, 1, 2, \dots, \dots, \pm(2n)_{r-2} B_{r-3} \\ \dots, \dots, \dots, \dots, \dots, \dots \\ \dots, 1, 2, 1, \dots, \dots, -(2n)_3 B_3 \\ \dots, \dots, 1, 2, 1, \dots, (2n)_2 B_2 \\ \dots, \dots, \dots, 1, 2, \dots, -(2n) B_0 \\ \dots, \dots, \dots, \dots, 1, \dots, 1 \end{array} \right| (r+1)^{\text{th}} \text{ order},$$

the signs of the terms in the last column being again alternately + and -, commencing with the lowest as +, and reading upwards.

Finally,

$$b_r = (r+1) + r(2n) B_0 + (r-1)(2n)_2 B_1 + (r-2)(2n)_3 B_2 + \dots \\ \dots + 2(2n)_{r-1} B_{r-2} + 1(2n)_r B_{r-1} \dots \dots \dots (4).$$

Now assume

$$\phi(z, 2n-1) = \phi(z, 3) \{c_0 \phi(z, 2n-4) + c_1 \phi(z, 2n-5) + \dots \\ \dots + c_{2n-6} \phi(z, 2) + c_{2n-5} \phi(z, 1) + c_{2n-4}\}, \\ \phi(z, 2n) = \phi(z, 4) \{d_0 \phi(z, 2n-4) + d_1 \phi(z, 2n-5) + \dots \\ \dots + d_{2n-6} \phi(z, 2) + d_{2n-5} \phi(z, 1) + d_{2n-4}\};$$

it is required to find the connections between the a 's and the c 's, and between the b 's and the d 's. To do this it will be sufficient to take a special case, as the method is general and the result simple. As soon as we commence the investigation of these connections, on remembering that the *even* Bernoulli's numbers vanish, we find that all the *odd* c 's and d 's vanish. Taking advantage of this simplification, let, for example,

$$\phi(z, 13) = \phi(z, 3) \{c_0 \phi(z, 10) + c_2 \phi(z, 8) + c_4 \phi(z, 6) + c_6 \phi(z, 4) \\ + c_8 \phi(z, 2) + c_{10}\},$$

then the coefficient of $\phi(z, 3)$ may be expanded in powers of z and

tabulated thus—

	z^{10}	z^9	z^8	z^7	z^6	z^5	z^4	z^3	z^2	z	1
$c_0\phi(z, 10)$	c_0	$c_0 10_1 B_0$	$c_0 10_2 B_1$		$c_0 10_4 B_3$		$c_0 10_6 B_5$		$c_0 10_8 B_7$		
$c_2\phi(z, 8)$			c_2	$c_2 8_1 B_0$	$c_2 8_2 B_1$		$c_2 8_4 B_3$		$c_2 8_6 B_5$		
$c_4\phi(z, 6)$					c_4	$c_4 6_1 B_0$	$c_4 6_2 B_1$		$c_4 6_4 B_3$		
$c_6\phi(z, 4)$							c_6	$c_6 4_1 B_0$	$c_6 4_2 B_1$		
$c_8\phi(z, 2)$									c_8	$c_8 2_1 B_0$	
c_{10}											c_{10}

a function which is to be identical with the coefficient of $\phi(z, 3)$ in equation (1), when $n = 7$.

$$\text{Hence} \quad c_0 10_1 B_0 = a_1 \quad \text{or} \quad c_0 = -\frac{a_1}{5} = a_0 = 1,$$

$$c_2 8_1 B_0 = a_3 \quad \text{or} \quad c_2 = -\frac{a_3}{4},$$

$$c_4 6_1 B_0 = a_5 \quad \text{or} \quad c_4 = -\frac{a_5}{3},$$

$$c_6 4_1 B_0 = a_7 \quad \text{or} \quad c_6 = -\frac{a_7}{2},$$

$$c_8 2_1 B_0 = a_9 \quad \text{or} \quad c_8 = -\frac{a_9}{1},$$

and

$$c_{10} = a_{10}.$$

In a similar manner, taking the Bernoullians of even degree, and for example, assuming

$$\phi(z, 14) = \phi(z, 4) \{ d_0\phi(z, 10) + d_2\phi(z, 8) + d_4\phi(z, 6) + d_6\phi(z, 4) \\ + d_8\phi(z, 2) + d_{10} \},$$

we shall find

$$d_0 10_1 B_0 = b_1 \quad \text{or} \quad d_0 = -\frac{b_1}{5} = b_0 = 1,$$

$$d_2 8_1 B_0 = b_3 \quad \text{or} \quad d_2 = -\frac{b_3}{4},$$

$$d_4 6_1 B_0 = b_5 \quad \text{or} \quad d_4 = -\frac{b_5}{3},$$

$$d_0 4_1 B_0 = b_7 \quad \text{or} \quad d_0 = -\frac{b_7}{2},$$

$$d_8 2_1 B_0 = b_9 \quad \text{or} \quad d_8 = -\frac{b_9}{1},$$

$$d_{10} = b_{10}.$$

Examples—

$$\begin{aligned} \frac{\phi(z, 13)}{\phi(z, 3)} &= z^{10} - 5z^9 + 5z^8 + 10z^7 - \frac{34}{3}z^6 - 22z^5 + \frac{284}{21}z^4 \\ &\quad + \frac{219}{7}z^3 - \frac{41}{15}z^2 - \frac{691}{35}z - \frac{691}{105}, \\ &= \phi(z, 10) - \frac{5}{2}\phi(z, 8) + \frac{22}{3}\phi(z, 6) - \frac{219}{14}\phi(z, 4) + \frac{691}{35}\phi(z, 2) - \frac{691}{105}; \\ \frac{\phi(z, 14)}{\phi(z, 4)} &= z^{10} - 5z^9 + \frac{25}{6}z^8 + \frac{40}{3}z^7 - \frac{163}{15}z^6 - \frac{526}{15}z^5 + \frac{367}{30}z^4 \\ &\quad + \frac{893}{15}z^3 + \frac{101}{15}z^2 - \frac{691}{15}z - \frac{691}{30} \\ &= \phi(z, 10) - \frac{10}{3}\phi(z, 8) + \frac{526}{45}\phi(z, 6) - \frac{893}{30}\phi(z, 4) + \frac{691}{15}\phi(z, 2) - \frac{691}{30}. \end{aligned}$$

The theorem is finally stated thus, ϵ being 3 or 4 as the case may be :

