ART. XLVI.—*Applications of the Lorentz-FitzGerald Hypothesis to Dynamical and Gravitational Problems*; by H. A. BUMSTEAD.

THERE is at the present time a general consensus of opinion, among those best qualified to judge, that the fundamental facts of optics and electro-dynamics require us to assume that the ether does not partake (to any sensible extent) in the motion of material bodies which pass through it. The aberration of light is perhaps the most conspicuous of those phenomena which it has hitherto been found impossible to account for on any other hypothesis without becoming involved in serious difficulties.* All the phenomena in which there is relative motion of the source of light with respect to the observer, or of a material medium (through which light is propagated) with respect to source and observer, appear to require the above assumption, even those which at first seemed to lead to a conclusion somewhat different in form. Thus the experiment of Fizeau, in which he compared the velocities of light when going with, and against, a stream of water; was interpreted by Fresnel as indicating a certain entrainment of the ether. This interpretation was based on Fresnel's theory of refraction, which assumed that the ethereal density was increased in material media; and it is only the excess ether which must be carried by the matter. On the electron theory, however (and indeed on any resonance theory of dispersion and refraction) there is no excess density of the ether in ponderable bodies; and it is not difficult to see that Fizeau's experiment requires a stationary ether.† A result which leads to the same view has been obtained in electro-dynamics by H. A. Wilson‡ in measuring the electric force produced by moving an insulator in a magnetic field. All such experiments upon the effects of relative motion, so far as I know, give positive results which may be predicted from the hypothesis of a fixed ether, and the magnitude of the effects observed is in general of the same order as the fraction $\frac{v}{V}$, where v is the relative velocity involved, and V the velocity of light.

The theory of a stagnant ether leads us, however, in a no less direct manner to expect certain modifications in the phenomena of light and electricity when there is no relative motion of

* Larmor, *Aether and Matter*, p. 37. Lorentz, *Amst. Proc.*, p. 443, 1899: *Abhandlungen*, I, p. 454.

† See Lorentz, *Versuch einer Theorie der Elektrischen und Optischen Erscheinungen, in Bewegten Körpern*, § 68.

‡ *Proc. Roy. Soc.*, lxxiii, p. 490, 1904.

material objects, but when all the apparatus concerned as well as the observer are carried through the ether with the velocity

v . The effects to be expected are of the order of $\left(\frac{v}{V}\right)^2$; this

is a very small fraction even when v is the velocity of the earth in its orbit, but the possible accuracy of certain optical experiments is so great that these effects could certainly be found if they existed without some compensating effect to mask them. As is well known, these effects have never been found; the first conclusively negative results were obtained in the celebrated experiments of Michelson and Morley,* and several other optical investigations have also failed to show the expected results. On the electrical side the problem has been attacked by Trouton and Noble,† who hung up an electrical condenser by a torsion wire and looked for a torque which, on the theory of a stagnant ether, ought to exist when the condenser is carried along by the earth. Although the sensitiveness of their experimental arrangement was ample for the observation of the expected second order effect, their result was also negative.

The most obvious interpretation of these results is that the ether near the earth has the same velocity as the earth; but, as has been stated, it appears to be impossible to reconcile this view with the great mass of optical and electro-dynamic evidence. The only satisfactory way out of this difficulty which has hitherto been suggested is a hypothesis put forward in 1892 by Lorentz,‡ and which had been independently suggested but not published by FitzGerald. According to this hypothesis, when any material body moves relatively to the ether its linear dimensions parallel to the direction of motion

are contracted in the ratio of $\sqrt{1 - \frac{v^2}{V^2}}$ to 1, while the dimen-

sions perpendicular to the direction of motion remain unchanged. If this contraction takes place in the interferometer of Michelson and Morley and in the condenser of Trouton and Noble, their negative results are entirely explained on the theory of a stationary ether.§ As Lorentz points out, this contraction will be very small in any motions of material bodies which we can observe; for example the diameter of the earth in the direction of its orbital path will be diminished by only 6.5^{cm} by its motion. It would moreover be impossible to detect the shrinkage, however great it might be, by ordinary

* This Journal, xxxiv, p. 333, 1887.

† Phil. Trans. R. S. (A), ccii. p. 165, 1903.

‡ Versl. Akad. Wet. Amsterdam. 1892-3.

