On a Peculiarity of the Wave-System due to the Free Vibrations of a Nucleus in an Extended Medium. By Prof. HORACE LAMB, F.R.S. Read May 10th, 1900. Received September 21st, 1900.

In a number of physical problems we have to do with a central body, or nucleus (usually taken to be spherical for mathematical simplicity), surrounded by a medium capable of transmitting energy in the form of waves. If an agitation be communicated to the nucleus in any way, waves will be started in the medium, and, owing to the energy which these carry off, the vibrations of the nucleus, if this be left to itself, will gradually subside. As instances, we may cite the electrical oscillations on a spherical conductor,* or in a dielectric sphere of great inductive capacity,† and the vibrations of a pendulum or of a deformable sphere in a gaseous‡ or an elastic-solid§ medium. The point to be here considered arises in the interpretation of the analytical expression for the waves thus generated. Though simple, it appears to have occasioned some perplexity, and it has been suggested that a brief elucidation of the matter might be acceptable.

In all problems of the above kind the usual method of procedure is to assume a time-factor e^{ikct}, c being the wave-velocity, and consequently $2\pi/k$ the wave-length (when k is real). The solution of the differential equations which hold throughout the medium then indicates that the amplitude of the disturbance at a distance r from the nucleus, great in comparison with the dimensions of the latter, is given by one or more terms of the type

$$\frac{A}{r} e^{ik(ct-r)} S_n, \tag{1}$$

where S_n involves the direction (only) of the radius vector r. The combination $ct+r$ would of course be equally admissible in the index,

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* J. J. Thomson, *Proc. Lond. Math. Soc.*, Vol. xv., p. 197 (1884); *Recent Researches*, § 308.

† *Camb. Trans.*, Vol. XVIII., p. 348.

‡ *Proc. Lond. Math. Soc.*, Vol. XXXII., p. 11.

§ *Ibid.*, Vol. XXXII., p. 120.

but is rejected for our present purpose as representing a system of waves travelling inwards. The admissible values of k are determined in each case by means of the special conditions to be satisfied at the boundary of the nucleus. It is found that k is either pure imaginary, or else complex with the imaginary part positive. In the latter case, writing

$$k = \kappa + im, \tag{2}$$

and substituting in (1), we obtain, on taking the real part, an expression of the form

$$\frac{C}{r} e^{-m(ct-r)} \cos \kappa \, (ct - r + a) \, S_n. \tag{3}$$

The circular function represents a train of simple harmonic waves of length $2\pi/\kappa$ travelling outwards with velocity c; the factor S_n represents the variation of intensity with direction; and the factor $1/r$ represents the attenuation due to spherical divergence. The factor e^{-mct} exhibits the decay of the vibration at any place as the original energy of the nucleus is gradually spent in the generation of waves. The difficulty which has been felt relates to the factor e^{mr}, m being positive. It is true that m is usually small compared with κ, but, however small it may be, the exponential indicates unlimited increase with r, which will ultimately outbalance altogether the weakening due to spherical divergence. For this reason it has been doubted whether a solution such as (3) really corresponds to any physical reality at all.

The explanation is that owing to the finite velocity of wave-propagation there is at any instant, in our unlimited medium, a region not yet reached by the waves, viz., that beyond a spherical surface of radius ct, where t is the time elapsed since the primitive disturbance of the nucleus. Up to the confines of this region the law of amplitude indicated by (3) is strictly applicable; but beyond them everything is as yet quiescent. And within the region occupied by the waves the amplitude at any point P will (except for spherical divergence) be less than that at a point Q further from the centre on the same radius vector in the ratio $e^{-m \cdot PQ}$, for the reason that it represents a disturbance which started later by an interval PQ/c, during which the vibration of the nucleus has been decaying according to the law e^{-mct}. It is clear in fact that the two parts of the exponential in (3) are necessary concomitants.

When k is pure imaginary, the solution (1) takes the form

$$\frac{A}{r} e^{-m(ct-r)}. \tag{4}$$

The only novelty is that the vibration which is sooner or later started at any point is "aperiodic."