If

$$\phi(z, 2m + \epsilon) = \phi(z, \epsilon) \{ e_0 z^{2m} + e_1 z^{2m-1} + e_2 z^{2m-2} + \dots + e_{2m-2} z^2 + e_{2m-1} z + e_{2m} \},$$

where, when $\epsilon = 3$,

$$\begin{aligned} e_r &= \frac{2^{r+1}-1}{2^r} + \frac{2^r-1}{2^{r-1}}(2m+\epsilon)B_0 + \frac{2^{r-1}-1}{2^{r-2}}(2m+\epsilon)_2 B_1 \\ &\quad + \frac{2^{r-2}-1}{2^{r-3}}(2m+\epsilon)_3 B_2 + \dots, \end{aligned}$$

and when $\epsilon = 4$,

$$e_r = (r+1) + r(2m+\epsilon)B_0 + (r-1)(2m+\epsilon)_2 B_1 + (r-2)(2m+\epsilon)_3 B_2 + \dots$$

$$\begin{aligned} \text{then} \quad \phi(z, 2m+\epsilon) &= \phi(z, \epsilon) \left\{ \phi(z, 2m) - \frac{e_8}{m-1}\phi(z, 2m-2) \right. \\ &\quad \left. - \frac{e_5}{m-2}\phi(z, 2m-4) - \dots - \frac{e_{2m-3}}{2}\phi(z, 4) - \frac{e_{2m-1}}{1}\phi(z, 2) + e_{2m} \right\}. \end{aligned}$$

The first 14 Bernoullians expressed in terms of lower Bernoullians are as follows :—

$$\phi(z, 5) = \phi(z, 3) \left\{ \phi(z, 2) - \frac{1}{3} \right\}$$

$$\phi(z, 6) = \phi(z, 4) \left\{ \phi(z, 2) - \frac{1}{2} \right\},$$

$$\phi(z, 7) = \phi(z, 3) \left\{ \phi(z, 4) - \phi(z, 2) + \frac{1}{3} \right\},$$

$$\phi(z, 8) = \phi(z, 4) \left\{ \phi(z, 4) - \frac{4}{3} \phi(z, 2) + \frac{2}{3} \right\},$$

$$\phi(z, 9) = \phi(z, 3) \left\{ \phi(z, 6) - \frac{3}{2} \phi(z, 4) + \frac{9}{5} \phi(z, 2) - \frac{3}{5} \right\},$$

$$\phi(z, 10) = \phi(z, 4) \left\{ \phi(z, 6) - 2 \phi(z, 4) + 3 \phi(z, 2) - \frac{3}{2} \right\},$$

$$\phi(z, 11) = \phi(z, 3) \left\{ \phi(z, 8) - 2 \phi(z, 6) + 4 \phi(z, 4) - 5 \phi(z, 2) + \frac{5}{3} \right\},$$

$$\phi(z, 12) = \phi(z, 4) \left\{ \phi(z, 8) - \frac{8}{3} \phi(z, 6) + \frac{13}{2} \phi(z, 4) - 10 \phi(z, 2) + 5 \right\},$$

$$\begin{aligned} \phi(z, 13) = \phi(z, 3) \left\{ \phi(z, 10) - \frac{5}{2} \phi(z, 8) + \frac{22}{3} \phi(z, 6) - \frac{219}{14} \phi(z, 4) \right. \\ \left. + \frac{691}{35} \phi(z, 2) - \frac{691}{105} \right\}, \end{aligned}$$

$$\begin{aligned} \phi(z, 14) = \phi(z, 4) \left\{ \phi(z, 10) - \frac{10}{3} \phi(z, 8) + \frac{526}{45} \phi(z, 6) - \frac{893}{30} \phi(z, 4) \right. \\ \left. + \frac{691}{15} \phi(z, 2) - \frac{691}{30} \right\}. \end{aligned}$$

We may note that the a 's and b 's are connected by the relation

$$(2n-r) b_r - (2n-r-1) b_{r-1} = 2na_r - na_{r-1} \dots\dots\dots(5).$$

For, by obvious steps, it is easily seen that ($m = 2n$)

$$\begin{aligned}
 b_r &= \text{coefficient of } z^r \text{ in } \frac{z^{r+1}-1}{z-1} \frac{d}{dz} \left\{ \frac{\phi(z, m)}{z^{m-r-1}} \right\}, \\
 &= \quad , \quad z^m \text{ in } \frac{z^{r+1}-1}{z-1} \{ m z \phi(z, m-1) - (m-r-1) \phi(z, m) \}, \\
 &= \quad , \quad z^m \text{ in } m z (z^{r+1}-1) z (z-\tfrac{1}{2}) \frac{\phi(z, m-1)}{\phi(z, 3)} \\
 &\quad \quad \quad - (m-r-1)(z^{r+1}-1) z^2 (z-1) \frac{\phi(z, m)}{\phi(z, 4)}, \\
 &= \quad , \quad z^m \text{ in } (m z^{r+4} - \tfrac{1}{2} m z^{r+3} - m z^3 + \tfrac{1}{2} m z^2) \\
 &\quad \quad \quad \times (\dots + a_{r-1} z^{m-r-5} + a_r z^{m-r-4} + \dots) \\
 &\quad \quad \quad - (m-r-1)(z^{r+4} - z^{r+3} - z^3 + z^2)(\dots + b_{r-1} z^{m-r-3} + b_r z^{m-r-4} + \dots),
 \end{aligned}$$

therefore $b_r = m a_r - \tfrac{1}{2} m a_{r-1} - (m-r-1) b_r + (m-r-1) b_{r-1}$,

whence the required result, viz.,

$$(2n-r) b_r - (2n-r-1) b_{r-1} - 2n a_r + n a_{r-1} = 0.$$

It is interesting to observe that this equation is true, not on account of any relation among Bernoulli's numbers, but because the coefficient of each Bernoulli's number vanishes. Thus the term containing B_p in this equation is

$$\left\{ (2n-r)(r-p)(2n)_{p+1} - (2n-r-1)(r-p-1)(2n)_{p+1} \right. \\
 \left. - 2n \frac{2^{r-p}-1}{2^{r-p-1}} (2n-1)_{p+1} + n \frac{2^{r-p-1}-1}{2^{r-p-2}} (2n-1)_{p+1} \right\} B_p,$$

which may be written in the form

$$\frac{(2n-p-1)(2n)_{p+1}}{2^{r-p-1}} B_p (2^{r-p-1} - 2^{r-p} + 1 + 2^{r-p-1} - 1),$$

an expression which vanishes identically.

In fact, equation (5) is merely the arithmetical equivalent of the differential identity

$$\frac{d}{dz} \left\{ \frac{\phi(z, 2n)}{\phi(z, 4)} \right\} + 4 \frac{\phi(z, 3)}{\phi(z, 4)} \frac{\phi(z, 2n)}{\phi(z, 4)} = 2n \frac{\phi(z, 2n-1)}{\phi(z, 4)}.$$

*On Deep-water Waves resulting from a Limited Original
Disturbance.* Mr. W. BURNSIDE.