§ See Lorentz, *Versuch einer Theorie*, etc., § 89. Amsterdam Proceedings, 1903-4, p. 809, reprinted in *Ions, Electrons, Corpuscles*, vol. i, p. 477. See also Larmor in FitzGerald's *Collected Papers*, p. 566.

measurements, since the standards of length must shrink in the same ratio as the bodies to be measured.

It would be quite misleading, however, to leave the impression that this hypothesis depends for its credibility altogether upon the fact that it enables us to evade a serious difficulty and that it cannot be disproved by ordinary means. The electrical forces between charged bodies (electrons) are modified by motion through the ether; and they are modified in precisely such a way that if a given system of charges were in equilibrium under these forces in a certain configuration when at rest, it would when in motion be in equilibrium in a configuration obtained from the first by the application of the Lorentz-FitzGerald shrinkage. Now it is a fundamental theorem in electrostatics, that a charged system cannot be in equilibrium under the electrical forces alone; in the case of a collection of electrons or atoms in equilibrium, the electrical forces must be balanced by other forces. If these inter-electronic forces are ethereal in origin and subject to the same laws as electro-magnetic forces, then the Lorentz-FitzGerald contraction would be expected *à priori*; and from this point of view the absence of the second order effects is evidence for the ethereal nature of inter-atomic and inter-molecular forces.

Forces of this character would suffice to account for the changed dimensions of moving bodies even if the electrons themselves were left unaltered by the motion. But, as Lorentz has pointed out,* we must also bring in dynamical considerations which show that for complete absence of second-order effects the electrons themselves must suffer the same contraction. The experiments of Lord Rayleigh† and of Brace‡ have shown that there is no double refraction due to the convection of transparent bodies by the earth. This implies that the periods of vibration of the electrons in the line of motion and perpendicular to it must be equal; and in order that this may be so, the longitudinal and the transverse masses of the electron must be altered by the motion in the same manner as the forces in these directions. An electron which does not change its shape (such as the rigid spherical electron of Abraham) will not have this property; nor will an electron which alters its form in any other manner than that described above for material bodies (such as the constant-volume electron of Bucherer). The electron proposed by Lorentz obviates these difficulties. If we assume that it is, when at rest, a sphere of radius, a , it must when in motion with velocity v , become an ellipsoid of revolution with its shorter axis in the direction of the motion and equal to $a \sqrt{1 - \frac{v^2}{V^2}}$, the dimen-

* Ions, Electrons, Corpuscles, vol. i, p. 477.

† Phil. Mag., vol. iv. p. 678, 1902.

‡ Phil. Mag., vol. vii. p. 317, 1904.

sions perpendicular to the motion remaining the same. If m_1 and m_2 are its longitudinal and transverse masses, and m_0 the mass for infinitesimal velocities, we shall have

$$m_1 = m_0 \frac{1}{(1-\beta^2)^{\frac{3}{2}}}$$

$$m_2 = m_0 \frac{1}{\sqrt{1-\beta^2}}$$

where, for brevity, β has been put for $\frac{v}{V}$. With this electron Lorentz has shown that no optical or electrical effects of motion through the ether can be detected.

The subject has been approached from a different standpoint, and treated in a very interesting and instructive manner by Einstein.* His fundamental postulate amounts to a denial that it is possible to observe any effects of uniform convection through the ether in which all the bodies concerned (including the observer) take part. This he calls the Principle of Relativity; the significance of the name is that only relative motion of one portion of matter with respect to another, or of one electrical charge with respect to another, can produce any observable effect; uniform motion, relative to the ether alone, becomes as impotent, if not as meaningless, as absolute motion.

Einstein considers two sets of coördinate axes, one at rest in the ether (x, y, z), while the other moves with the constant velocity v in the x direction (ξ, η, ζ). He defines carefully the meaning of "time" (t in the fixed system, τ in the moving system) by means of clocks distributed at various points, some at rest with the fixed axes, and some moving with the moving axes. The clocks are supposed to be synchronized by light signals. By kinematic considerations he shows that, in order for the principle of relativity to hold, we must have,

$$\xi = \frac{1}{\sqrt{1-\beta^2}} (x - vt)$$

$$\eta = y$$

$$\zeta = z$$

$$\tau = \frac{1}{\sqrt{1-\beta^2}} (t - \frac{v}{V^2} x)$$

where, as before, $\beta = \frac{v}{V}$.