To reduce the question to its simplest form, we may take a one-dimensional analogy. Let us suppose that we have an infinitely long tense string of uniform density ρ, to which a mass M is attached at the origin ($x = 0$); and let us further suppose that any lateral displacement of M is resisted not only by the tension (T) of the string, but also by springs fastened to fixed supports; and let $2\pi/\sigma$ be the period in which M would oscillate if the string were absent. Everything being at rest to begin with, let us trace the effect of a sudden blow given to M at time $t = 0$. The equation of motion of the string will be of the form

$$\frac{d^2y}{dt^2} = c^2 \frac{d^2y}{dx^2}, \tag{5}$$

where $c^2 = T/\rho$, and that of M

$$M\left(\frac{d^2y_0}{dt^2} + \sigma^2 y_0\right) = -2T\left(\frac{dy}{dx}\right)_0, \tag{6}$$

where the limit of dy/dx, when x approaches 0 from the positive side of the origin, is to be understood. For $x > 0$, the appropriate solution of (5) is of the form

$$y = f(ct - x). \tag{7}$$

If we put $M = 2\rho b$, the equation (6) gives

$$f''(u) + \frac{1}{b} f'(u) + \frac{\sigma^2}{c^2} f(u) = 0, \tag{8}$$

to be satisfied *for positive values of* u. Since when $t = 0$ we have $y = 0$ for all values of x, it appears that for all negative values of u we must have $f(u) = 0$. The solution of (8) is

$$f(u) = Ce^{-u/2b} \cos \kappa (u + \alpha), \tag{9}$$

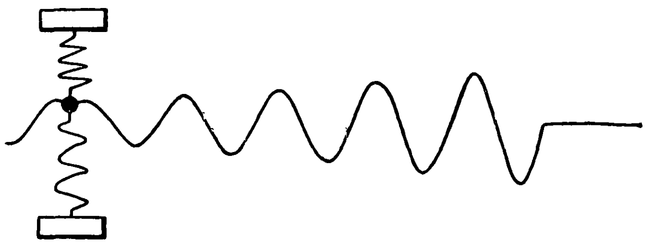
if

$$\kappa = \sqrt{\left(\frac{\sigma^2}{c^2} - \frac{1}{4b^2}\right)}, \tag{10}$$

the constants C and α being (so far) arbitrary. Adjusting α so as to make $f(0)=0$, since M starts from its equilibrium position, we have

$$\left.\begin{array}{ll} \text{for} \quad x < ct, & y = Ce^{-(ct-x)/2b}\sin\kappa\,(ct-x) \\ \text{and} \quad \text{for} \quad x > ct, & y = 0 \end{array}\right\}. \tag{11}$$

The formulæ relate to the positive side of the origin. The circumstances are, of course, symmetrical on the two sides. The figure is intended as a rough illustration, the amplitude being of course greatly exaggerated.



We have assumed that $\sigma > c/2b$. In the opposite event, the solution of (8) is

$$f(u) = Ae^{-m_1 u} + Be^{-m_2 u}, \tag{12}$$

where

$$\left.\begin{array}{l} m_1 \\ m_2 \end{array}\right\} = -\frac{1}{2b} \pm \sqrt{\left(\frac{1}{4b^2} - \frac{\sigma^2}{c^2}\right)}. \tag{13}$$

Adjusting the constants, we have

$$\left.\begin{array}{ll} \text{for} \quad x < ct, & y = A\left(e^{-m_1(ct-x)} - e^{-m_2(ct-x)}\right) \\ \text{and} \quad \text{for} \quad x > ct, & y = 0 \end{array}\right\}. \tag{14}$$

Thursday, June 14th, 1900.

Lord KELVIN, G.C.V.O., F.R.S., President, in the Chair.

Twenty-one ordinary members, three honorary members, and a visitor present.