[Read Nov. 8th, 1888.]

Cauchy,* in his "Memoire sur la Théorie des Ondes," determines under certain limitations the disturbance at any point of the surface of deep water and at any time, due to an initial surface displacement over a given limited area.

Reckoning t from the time of the original disturbance and x along the surface from its centre, his result, when the motion is in two dimensions, is that the displacement

$$\propto \frac{t}{x^{\frac{3}{2}}} \text{si} \left(\frac{gt^2}{4x} + \frac{\pi}{4} \right).$$

The limitations mentioned are that l/x and $gt^2l/4x^3$ should both be very small quantities, l being the length of that part of the surface in which the original displacement is sensible.

The latter limitation robs the result of most of its interest, for it will be seen later that it is not until $gt^2l/4x^3$ has a finite value (in the particular case first treated it is $\frac{1}{2}$) for a given value of x that the displacement at the corresponding place reaches its maximum.

Cauchy's expression is obtained as follows:

If the original vertical displacement at any point (the motion being supposed to start from rest) is $f(x)$, the form of the surface at any subsequent time is given by

$$y = \frac{1}{\pi} \int_0^\infty dm \int_{-\infty}^\infty da \cos m^{\frac{1}{2}}gt \cos m(a-x)f(a).$$

If $f(a)$ vanishes for all but very small values of a ,

$$\int_{-\infty}^\infty \cos m(a-x)f(a) da = H \cos mx \text{ approximately,}$$

where H is some constant quantity. (Cauchy shews that it is here that the limitation of $gt^2l/4x^3$ to small values is introduced.)

Therefore
$$y = \frac{H}{\pi} \int_0^\infty \cos m^{\frac{1}{2}}gt \cos mx dm.$$

* *Mém. des Sav. Etran.*, Vol. i., 1827.

Next, the value of the integral

$$\frac{1}{\pi} \int_0^{\infty} e^{-am} \cos m^{\frac{1}{2}} g^{\frac{1}{2}} t \cos mx \, dm$$

is calculated in the form of an infinite series, and in the result zero is written for α . If this is called I , then finally

$$y = HI.$$

It will be seen that, though he starts from a perfectly general form for the original surface displacement, Cauchy, as a matter of fact, introduces in the course of his work a special form, and that his result holds only for the latter under certain conditions.

Indeed his final equation may be written

$$y = \frac{H}{\pi} L_{\alpha=0}^t \int_0^{\infty} e^{-am} \cos m^{\frac{1}{2}} g^{\frac{1}{2}} t \cos mx \, dm,$$

and hence the initial surface displacement is

$$\begin{aligned} y_0 &= \frac{H}{\pi} L_{\alpha=0}^t \int_0^{\infty} e^{-am} \cos mx \, dm \\ &= \frac{H}{\pi} L_{\alpha=0}^t \frac{\alpha}{\alpha^2 + x^2}. \end{aligned}$$

Strictly, then, the case investigated is one in which the original displacement $\propto 1/\alpha^2 + x^2$, and the result is obtained for such times and places that zero may be written for α at a certain stage without sensible error.

In what follows I have attempted, starting first from the same initial form, but retaining the quantity α throughout, to obtain an expression for the displacement, which at a given place, shall hold through a much greater range of time than Cauchy's, and in particular from which it may be possible to determine when the displacement at a given place is a maximum and also its magnitude.

Taking c as the greatest initial displacement of any point of the surface, and l as the length of surface over which the initial disturbance is sensible, I shall speak of a term $c(l/x)^n$ in the expression for the displacement at any point as of the order $(l/x)^n$. The displacement will be only considered for points for which l/x is a small quantity.

$$\begin{aligned} \text{Putting, then,} \quad y_0 &= c\alpha \int_0^{\infty} e^{-am} \cos mx \, dm \\ &= \frac{c\alpha^2}{\alpha^2 + x^2}, \end{aligned}$$

the initial displacement at any distant point is of the order $(l/x)^2$, and this is to be considered vanishingly small.

The theory of deep-water waves shews that, if the surface be initially displaced but motionless, the typical solution is

$$y = A \cos mx \cos m^{\frac{1}{2}} g^{\frac{1}{2}} t,$$

the corresponding initial form being

$$y = A \cos mx;$$

and by an application of Fourier's theorem, any arbitrary form of free surface may be built up from this typical solution.

Hence at once the form of the surface at time t , corresponding to the above value y_0 , for the initial displacement, is

$$y = c\alpha \int_0^\infty e^{-\alpha m} \cos mx \cos m^{\frac{1}{2}} g^{\frac{1}{2}} t \, dm.$$

By expanding the term $\cos m^{\frac{1}{2}} g^{\frac{1}{2}} t$, and integrating term by term, this becomes

$$y = \frac{c\alpha}{(\alpha^2 + x^2)^{\frac{1}{2}}} \sum_0^\infty \frac{(-1)^n n!}{2n!} \frac{(gt^2)^n}{(\alpha^2 + x^2)^{\frac{1}{2}}} \cos \left(n+1 \tan^{-1} \frac{x}{\alpha} \right),$$

or, if
$$\frac{gt^2}{(\alpha^2 + x^2)^{\frac{1}{2}}} = 2z, \quad \tan^{-1} \frac{x}{\alpha} = \beta,$$

$$y = c \cos \beta \left[\cos \beta - z \cos 2\beta + \frac{z^2}{1 \cdot 3} \cos 3\beta - \frac{z^3}{1 \cdot 3 \cdot 5} \cos 4\beta + \dots \right].$$

Now, it may be easily verified, by integrating by parts, that

$$\begin{aligned} 2 \cos \beta \left[z \cos 2\beta - \frac{z^2}{1 \cdot 3} \cos 3\beta + \frac{z^3}{1 \cdot 3 \cdot 5} \cos 4\beta - \dots \right] \\ = (z \cos \beta)^{\frac{1}{2}} \int_0^{z \cos \beta} \frac{e^{-y/2} \cos \left(2\beta - \frac{1}{2} y \tan \beta \right)}{(z \cos \beta - y)^{\frac{1}{2}}} dy, \end{aligned}$$

and hence

$$y = c \cos^2 \beta - \frac{1}{2} c (z \cos \beta)^{\frac{1}{2}} \int_0^{z \cos \beta} \frac{e^{-y/2} \cos \left(2\beta - \frac{1}{2} y \tan \beta \right)}{(z \cos \beta - y)^{\frac{1}{2}}} dy.$$