* Ann. d. Phys., xvii, p. 891, 1905.

The distances x, y, z are measured by standards at rest, ξ, η, ζ , by standards in motion. The distance between two points (say on the x -axis) when measured by the first is $x_2 - x_1$; when measured by the second, it is $\xi_2 - \xi_1 = \frac{x_2 - x_1}{\sqrt{1 - \beta^2}}$. The length of the moving standards, when parallel to the axis of x are thus $\sqrt{1 - \beta^2}$ times the fixed standards; when perpendicular to x , they have the same length as the fixed standards. In order therefore that Einstein's principle should hold, it is necessary that all moving objects should suffer the Lorentz-FitzGerald contraction.

It is easy to compare the rates of the fixed and moving clocks by considering two events whose difference in time as measured by the fixed clocks is $t_2 - t_1$; as measured by a moving clock whose coördinate is ξ' , let the interval be $\tau_2 - \tau_1$. Then

$$\tau_2 - \tau_1 = \frac{1}{\sqrt{1 - \beta^2}} \left[t_2 - t_1 - \frac{v}{V_2} (x_2 - x_1) \right]$$

$$\text{But } x_2 = \sqrt{1 - \beta^2} \xi' - vt_2 \text{ and } x_1 = \sqrt{1 - \beta^2} \xi' - vt_1$$

$$\text{So that } \tau_2 - \tau_1 = \sqrt{1 - \beta^2} (t_2 - t_1).$$

Thus the moving clocks run slower than the fixed ones; if a clock at rest beats seconds, it must, when in motion, have a period of $\frac{1}{\sqrt{1 - \beta^2}}$ seconds.*

It is possible that the principle of relativity may come to be regarded as one of the fundamental empirical laws of Physics, occupying a position analogous to that of the Second Law of Thermodynamics. It rests on a similar basis, in that no deviations from it have been observed. Indeed the analogy may be made more complete by showing that the denial of the principle leads to a third kind of perpetual motion, by which the kinetic energy of any body might be exhausted and the body be brought to rest with reference to the ether.† There is however an enormous difference in the breadth of the evidence on which the two principles rest. Violations of the principle of relativity lead only to minute effects which must be sought for in difficult and recondite experiments. The fact remains however that, so far as our knowledge extends, the principle holds; the most reasonable course in regard to it, and that which promises to be most fertile in results, is to accept it provisionally and to develop its conse-

* This relation between the time in fixed and moving systems was also taken into account by Lorentz, by means of a variable which he calls local time. Versuch einer Theorie, § 31.

† Larmor in FitzGerald's Papers, p. 566.

quences. This will doubtless lead to further experimental tests; and even apart from direct tests, one may regard the evidence for the principle as being strengthened if it introduces simplification and harmony into the theory of phenomena which are apparently remote from those that led originally to its adoption.

As the dimensions of all bodies are altered by motion through the ether, it is plain that such motion must be taken into account in the exact theory of even purely dynamical phenomena. As such applications are not very familiar and present some points of interest, it seems not altogether superfluous to consider a few very simple dynamical cases from this point of view.

The Torsion Pendulum.

Suppose a bar of length L (when at rest) hung up by a torsion wire in the ordinary way. Let the apparatus be car-

FIG. 1.

ried by the earth through the ether with the velocity v in a direction perpendicular to the wire; and let us consider the period of the pendulum when the bar is clamped to the wire in two different positions: (1) with its length perpendicular to the earth's motion, and (2) parallel to the direction of motion. By the principle of relativity the two periods must be equal. As the length of the bar in the first position is L and in the second position $\sqrt{1-\beta^2} L$, it appears at first sight that the mass of every particle of the bar should be greater in position (2) when it is moving perpendicularly to the earth's motion than in (1) when it is moving parallel to it. This would make the

transverse mass greater than the longitudinal, whereas the opposite is the case with the apparent mass due to electrical charges. A closer consideration however shows that this is an error arising from the application of the ideas of rigid dynamics to a body which is changing its shape.