At the request of the President communications were made by Prof. Klein ("On the Continuation of the Edition of Gauss's Collected Works"); by Prof. Darboux ("Sur différents Problèmes relatifs aux

Transformations de l'Espace et aux Déformations finis de la Matière et sur leur rapports avec la Théorie des Systèmes triples orthogonaux); and by Prof. Poincaré ("Sur quelques Théorèmes relatifs à l'Analysis Situs et sur les propriétés des Polyèdres dans l'Espace à plus de trois dimensions"). Professors Darboux and Poincaré were admitted by acclamation, and thanks were voted to all three gentlemen for their communications.

Thanks were also voted to Prof. Stringham, of California, for his remarks on "A Proof by non-Euclidean Geometry of the Focus and Directrix Property of Plane Sections of a Cone."

Prof. Elliott communicated some notes on "Concomitants of Binary Quantics."

Lord Kelvin communicated the following papers by reading out their titles:—

Some Multiform Solutions of the Partial Differential Equations of Physical Mathematics and their Applications, Part ii., by H. S. Carslaw.

Some Quadrature Formulæ, by W. F. Sheppard.

Extensions of the Riemann-Roch Theorem in Plane Geometry, by Dr. F. S. Macaulay.

On the Invariants of a certain Differential Expression connected with the Theory of Geodesics, by J. E. Campbell.

On the Transitive Groups of degree n and class $n-1$, by Prof. W. Burnside.

The Invariant Syzygies of Lowest Order for any Number of Quartics, by A. Young.

Canonical Reduction of Bilinear Forms, by T. J. I'A. Bromwich.

The Energy Function of a Continuous Medium, by H. M. Macdonald.

Note on the Representation of a Circle by a Linear Equation, by J. Griffiths.

The Stress in an Æolotropic Elastic Solid with an Infinite Plane Boundary, by J. H. Michell.

The following presents were made to the Library:—

"Educational Times," June, 1900.

"Indian Engineering," Vol. xxvii., Nos. 16-20, April 21-May 19, 1900.

"Mittheilungen der Mathematischen Gesellschaft," Bd. iii., Heft 10; Hamburg, 1900.

"Proceedings of the American Philosophical Society," Vol. xxxviii., No. 160, December, 1897; Philadelphia.

"Mathematical Gazette," Vol. I., No. 21; May, 1900.

"Periodico di Matematica," Serie 2, Vol. II., Fasc. 6; Livorno, 1900.

"Transactions of the American Mathematical Society," Vol. I., No. 2; New York, 1900.

The following exchanges were received:—

"Proceedings of the Royal Society," Vol. LXVI., No. 429; 1900.

"Rendiconti del Circolo Matematico di Palermo," Tomo XIV., Fasc. 3, 4; 1900.

"Bulletin of the American Mathematical Society," Series 2, Vol. VI., No. 8, May, 1900; New York.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 3, Vol. VI., Fasc. 3, 4; Napoli, 1900.

"Journal für die reine und angewandte Mathematik," Band CXXII., Hefte 1, 2; Berlin, 1900.

"Archives Néerlandaises," Serie 2, Tome III., Liv. 5; La Haye, 1900.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. IX., Fasc. 8, 9, 10; Roma, 1900.

"Nyt Tidsskrift for Matematik," B. Aargang II., Nr. 2; Copenhagen, 1900.

"Proceedings of the Physical Society," Vol. XVII., Pt. 1; April, 1900.

"Sitzungsberichte der Königl. Preuss. Akademie der Wissenschaften zu Berlin," 1–22; 1900.

"Proceedings of the Cambridge Philosophical Society," Vol. X., Pt. 5 (Lent Term); 1900.

"Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen," Heft 3, Math.-Physikalische Klasse; 1899.

Notes on Concomitants of Binary Quantics. By E. B. Elliott.

Read and received June 14th, 1900. Received, in recast form, July 30th, 1900.

1. Too little attention has, perhaps, been devoted by expositors of the English method to any concomitants of binary quantics except covariants and invariants; the fact that the connexion between cogrediency and contragrediency in the case of pairs of variables is of a very simple nature, and that, for instance, any homogeneous contravariant is made a covariant by a mere substitution for the variables, having been regarded as making the consideration of other concomitants than covariants a matter of secondary interest and importance. Thus, while in respect to a *ternary* quantic it has been made a fundamental