If in the integral the variable be changed by the substitution

$$y = z \cos \beta - 2y' \cot \beta,$$

we have, after reduction,

$$y = c \cos^3 \beta - c \cos \beta \left(\frac{z}{2 \sin \beta} \right)^4 e^{-\frac{1}{2} \cos \beta} \\ \times \left[\cos \left(2\beta - \frac{z \sin \beta}{2} \right) \int_0^{\frac{1}{2} \sin \beta} \frac{e^{y' \cot \beta} \cos y'}{y'^4} dy' \right. \\ \left. - \sin \left(2\beta - \frac{z \sin \beta}{2} \right) \int_0^{\frac{1}{2} \sin \beta} \frac{e^{y' \cot \beta} \sin y'}{y'^4} dy' \right].$$

The integrals now to be evaluated are of the form

$$\int_0^p \frac{e^{qx} \cos x}{x^4} dx,$$

where q is a very small numerical quantity. By expanding the exponential and integrating each term by parts, the following expressions are at once obtained for the two integrals:—

$$\int_0^p \frac{e^{qx} \cos x}{x^4} dx = \left(1 - \frac{1 \cdot 3}{2 \cdot 4} q^2 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} q^4 - \dots \right) \int_0^p \frac{\cos x}{x^4} dx \\ + \left(-\frac{1}{2} q + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} q^3 - \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} q^5 + \dots \right) \int_0^p \frac{\sin x}{x^4} dx \\ + \frac{\sin p}{p^4} \left(pq + \frac{p^2 q^2}{2!} + \frac{p^3 q^3}{3!} + \dots \right) \\ + \frac{\cos p}{p^3} \left(\frac{3}{2} \frac{p^2 q^2}{2!} + \frac{5}{2} \frac{p^3 q^3}{3!} + \frac{7}{2} \frac{p^4 q^4}{4!} + \dots \right) \\ - \frac{\sin p}{p^2} \left(\frac{3 \cdot 5}{2 \cdot 2} \frac{p^3 q^3}{3!} + \frac{5 \cdot 7}{2 \cdot 2} \frac{p^4 q^4}{4!} + \dots \right) \\ - \frac{\cos p}{p^4} \left(\frac{3 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 2} \frac{p^4 q^4}{4!} + \frac{5 \cdot 7 \cdot 9}{2 \cdot 2 \cdot 2} \frac{p^5 q^5}{5!} + \dots \right) + \text{etc.}, \\ \int_0^p \frac{e^{qx} \sin x}{x^4} dx = \left(1 - \frac{1 \cdot 3}{2 \cdot 4} q^2 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} q^4 - \dots \right) \int_0^p \frac{\sin x}{x^4} dx \\ + \left(\frac{1}{2} q - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} q^3 + \dots \right) \int_0^p \frac{\cos x}{x^4} dx \\ - \frac{\cos p}{p^4} \left(pq + \frac{p^2 q^2}{2!} + \dots \right) \\ + \frac{\sin p}{p^3} \left(\frac{3}{2} \frac{p^2 q^2}{2!} + \frac{5}{2} \frac{p^3 q^3}{3!} + \dots \right) \\ + \frac{\cos p}{p^2} \left(\frac{3 \cdot 5}{2 \cdot 2} \frac{p^3 q^3}{3!} + \frac{5 \cdot 7}{2 \cdot 2} \frac{p^4 q^4}{4!} + \dots \right) \\ - \frac{\sin p}{p^4} \left(\frac{3 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 2} \frac{p^4 q^4}{4!} + \frac{5 \cdot 7 \cdot 9}{2 \cdot 2 \cdot 2} \frac{p^5 q^5}{5!} + \dots \right) - \text{etc.}$$

The general term of the part occurring after the first two lines in either of these integrals is

$$\pm \frac{\sin p}{p^{\frac{1}{2}(2n-1)}} \left[\frac{2n-1 \cdot 2n-3 \dots 3}{2^{n-1}} \frac{(pq)^n}{n!} + \frac{2n+1 \dots 5}{2^{n-1}} \frac{(pq)^{n+1}}{n+1!} + \dots \right],$$

and the quantity in brackets is clearly less than

$$(pq)^{n-1} (e^{pq} - 1).$$

The general term of the series is therefore in magnitude less than

$$\frac{\sin p}{p^{\frac{1}{2}}} (e^{pq} - 1) q^{n-1}.$$

Hence, q being very small, we have as first approximations

$$\begin{aligned} \int_0^p \frac{e^{qx} \cos x}{x^{\frac{1}{2}}} dx &= \int_0^p \frac{\cos x}{x^{\frac{1}{2}}} dx + \frac{\sin p}{p^{\frac{1}{2}}} (e^{pq} - 1), \\ \int_0^p \frac{e^{qx} \sin x}{x^{\frac{1}{2}}} dx &= \int_0^p \frac{\sin x}{x^{\frac{1}{2}}} dx - \frac{\cos p}{p^{\frac{1}{2}}} (e^{pq} - 1), \end{aligned}$$

the next terms of either part of either integral being of the order q times those written.

Also $(e^{pq} - 1)/p^{\frac{1}{2}}$ increases continually with p , and for moderate values of pq (which will be found to correspond to the case here to be considered) is of the order $q^{\frac{1}{2}}$, and therefore in the expression of each integral the second terms written are intermediate in magnitude between the first and those omitted.

Finally, as is well known,

$$\begin{aligned} \int_0^p \frac{\cos x}{x^{\frac{1}{2}}} dx &= \sqrt{\frac{\pi}{2}} + \frac{\sin p}{p^{\frac{1}{2}}} - \frac{1}{2} \frac{\cos p}{p^{\frac{1}{2}}} - \frac{1 \cdot 3}{2 \cdot 2} \frac{\sin p}{p^{\frac{3}{2}}} + \dots, \\ \int_0^p \frac{\sin x}{x^{\frac{1}{2}}} dx &= \sqrt{\frac{\pi}{2}} - \frac{\cos p}{p^{\frac{1}{2}}} - \frac{1}{2} \frac{\sin p}{p^{\frac{1}{2}}} + \frac{1 \cdot 3}{2 \cdot 2} \frac{\cos p}{p^{\frac{3}{2}}} + \dots, \end{aligned}$$

where the divergent series may be used as approximations by stopping at or before the smallest terms.