The path of any particle of the bar, if measured by a scale carried along with the earth, will appear to be the circle $AB'C$; if measured with reference to a scale at rest however it will be the ellipse ABC in which $OB = \sqrt{1-\beta^2} OA$. For brevity we shall refer to these as the "apparent" and the "true" paths. In case (1), let P_1 be the true position of the particle, P_1' its apparent position; let $OM_1 = x_1$; $M_1P_1 = y_1$; $\angle AOP_1 = \theta_1$; $\angle AOP_1' = \theta_1'$. In case (2), let $OM_2 = x_2$; $M_2P_2 = y_2$; $\angle BOP_2 = \theta_2$; $\angle BOP_2' = \theta_2'$. The potential energy of the twisted wire in either case depends on the *apparent* angle θ_1' or θ_2' . This is seen if we consider two pointers attached to the wire, one along OA when the wire is untwisted and the other along OB ; if the wire is now given any twist the two apparent angles θ_1' and θ_2' will be the same, but the real angles θ_1 and θ_2 will be different as well as the two elliptical arcs traced out by the ends of the pointers. As the apparent motion is isochronous we may put the potential energy equal to $\frac{1}{2} k \theta'^2$.

In position (1) we have

$$y_1 = x_1 \sqrt{1-\beta^2} \tan \theta',$$

For small oscillations, $x_1 = a$; $\tan \theta'_1 = \theta'_1$ and

$$y_1 = a \sqrt{1-\beta^2} \theta'_1$$

Thus the potential energy is $\frac{1}{2} \frac{k}{a^2 (1-\beta^2)} y_1^2$; the equation of motion of the particle becomes

$$m_1 \ddot{y}_1 = - \frac{k}{a^2 (1-\beta^2)} y_1$$

and the period of oscillation

$$T_1 = 2\pi \sqrt{\frac{m_1 (1-\beta^2) a^2}{k}}$$

In case (2)

$x_2 = a \sin \theta'_2 = a \theta'_2$ for small oscillations. The potential

energy is thus $\frac{1}{2} \frac{k}{a^2} x_2^2$, and the period

$$T_2 = 2\pi \sqrt{\frac{m_2 a^2}{k}}$$

In order for these periods to be equal we must have

$$m_2 = (1 - \beta^2) m_1$$

which is the same relation as that between the longitudinal and transverse masses of Lorentz's electron. That the variation with the velocity of m_1 or m_2 for ordinary matter is also the same as for Lorentz's electron may be shown in many ways; the following simple example will suffice for the purpose.

Consider an elastic rod with its length perpendicular to the motion of the earth and making longitudinal vibrations. If its period of vibration is T we shall have

$$T \propto \sqrt{\frac{m_2}{\kappa}}$$

where m_2 is the transverse mass of any particle and κ is the coefficient of stretching of the rod. We must also have, by Einstein's transformation,

$$T = \frac{T_0}{\sqrt{1 - \beta^2}}$$

where T_0 is the period of the rod when at rest.*

The constant κ depends on the intermolecular forces in the direction of the length of the rod, that is perpendicular to the earth's motion; and these must vary with the velocity in the same manner as electrical forces. If we have two point charges moving through the ether in a direction perpendicular to the line joining them, the force between them is

$$E = E_0 \sqrt{1 - \beta^2}$$

where E_0 is the force when they are at rest.† Thus we have

$$\kappa = \kappa_0 \sqrt{1 - \beta^2}$$

and

$$\sqrt{\frac{m_2}{\kappa_0 \sqrt{1 - \beta^2}}} = \sqrt{\frac{m_0}{\kappa_0} \frac{1}{\sqrt{1 - \beta^2}}}$$

whence

$$m_2 = m_0 \frac{1}{\sqrt{1 - \beta^2}}$$

It follows therefore from our hypothesis not only that all mass is electromagnetic but also that it varies with the speed in the specific manner of Lorentz's electron.

* If this relation did not hold for any time-keeper, the velocity of light measured in a moving system would be different from that measured in a system at rest, and thus the principle of relativity would be violated.

† See below, p. 503.

The Gravitational Pendulum.