Before substituting in the expression for the displacement the values of the integrals now found, it will be well to consider the magnitudes of some of the quantities involved, with a view to neglecting systematically terms q times those retained.

$$\text{Thus} \quad \cos \left(2\beta - \frac{z \sin \beta}{2} \right) \quad \text{and} \quad \sin \left(2\beta - \frac{z \sin \beta}{2} \right)$$

differ from $-\cos\left(\frac{z \sin \beta}{2}\right)$ and $\sin\left(\frac{z \sin \beta}{2}\right)$,

by quantities of the order g times these latter quantities; also

$$p = \frac{z \sin \beta}{2} = \frac{gt^2 x}{4(x^2 + a^2)},$$

and hence, except in the circular functions $gt^2/4x$, may certainly be written for p ; so also

$$\frac{z \cos \beta}{2} (= pq) = \frac{gt^2 a}{4x^2}, \quad \cos \beta = \frac{a}{x},$$

and

$$\frac{z}{2 \sin \beta} = \frac{gt^2}{4x} \text{ approximately.}$$

Hence, retaining p for $\frac{1}{2}z \sin \beta$ in the circular functions, we get

$$\begin{aligned} y &= \frac{ca}{x} \left(\frac{gt^2}{4x}\right)^{\frac{1}{2}} e^{-gt^2 a/4x^2} \\ &\times \left[\cos p \left\{ \sqrt{\frac{\pi}{2}} + \left(\frac{gt^2}{4x}\right)^{-\frac{1}{2}} e^{gt^2 a/4x^2} \sin p - \frac{1}{2} \left(\frac{gt^2}{4x}\right)^{-\frac{1}{2}} \cos p - \text{etc.} \right\} \right. \\ &\quad \left. + \sin p \left\{ \sqrt{\frac{\pi}{2}} - \left(\frac{gt^2}{4x}\right)^{-\frac{1}{2}} e^{gt^2 a/4x^2} \cos p - \frac{1}{2} \left(\frac{gt^2}{4x}\right)^{-\frac{1}{2}} \sin p + \text{etc.} \right\} \right] \\ &= \frac{ca}{x} \left(\frac{gt^2}{4x}\right)^{\frac{1}{2}} e^{-gt^2 a/4x^2} \\ &\times \left[\sqrt{\pi} \sin \left(p + \frac{\pi}{4}\right) - \frac{1}{2} \left(\frac{gt^2}{4x}\right)^{-\frac{1}{2}} + \frac{1.3.5}{2.2.2} \left(\frac{gt^2}{4x}\right)^{-\frac{3}{2}} - \dots \right]. \end{aligned}$$

With the same coefficient outside the bracket, the most important of the omitted periodic terms inside is

$$\frac{3\sqrt{\pi}}{2} \frac{a}{x} \cos \left(p + \frac{\pi}{4}\right).$$

The ultimately divergent series following the first term inside the bracket, may, as in the case of the two from which it is derived, be used as an approximation by stopping before the smallest term; and, as it converges for three terms if $gt^2/4x$ is greater than 4, the above expression will serve for the calculation of y at a given point for any values of the time not less than $4\sqrt{x/g}$.

For small values of t the original series is certainly the best form from which to calculate y ; and from it may be seen that for such values y is of the order $(a/x)^2$, as it is initially. The expression for y now found shows that, when $gt^2/4x$ is a moderately large number, y is of the

order a/x ; but also indicates that, when $gt^2/4x$ is comparable with x/a , y is of the order $\sqrt{a/x}$. For values of t such as this all terms following $\sqrt{\pi} \sin(p + \pi/4)$ may certainly be omitted as smaller than terms already neglected. Moreover, when $gt^2/4x$ is comparable with x^2/a^2 , the value of y will, owing to the factor $e^{-gt^2/4x}$, be again extremely small, and hence in the term $\sin(p + \pi/4)$ we may safely write $gt^2/4x$ for p . We have finally then, as an expression for the displacement at a given place, as long as its magnitude is comparable with its maximum value,

$$y = \sqrt{\pi} \frac{ca}{x} \left(\frac{gt^2}{4x} \right)^{\frac{1}{2}} e^{-gt^2/4x} \sin \left(\frac{gt^2}{4x} + \frac{\pi}{4} \right).$$

The greatest value of

$$\left(\frac{gt^2}{4x} \right)^{\frac{1}{2}} e^{-gt^2/4x}$$

for a particular value of x , is given by

$$gt^2 a = 2x^2,$$

and the ratio of the logarithmic differential coefficient of $\left(\frac{gt^2}{4x} \right)^{\frac{1}{2}} e^{-gt^2/4x}$ to that of $gt^2/4x$ is

$$1 - \frac{gt^2 a}{2x^2};$$

hence, when, for a given value of x , y is near its greatest value, $\sin \left(\frac{gt^2}{4x} + \frac{\pi}{4} \right)$ changes very rapidly compared to its coefficient; or, in other words, to determine the maximum value of y , the coefficient only need be considered.

The greatest amplitude of displacement at the point x is therefore

$$\sqrt{\frac{\pi}{2}} c e^{-\frac{1}{2}} \left(\frac{a}{x} \right)^{\frac{1}{2}},$$

and the amplitude has this value when, as seen above,

$$t = x \sqrt{\frac{2}{ga}}.$$

Also, if $gt^2/4x$ increases by 2π between times t and t' , then

$$t' - t = \frac{8\pi x}{g(t' + t)};$$

or, in words, the period of the wave motion at x , when the amplitude is greatest, is

$$2\pi \sqrt{\frac{2a}{g}},$$

and the length of the waves is $4\pi a$.

For places at which the displacement has not reached its maximum the period and wave-length are greater, and for those for which the greatest displacement is past they are less than those quantities.

The energy of the initial displacement varies as $c^2\alpha$, and the extent of surface over which it is sensible varies as α . The above results may then be expressed as follows.

For the same *form* of initial displacement—

(i.) The greatest amplitude of displacement at any point varies as the square root of the whole energy of the motion, and inversely as the square root of the distance from the originally disturbed area.

(ii.) The greatest disturbance is propagated with a uniform velocity, varying as the square root of the space over which the original disturbance is sensible; and

(iii.) The wave-length of the motion, when greatest, varies as the space over which the original disturbance is sensible.

These results are proved only for the case now considered; but, from the nature of the problem, there can be little doubt but that similar statements may be made with regard to the motions resulting from any other *form* of initial displacement. Indeed this is shown to be the case later for two other forms.

To represent the results in a complete way graphically would involve very considerable labour; but, omitting the periodic term in the expression for the displacement, the magnitude of the motion only may be represented as follows.

If in the general expression for y the periodic factor be omitted, then, when x is regarded as constant,

$$y = Ate^{-\alpha t^2},$$

and, when t is regarded as constant,

$$y = \frac{B}{x^{\frac{1}{2}}} e^{-\beta/x^2},$$

where A , B , α , β are constant quantities. The general forms of the curves with these equations will not depend on the values of the constants, which affect only the relative scales parallel to the two axes on which they are drawn.

Hence the curves $y = 5xe^{-x^2}$ (i.),

$$y = \frac{5}{x^{\frac{1}{2}}} e^{-1/x^2} \text{ (ii.),}$$

(where the numerical factor is introduced to give suitable propor-

tions to the figures,) will show graphically—the first, how the amplitude of the displacement varies with the time at a particular place; the second, how the amplitudes vary from place to place at a given time.

FIG. 1.