As a further example, consider a simple pendulum at a point on the earth's surface 90° from the pole of its motion, so that the string is perpendicular to the direction of motion. When it vibrates in a plane at right angles to the motion the path of the bob is a circular arc and the period is

$$T = 2\pi \sqrt{\frac{m_2 L}{G}}$$

where G is the force with which the earth attracts the bob. When it vibrates in the plane of motion its path is the arc of an ellipse whose axes are L and $L \sqrt{1 - \beta^2}$; for the same vertical height (that is for the same potential energy), the infinitesimal arc described will be in this case less than in the other in the ratio of $\sqrt{1 - \beta^2}$ to unity. So that the period is

$$T = 2\pi \sqrt{\frac{m_1 (1 - \beta^2) L}{G}}$$

giving the same ratio of masses as before.

Comparing, say, the first of these with the period which the pendulum would have if the earth were at rest, we have

$$\sqrt{1 - \beta^2} \sqrt{\frac{m_2 L}{G}} = \sqrt{\frac{m_0 L}{G_0}}$$

and since

$$m_2 = m_0 \frac{1}{\sqrt{1 - \beta^2}},$$

$$G = \sqrt{1 - \beta^2} G_0$$

Thus the gravitational force between two bodies moving at right angles to the line joining them is the same function of the velocity as the electric force between two moving charges in a corresponding position.*

If we imagine the pendulum suspended at the place on the earth which is foremost or rearmost in its motion, the length of the string will be $L \sqrt{1 - \beta^2}$ and the period

$$T = 2\pi \sqrt{\frac{m_2 L \sqrt{1 - \beta^2}}{G'}}$$

whence

$$G_1 = (1 - \beta^2) G_0$$

which again corresponds to the electrical case when the line joining the charges is parallel to the motion.†

* See p. 503.

† See below p. 503.

It is scarcely necessary to point out that such problems as we have been considering do not lead to any practicable experimental tests. In order to detect deviations, it would be necessary to measure the periods in question with an accuracy such that the errors should be less than 10^{-6} , which is quite out of the question at present. This does not however affect the legitimacy of the use of such methods in following out the consequences of the principle; just as the impossibility of actually constructing a reversible engine does not invalidate that method of applying the second law of thermodynamics. More general methods might be used; but there is some advantage, especially in a comparatively new subject, in the simplicity and concreteness of ideas derived from the consideration of special problems.

Applications to Gravitation.

A promising direction in which to look for possible tests of the hypothesis is among the consequences of the deduction that gravitational forces must vary with motion through the ether in the same manner as electrical forces. If we have a

FIG. 2.

point charge e moving along the axis of x , with the uniform velocity v , the electric intensity at a point P whose coördinates are r, θ is

$$\mathbf{E} = V^2 \frac{e (1 - \beta^2)}{r^2 (1 - \beta^2 \sin^2 \theta)^{\frac{3}{2}}} \mathbf{r}_1 \quad (1)$$

where $\mathbf{r}_1$ is a unit vector in the direction of r and $\beta = \frac{v}{V}$.*

The magnetic force is perpendicular to the plane of $\mathbf{r}$ and $\mathbf{v}$ and has the magnitude

$$H = \frac{v}{V^2} E \sin \theta = \frac{ve (1 - \beta^2) \sin \theta}{r^2 (1 - \beta^2 \sin^2 \theta)^{\frac{3}{2}}} \quad (2)$$

* Electromagnetic units are used.

These are also the values of the electric and magnetic forces produced at points outside, by Lorentz's electron, or by any charged system in which, when at rest, the charge is distributed with spherical symmetry and which, when in motion, suffers the Lorentz-FitzGerald contraction. $\mathbf{E}$ is the force exerted by the moving charge e , upon a unit charge *which is at rest* at the point P. If the unit charge at P is in motion with the velocity $\mathbf{u}$, then the force exerted upon it, which we may call $(\mathbf{E})$, is

$$(\mathbf{E}) = \mathbf{E} + \mathbf{u} \times \mathbf{H} \quad (3)$$

where $\mathbf{u} \times \mathbf{H}$ represents the vector product. Thus the force on a charge at rest at the point, P, is in the direction of $\mathbf{r}$, but this is not true in general if it is in motion.

Let us consider first the special case when the two charges have the same velocity, $\mathbf{u} = \mathbf{v}$. Let the two components of $\mathbf{E}$ parallel and perpendicular to $\mathbf{v}$ be E_1 and E_2 , respectively. The force $\mathbf{v} \times \mathbf{H}$ will be parallel to E_2 and in the opposite direction and its magnitude will be vH . So that the corresponding components of $(\mathbf{E})$ are

$$(E)_1 = E \cos \theta \quad (4)$$

and

$$(E)_2 = E \sin \theta - vH$$

or since

$$H = \frac{v}{V^2} E \sin \theta,$$

$$(E)_2 = E \sin \theta (1 - \beta^2) \quad (5)$$

These are the components of the actual force on the moving charge at P; if it is of opposite sign to the charge e , the force will have the direction given in fig. 2.

When $\theta = 0$, $(E)_2 = 0$ and

$$(E)_1 = E = V^2 \frac{e (1 - \beta^2)}{r^2}$$

which is $(1 - \beta^2)$ times the value of the electrostatic force when the charges are at rest: this corresponds to the gravitational case of p. 501 when the force was in the direction of motion.

When $\theta = \frac{\pi}{2}$, $(E)_1 = 0$, and

$$(E)_2 = E (1 - \beta^2) = V^2 \frac{e}{r^2} \sqrt{1 - \beta^2}$$

which also agrees with the corresponding case for gravitation.

If we apply this electromagnetic law of force to gravitation we are at first sight confronted with the difficulty that the magnitude of the force varies not only with the distance but

also with the angle θ ; and there is also an aberration in the direction of the force. It is important, however, to notice that the variation and aberration of the force is of the second

order in the small fraction $\frac{v}{V}$,* instead of the first order as has often been assumed in discussing the possible speed of propagation of gravitational force.†

In the special case before us, the principle of relativity relieves us entirely from the difficulty of even these small variations from the Newtonian law. This is apparent from the general statement of the principle; but it is of some interest to see how the matter works out in detail. What is subject to observation is not the force but the acceleration; if we let f_1 and f_2 be the components of the acceleration parallel and perpendicular to the common motion of the two bodies, we shall have

$$\begin{aligned} f_1 &= \frac{(E)_1}{m_1} = \frac{(1 - \beta^2)^{\frac{3}{2}}}{m_0} E \cos \theta \\ f_2 &= \frac{(E)_2}{m_2} = \frac{(1 - \beta^2)^{\frac{1}{2}}}{m_0} E \sin \theta (1 - \beta^2) \end{aligned} \quad (6)$$

and the resultant of these is along r , so that there is no aberration of the acceleration. With regard to the variation of the acceleration with the distance, it must be remembered that, to an observer moving with the system, apparent distances in the direction of motion, (x), are greater than their true values in

the ratio $\frac{1}{\sqrt{1 - \beta^2}}$. Thus if the "true" coördinates of P (fig. 2) are x, y , the "apparent" coördinates will be x', y , where $x' = \frac{x}{\sqrt{1 - \beta^2}}$. The "true" distance, r , will be the radius vector of an ellipse whose major axis is the "apparent" distance, r' , and whose minor axis is $\sqrt{1 - \beta^2} r'$; the polar equation of the ellipse (θ being measured from the minor axis) gives

$$\begin{aligned} r^2 &= \frac{r'^2 (1 - \beta^2)}{1 - \beta^2 \sin^2 \theta}; \\ x'^2 + y^2 &= r'^2. \end{aligned} \quad (7)$$

* This was pointed out by Heaviside, who was the first, so far as I know, to apply the modern electrodynamics to gravitation. *Electrician*, 1893, July 14 and Aug. 4. *Electromagnetic Theory*, Vol. I, Appendix B.

† The general reason for this has been put very clearly by Lorentz, *Amsterdam Proceedings*, II, p. 573, 1900.

We must also observe that the "apparent" acceleration (f'_1, f'_2) differs from the "true" acceleration not only on account of the different scale of length in the x direction, but also because of the larger unit of time given by a moving clock. Thus

$$f'_1 = \frac{1}{1 - \beta^2} f_1 \frac{x'}{x}$$

$$f'_2 = \frac{1}{1 - \beta^2} f_2$$
(8)

In equations (6) put $\frac{x}{r}$ for $\cos \theta$, and $\frac{y}{r}$ for $\sin \theta$; put for E its value from (1) and for r its value from (7); substituting in (8) the values thus obtained for f_1 and f_2 , we obtain.

$$f'_1 = V^2 \frac{e}{r'^2} x'$$

$$f'_2 = V^2 \frac{e}{r'^2} y$$

The resultant "apparent" acceleration will thus be

$$\mathbf{f}' = V^2 \frac{e}{r'^2} \mathbf{r}'_1$$

When $\mathbf{r}'_1$ is an "apparent" unit vector in the direction r' .