FIG. 2.

These curves enable us to further interpret the formula for y in a general way as follows.

At a particular point the disturbance dies away from its maximum value at a slower rate than it grows up to it; though, the first curve not being very unsymmetrical on either side of its maximum ordinate, the rise and dying away of the disturbance at a particular point will not be very dissimilar.

At a particular time, the greater part of the surface sensibly disturbed will be beyond the point of maximum disturbance, on the near side of which the amplitude rapidly diminishes to an insensible amount.

These results, which at first sight appear almost to contradict

each other, are reconciled at once when it is noticed that at any instant the extent of sensibly disturbed surface (*i.e.*, the amount of surface in which the amplitude of the disturbance is not less than a given sub-multiple of the existing greatest disturbance) increases in direct proportion with the distance of the point of greatest disturbance from the origin.

It appears certain that, for the particular form of original disturbance hitherto considered, y cannot be *exactly* expressed in terms of x and t otherwise than by an infinite series. It is not, however, difficult artificially to build up forms for the original disturbance, such that the value of y for any values of x and t can be expressed in finite terms; and if these are chosen so that the extent of surface over which the original disturbance is sensible is limited, the results will serve as a sort of test of the accuracy of the approximate value of y obtained above, and of the law of the motion deduced from it.

In particular, if

$$y_0 = A \int_0^{\infty} m^{i+\frac{1}{2}} e^{-am} \cos mx \, dm,$$

where i is a positive integer, the resulting value of y may be expressed in finite terms.

Taking the case of i zero, put

$$\begin{aligned} y_0 &= \frac{2c}{\sqrt{\pi}} a^{\frac{1}{2}} \int_0^{\infty} m^{\frac{1}{2}} e^{-am} \cos mx \, dm \\ &= c \left(\frac{a^{\frac{1}{2}}}{a^2 + x^2} \right)^{\frac{1}{2}} \cos \left(\frac{3}{2} \tan^{-1} \frac{x}{a} \right). \end{aligned}$$

Then
$$y = \frac{2ca^{\frac{3}{2}}}{\sqrt{\pi}} \int_0^{\infty} m^{\frac{1}{2}} e^{-am} \cos mx \cos m^{\frac{1}{2}} g^{\frac{1}{2}} t \, dm.$$

Put
$$mx = m'^2,$$

$$\tan^{-1} \frac{x}{a} = \beta,$$

$$\frac{gt^2}{4x} = p,$$

and therefore

$$y = \frac{4c}{\sqrt{\pi}} \left(\frac{a}{x} \right)^{\frac{1}{2}} \int_0^{\infty} m'^2 e^{-m'^2 \cot \beta} \cos m'^2 \cos 2mp^{\frac{1}{2}} \, dm'.$$

This integral may be deduced from the known form

$$\int_0^\infty e^{-qm^2} \cos 2mp^{\frac{1}{2}} dm = \frac{\sqrt{\pi}}{2} \frac{1}{q^{\frac{1}{2}}} e^{-p/q}.$$

Differentiate this with respect to q , giving

$$\int_0^\infty m^2 e^{-qm^2} \cos 2mp^{\frac{1}{2}} dm = \frac{\sqrt{\pi}}{4} \left[\frac{1}{q^{\frac{3}{2}}} - \frac{2p}{q^{\frac{5}{2}}} \right] e^{-p/q}.$$

Now, for q write $\cot \beta + i$, and equate real parts; then

$$\begin{aligned} & \int_0^\infty m^2 e^{-m^2 \cot \beta} \cos m^2 \cos 2mp^{\frac{1}{2}} dm \\ &= \text{real part of } \frac{\sqrt{\pi}}{4} \left[\sin^{\frac{1}{2}} \beta e^{-3i\beta/2} - 2p \sin^{\frac{1}{2}} \beta e^{-5i\beta/2} \right] e^{-p \sin \beta (\cos \beta - i \sin \beta)} \\ &= \frac{\sqrt{\pi}}{4} \sin^{\frac{1}{2}} \beta e^{-p \sin \beta \cos \beta} \left[\cos \left(p \sin^2 \beta - \frac{3\beta}{2} \right) \right. \\ & \quad \left. - 2p \sin \beta \cos \left(p \sin^2 \beta - \frac{5\beta}{2} \right) \right]. \end{aligned}$$

[The substitution of a complex value for q may be justified as follows:—The integral

$$\int_0^\infty m^2 e^{-qm^2} \cos 2mp^{\frac{1}{2}} dm$$

is the sum of a series of integrals of the form

$$\int_0^\infty m^{2n} e^{-qm^2} dm.$$

The result of writing $\cot \beta + i$ for q is the same, neglecting a factor in the integrand independent of m , as that of writing

$$\left(\cos \frac{\beta}{2} + i \sin \frac{\beta}{2} \right) m' \text{ for } m \sqrt{q \sin \beta},$$

and since $\beta/2$ is necessarily less than $\pi/4$, the known theory shows at once that the result of the substitution is correct.]

Finally, then, neglecting terms of the order a/x multiplied by those retained; so as to compare this result directly with the former one,

$$y = c \left(\frac{a}{x} \right)^{\frac{1}{2}} e^{-g t^2 a / 4 x^2} \left[\frac{g t^2}{2x} \sin \left(\frac{g t^2}{4x} + \frac{\pi}{4} \right) - \cos \left(\frac{g t^2}{4x} + \frac{\pi}{4} \right) \right],$$

and for values of y at any place comparable with its maximum value, the second term inside the bracket may be neglected in comparison with the first.

It will be clear, on inspection of the value of y just obtained, that, with the new form of initial disturbance now considered, the general laws of the motion are the same as those already given. When, however, the particular results for the two given forms are compared, it will be seen that the initial form seems to have a decided effect on the resulting motion.

Initial ordinate	$\frac{ca^2}{a^2+x^2}$	$c \left(\frac{a^2}{a^2+x^2} \right)^{\frac{3}{2}} \cos \left(\frac{3}{2} \tan^{-1} \frac{x}{a} \right)$
Whole energy of } motion	$\frac{\pi}{4} gc^2a$	$\frac{1}{2} gc^2a$
Space in which initial ordinate diminishes to about $\frac{1}{100}$ th of its greatest value } ...	$10a$	$21a$
Maximum amplitude at place x } ...	$\sqrt{\frac{\pi}{2}} ce^{-1} \left(\frac{a}{x} \right)^{\frac{1}{2}}$	$2ce^{-1} \left(\frac{a}{x} \right)^{\frac{1}{2}}$
Rate of propagation of greatest disturbance } ...	$\sqrt{ga/2}$	$\sqrt{ga/4}$
Wave-length of motion when amplitude is greatest } ...	$4\pi a$	$2\pi a$

The two cases hitherto considered are alike in this, that, the initial displacement being continuous, the extent of surface over which they are sensible can only be assigned by a convention as to what small quantities are to be considered negligible. It can hardly be doubted but that such a phenomenon as the Krakatao waves would be represented more nearly in symbols by a discontinuous initial form, the ordinate of which is strictly zero beyond a certain point. It is interesting, too, to notice how far the analytical expression for the displacement in such a case differs from those already considered. Suppose then that, for values of x between $+a$ and $-a$, the initial displacement is c , and that for all other values of x it is zero.

Then
$$y_0 = \frac{c}{\pi} \int_0^\infty \frac{\sin(x+a)m - \sin(x-a)m}{m} d\tilde{m},$$

and
$$y = \frac{c}{\pi} \int_0^\infty \frac{\sin(x+a)m - \sin(x-a)m}{m} \cos m^{\frac{1}{2}} g^{\frac{1}{2}} t d\tilde{m}.$$

The integral to whose value it is necessary to approximate in this case is

$$\int_0^\infty \frac{\sin m \cos m^{\frac{1}{2}} p^{\frac{1}{2}}}{m} dm,$$

where

$$p = \frac{gt^{\frac{3}{2}}}{x \pm a},$$

and it may be readily expressed in a series of ascending powers of p from which, for moderate values of p , the displacement may be calculated.

Thus, if $f(p)$ denote the integral,

$$\begin{aligned} f(p) &= \int_0^\infty dv \int_0^\infty e^{-mv} \sin m \sum_0^\infty (-1)^n \frac{p^n}{2n!} m^n dv \\ &= \frac{1}{2i} \int_0^\infty dv \sum_0^\infty (-1)^n \frac{n!}{2n!} p^n \left(\frac{1}{(v-i)^{n+1}} - \frac{1}{(v+i)^{n+1}} \right) \\ &= \frac{\pi}{2} + \sum_1^\infty (-1)^{n+1} \frac{2n!}{4n+2!} p^{2n+1}, \end{aligned}$$

and

$$y = \frac{c}{\pi} \left[f\left(\frac{gt^{\frac{3}{2}}}{x+a}\right) - f\left(\frac{gt^{\frac{3}{2}}}{x-a}\right) \right].$$

As long as $gt^{\frac{3}{2}}/x$ is small compared with x/a the above value of y is clearly of the order a/x , and as we may now anticipate that the greatest displacement at any point is of the order $(a/x)^{\frac{1}{2}}$, it is necessary to consider the value of $f(p)$ for large values of p for which purpose the series is of little use.

Writing v^2 for mp in the integral, we have

$$\begin{aligned} f(p) &= 2 \int_0^\infty \frac{\sin v^2/p \cos v}{v} dv \\ &= 2 \left[\frac{\sin v^2/p \sin v}{v} \right]_0^\infty - \frac{4}{p} \int_0^\infty \cos v^2/p \sin v dv + 2 \int_0^\infty \frac{\sin v^2/p \sin v}{v^2} dv. \end{aligned}$$

The integrated term vanishes at the limits, and the last term

$$= 2 \int_p^\infty \frac{dp}{p^{\frac{3}{2}}} \int_0^\infty \cos v^2/p \sin v dv,$$

therefore
$$f(p) = \left[-\frac{4}{p} + 2 \int_p^\infty \frac{dp}{p^{\frac{3}{2}}} \right] \int_0^\infty \cos v^2/p \sin v dv.$$

$$\begin{aligned}
\text{Again, } \int_0^\infty \cos v^2/p \sin v \, dv &= \frac{1}{2} \int_0^\infty \sin \left[\left(\frac{v}{p^{\frac{1}{4}}} + \frac{p^{\frac{1}{4}}}{2} \right)^2 - \frac{p}{4} \right] dv \\
&\quad - \frac{1}{2} \int_0^\infty \sin \left[\left(\frac{v}{p^{\frac{1}{4}}} - \frac{p^{\frac{1}{4}}}{2} \right)^2 - \frac{p}{4} \right] dv \\
&= \frac{1}{2} p^{\frac{1}{4}} \cos \frac{p}{4} \left[\int_{\frac{1}{2}p^{\frac{1}{4}}}^\infty \sin u^2 du - \int_{-\frac{1}{2}p^{\frac{1}{4}}}^\infty \sin u^2 du \right] \\
&\quad - \frac{1}{2} p^{\frac{1}{4}} \sin \frac{p}{4} \left[\int_{\frac{1}{2}p^{\frac{1}{4}}}^\infty \cos u^2 du - \int_{-\frac{1}{2}p^{\frac{1}{4}}}^\infty \cos u^2 du \right] \\
&= \frac{1}{2} p^{\frac{1}{4}} \cos \frac{p}{4} \left[-\sqrt{\frac{\pi}{2}} + 2 \int_{\frac{1}{2}p^{\frac{1}{4}}}^\infty \sin u^2 du \right] \\
&\quad - \frac{1}{2} p^{\frac{1}{4}} \sin \frac{p}{4} \left[-\sqrt{\frac{\pi}{2}} + 2 \int_{\frac{1}{2}p^{\frac{1}{4}}}^\infty \cos u^2 du \right];
\end{aligned}$$

and, using the values of the integrals already given in a slightly different form,

$$\begin{aligned}
&= \frac{1}{2} p^{\frac{1}{4}} \cos \frac{p}{4} \left[-\sqrt{\frac{\pi}{2}} + \frac{\cos p/4}{(p/4)^{\frac{1}{4}}} + \frac{1}{2} \frac{\sin p/4}{(p/4)^{\frac{3}{4}}} - \frac{1 \cdot 3}{2 \cdot 2} \frac{\cos p/4}{(p/4)^{\frac{5}{4}}} - \dots \right] \\
&\quad - \frac{1}{2} p^{\frac{1}{4}} \sin \frac{p}{4} \left[-\sqrt{\frac{\pi}{2}} - \frac{\sin p/4}{(p/4)^{\frac{1}{4}}} + \frac{1}{2} \frac{\cos p/4}{(p/4)^{\frac{3}{4}}} + \frac{1 \cdot 3}{2 \cdot 2} \frac{\sin p/4}{(p/4)^{\frac{5}{4}}} - \dots \right] \\
&= -\frac{\sqrt{\pi}}{2} p^{\frac{1}{4}} \cos \left(\frac{p}{4} + \frac{\pi}{4} \right) + 1 - \frac{1 \cdot 3}{2 \cdot 2} \left(\frac{4}{p} \right)^{\frac{1}{4}} + \dots,
\end{aligned}$$

where, as before, the divergent series may be used as an approximation by stopping before the smallest term.