When there is *relative* motion of the planet with respect to the sun, however, the compensation is not perfect. In fact, deviations from the Newtonian law may be introduced which would not exist if the longitudinal and transverse masses were equal. This may be most easily seen when the attracting body is at rest in the ether with a planet moving about it; in this case the force given by electrical theory is the ordinary electrostatic force; it will be in the direction of the radius vector and will vary according to the inverse square of the distance. But the resultant acceleration will not be along the radius vector if the longitudinal and transverse masses are different. Let ϕ be the angle between the radius vector and the tangent to the path; and let the forces and accelerations, tangential and normal to the path, be respectively F_t, F_n, f_t, f_n . Then

$$F_t = V^2 \frac{e}{r^2} \cos \phi; F_n = V^2 \frac{e}{r^2} \sin \phi$$

$$f_t = \frac{(1 - \beta^2)^{\frac{3}{2}}}{m_0} F_t = \frac{V^2 e}{m_0 r^2} (1 - \beta^2)^{\frac{3}{2}} \cos \phi$$

$$f_n = \frac{(1 - \beta^2)^{\frac{1}{2}}}{m_0} F_n = \frac{V^2 e}{m_0 r^2} (1 - \beta^2)^{\frac{1}{2}} \sin \phi$$

The acceleration along the radius vector is

$$f_n \sin \phi + f_t \cos \phi = \frac{V^2 e}{m_0 r^2} (1 - \beta^2)^{\frac{1}{2}} (1 - \beta^2 \cos^2 \phi)$$

The acceleration perpendicular to the radius vector is

$$f_n \cos \phi - f_t \sin \phi = \frac{V^2 e}{m_0 r^2} (1 - \beta^2)^{\frac{1}{2}} \sin \phi \cos \phi \cdot \beta^2$$

If we take the earth as a numerical example, this perpendicular acceleration is very small. Its maximum value will occur when the earth is at the extremities of the minor axis of its orbit; at this point

$$\cos \phi = \epsilon; \quad \sin \phi = \sqrt{1 - \epsilon^2};$$

where ϵ is the eccentricity of the orbit. Taking $\epsilon = 1.7 \times 10^{-2}$ and $\beta^2 = 10^{-3}$ we find

$$\text{Acceleration along } r = \frac{V^2 e}{m_0 r^2} (1 - \beta^2)^{\frac{1}{2}} [1 - 2.9 \times 10^{-12}]$$

$$\text{Acceleration perpendicular to } r = \frac{V^2 e}{m_0 r^2} (1 - \beta^2)^{\frac{1}{2}} [1.7 \times 10^{-10}]$$

I am not sufficiently familiar with the details of astronomical calculations to be able to say with entire confidence whether or not such an acceleration perpendicular to r could be detected. It seems, however, unlikely. The maximum effect is of the same order as would be produced by a perturbing body at a distance equal to that of the sun, and whose mass was only $\frac{1}{200,000}$ that of the earth. The perturbation, moreover, would be periodic, vanishing at perihelion and aphelion and accelerating the earth's motion in one-half the orbit, retarding it in the other half.

When the sun is also moving, the problem becomes more complicated. For the present purpose it will be sufficient to obtain the order of magnitude of the acceleration perpendicular to the radius vector. Let $\mathbf{v}$ be the velocity of the sun, and $\mathbf{u}$ that of the planet *relative to the sun*. Then the force on the planet is

$$(\mathbf{E}) = \mathbf{E} + (\mathbf{v} + \mathbf{u}) \times \mathbf{H}$$

where $\mathbf{E}$ is given by equation (1), in which β is now the ratio of the velocity of the sun to the velocity of light, and θ is the angle between the radius vector and the sun's path. The magnitude of $\mathbf{H}$ is given by equation (2). The force $\mathbf{E}$ is along the radius vector; the force $(\mathbf{v} + \mathbf{u}) \times \mathbf{H}$ is normal to the resultant path of the planet. Let ψ be the angle between r and the tangent to the resultant path of the planet, then