The leading part of

$$\int_p^\infty \frac{dp}{p^{\frac{3}{2}}} \int_0^\infty \cos v^2/p \sin v \, dv,$$

when p is great, will be

$$-\frac{1}{2} \sqrt{\pi} \int_p^\infty \frac{\cos (p+\pi)/4}{p^{\frac{3}{2}}} dp + \int_p^\infty \frac{dp}{p^{\frac{3}{2}}},$$

or $\frac{1}{p} + \text{terms of order } \frac{1}{p^{\frac{3}{2}}}, \text{ etc.};$

hence $f(p) = 2\sqrt{\pi} \frac{1}{p^{\frac{1}{4}}} \cos \frac{p+\pi}{4} - \frac{2}{p} + \text{terms of order } \frac{1}{p^{\frac{3}{2}}}, \text{ etc.}$

The resulting value of the displacement is

$$y = \frac{2c}{\sqrt{\pi}} \left[\left(\frac{x+a}{gt^2} \right)^{\frac{1}{2}} \cos \left(\frac{gt^2}{4(x+a)} + \frac{\pi}{4} \right) - \left(\frac{x-a}{gt^2} \right)^{\frac{1}{2}} \cos \left(\frac{gt^2}{4(x-a)} + \frac{\pi}{4} \right) \right] \\ - \frac{2c}{\pi} \left[\frac{x+a}{gt^2} - \frac{x-a}{gt^2} \right] \\ + \text{etc.};$$

and when, as before, terms of the order a/x multiplied into those retained are neglected,

$$y = \frac{2c}{\sqrt{\pi}} \left(\frac{x}{gt^2} \right)^{\frac{1}{2}} \left[\cos \left(\frac{gt^2}{4(x+a)} + \frac{\pi}{4} \right) - \cos \left(\frac{gt^2}{4(x-a)} + \frac{\pi}{4} \right) \right. \\ \left. + \frac{a}{2x} \left\{ \cos \left(\frac{gt^2}{4(x+a)} + \frac{\pi}{4} \right) + \cos \left(\frac{gt^2}{4(x-a)} + \frac{\pi}{4} \right) \right\} \right].$$

So long as gt^2/x , though considerable, is small compared with x/a , the two pairs of terms in the bracket are both of the order a/x . When, however, gt^2/x and x/a are of the same order of magnitude, the second pair of terms are of the order a/x compared with the first. Neglecting, therefore, the last pair, as continually before, and for reasons already given neglecting terms of the order gt^2a^2/x^3 in the arguments of the periodic terms, we have, finally,

$$y = \frac{4c}{\sqrt{\pi}} \left(\frac{x}{gt^2} \right)^{\frac{1}{2}} \sin \frac{gt^2a}{4x^2} \sin \left(\frac{gt^2}{4x} + \frac{\pi}{4} \right).$$

Here, again, $\sin \left(\frac{gt^2}{4x} + \frac{\pi}{4} \right)$ changes rapidly as compared with

$$\left(\frac{x}{gt^2} \right)^{\frac{1}{2}} \sin \frac{gt^2a}{4x^2},$$

and to determine the magnitude of the disturbance at any time the latter quantity need alone be considered. Its maximum values are given by

$$0 = \frac{1}{t^2} \sin \frac{gt^2a}{4x^2} - \frac{ga}{2x^3} \cos \frac{gt^2a}{4x^2},$$

$$\text{or} \quad \tan \theta = 2\theta,$$

$$\text{where} \quad \theta = \frac{gt^2a}{4x^2}.$$

The roots of this equation, omitting zero, form an ascending series

θ_1, θ_2 , etc., such that θ_n approximates rapidly to $(2n-1)\frac{1}{2}\pi$ as n increases. The first two roots are approximately

$$\theta_1 = 1.165,$$

$$\theta_2 = 4.622.$$

To each root of the equation corresponds a maximum value of the amplitude for the particular value of x considered, that corresponding to θ_n being

$$\frac{2c}{\sqrt{\pi}} \left(\frac{a}{x} \right)^{\frac{1}{2}} \frac{\sin \theta_n}{\theta_n^{\frac{1}{2}}}.$$

Now
$$\frac{\sin \theta_1}{\theta_1^{\frac{1}{2}}} = .851,$$

and
$$\frac{\sin \theta_2}{\theta_2^{\frac{1}{2}}} = -.463,$$

and
$$\frac{\sin \theta_n}{\theta_n^{\frac{1}{2}}} = (-1)^{n-1} \sqrt{\frac{2}{(2n-1)\pi}}, \text{ nearly,}$$

so that the first of these successive maxima is the greatest in absolute magnitude, and they diminish continually.

The maximum corresponding to each root of the equation

$$\tan \theta - 2\theta = 0$$

is propagated with uniform velocity, and the magnitude of each maximum amplitude is, as before, directly proportional to the square root of the whole energy, and inversely as the square root of the distance from the original disturbance.

The intervals between successive maxima are periods of time comparable with that elapsing between the original disturbance and the first maximum.

A second feature distinguishing this case from those before, is in the much greater duration, *ceteris paribus*, of the sensible disturbance at a particular point.

If t_0 is the time elapsing between the original disturbance and the chief maximum at the point considered, $t_0/\log n$ is an approximate measure of the time that elapses in the former case before the amplitude diminishes to $1/n^{\text{th}}$ of its greatest value; while in the case now under consideration $(n-1)t_0$ is the corresponding quantity.

The integrals which will represent the form of the surface at any time, when the disturbance is caused by impulsive pressures applied over a limited extent of surface, are generally of slightly different form to those arising from the cause considered in this paper; but similar methods may be used to approximate to their values. The forms obtained for the surface and the general laws which the motion follows are, as is to be expected, precisely similar to the results obtained here.

On a certain Atomic Hypothesis. By Prof. KARL PEARSON.

[Read Nov. 8th, 1888.]

1. In a paper written in 1883, and read before the Cambridge Philosophical Society in 1885 (*Camb. Phil. Trans.*, xiv., Part II., pp. 71—120), I supposed, as a first approximation, that the ultimate constituents of matter might be treated as spherical bodies capable of surface vibration and pulsation. I then endeavoured to show that the free and forced vibrations of such atoms were sufficient to produce in a fluid ether many of the phenomena of chemical affinity, cohesion, and gravitation. The assumptions really made were:

(i.) The pressures produced on the surface of one atom by the vibrations of another are, at any rate to a first approximation, identical with those which would be produced were the ether a perfect fluid.

(ii.) The ultimate atom may, to a first approximation, be treated as a vibrating sphere.

In the course of my work, I did not pre-suppose any special internal structure for the spherical atom, beyond a capacity for absorbing energy in the form of surface vibrations. I also gave some reasons for the belief that a simple pulsation must play a more important part than polar vibrations within the sphere of inter-atomic influence. Further, the more complex molecule, which forms the basis of the