$$F_t = E \cos \psi$$

$$F_n = E \sin \psi + |(\mathbf{v} + \mathbf{u}) \times \mathbf{H}|$$

in which the term enclosed by vertical lines represents the magnitude only of the vector. $\mathbf{H}$ is perpendicular to the plane containing $\mathbf{r}$ and $\mathbf{v}$; let u_t be the component of $\mathbf{u}$ in this plane and let w be the resultant of u_t and v . Then

$$|(\mathbf{v} + \mathbf{u}) \times \mathbf{H}| = wH = \frac{wv}{V^2} E \sin \theta$$

and

$$F_n = E \left(\sin \psi + \frac{wv}{V^2} \sin \theta \right).$$

Dividing F_t and F_n by the longitudinal and transverse masses respectively, we obtain for the accelerations,

$$f_t = \frac{(1 - \beta^2)^{\frac{3}{2}}}{m_0} E \cos \psi$$

$$f_n = \frac{(1 - \beta^2)^{\frac{1}{2}}}{m_0} E \left(\sin \psi + \frac{wv}{V^2} \sin \theta \right)$$

The acceleration perpendicular to the radius vector is

$$f_n \cos \psi - f_t \sin \psi = \frac{(1 - \beta^2)^{\frac{1}{2}}}{m_0} E \left(\beta^2 \sin \psi \cos \psi + \frac{wv}{V^2} \sin \theta \cos \psi \right).$$

Recent estimates make the sun's velocity about 20 kilometers per second, so that $\beta^2 = 0.45 \times 10^{-8}$; its direction makes an angle with the plane of the earth's orbit of about 55° . When $\mathbf{r}$ is perpendicular to the plane containing $\mathbf{v}$ and the normal to the plane of the orbit, $\cos \psi$ is nearly zero; it must in fact be less than ϵ (the eccentricity of the orbit) even in the favorable case when the minor axis falls in this position; with the major axis in this position it will be zero. In this position, therefore, the acceleration perpendicular to the radius vector cannot be as much as twice that which was found for the sun at rest. When $\mathbf{r}$ is in the plane containing $\mathbf{v}$ and the normal to the plane of the orbit, $\theta = 55^\circ$, $\psi < 55^\circ$ and $w = v$. So that the acceleration perpendicular to the radius vector will be less than

$$\frac{(1 - \beta^2)^{\frac{1}{2}}}{m_0} E \beta^2 \sin 110^\circ$$

that is its ratio to the acceleration in the direction of the radius will be less than 1.4×10^{-9} .

In order to be quite certain that astronomical facts are not in conflict with the principle of relativity, it will doubtless be necessary to make detailed comparisons between observation

and calculation based upon this hypothesis. The small magnitude of the departures from the Newtonian law, of which more or less rough estimates have been given above, render it probable that there would be no serious lack of agreement. This probability is strengthened by a calculation published some years ago by Lorentz.* In this he found the secular variations of the elements of the orbit of Mercury due to the substitution of electro-dynamic forces for the strictly Newtonian force. The variations in the angular elements amounted to only a few seconds of arc in a century and the change in the eccentricity to 0.000005. He did not, it is true, take into account the effects of variable mass, which had not at that time become prominent even in electrical theory. The introduction of electromagnetic mass will, in general, tend to diminish the effects of the sun's motion and to exaggerate the effects of the motion of the earth relative to the sun. But from a comparison of the theoretical accelerations in the two cases, it does not appear that the variations could be increased enough to produce a sensible discrepancy.

[*Note added in Proof, Oct. 12.* Since the above was written, two papers have come to my knowledge which bear upon this question. A. Wilkens (Phys. Zeitschr., vii, p. 846, 1906) has introduced electromagnetic mass in the ordinary Newtonian equations and has calculated the resulting secular variations in the elements of Mercury, Venus, the Earth, Mars, and Encke's comet. In all cases the variations are within the limits of accuracy of the observations. F. Wacker (Ibid., p. 300) considers the case when both force and mass are electromagnetic and, upon applying his equations to Mercury, finds for the motion of its perihelion a value less than one-fifth of that which is at present unaccounted for. The changes in the scales of length and time which would be introduced by the principle of relativity could affect these results very little; so that it seems quite certain that our present observational knowledge of gravitation is not sufficiently exact either to exclude the general application of the principle or to supply evidence in its favor.]

* Amsterdam Proc. II, p. 571, 1900.