

On the Isoclinical Lines of a Differential Equation of the First Order. By J. H. MacLagan-Wedderburn.  
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A differential equation may be regarded from two points of view, one purely analytical, the other geometrical. From the analytical point of view, a differential equation of the first order is merely a functional relation between  $x$ ,  $y$ , and  $p$  (where  $p = dy/dx$ ), and the problem of solving the equation is to find a function of  $x$ , say  $f(x)$ , such that if  $f(x)$  and  $df(x)/dx$  are substituted for  $y$  and  $p$  in the equation, the result is an identity in  $x$ . In the geometrical method, on the other hand,  $x$  and  $y$  are treated as the co-ordinates of a point in a plane and  $p$  as a direction. The differential equation then attaches to every point in the plane a certain direction, which may be conveniently represented by an arrow drawn through the point. The problem of integration then resolves itself into finding a family of curves, such that, at every point  $(x', y')$ , the direction of the curve at that point is the direction obtained by substituting  $x'$  and  $y'$  in the differential equation and solving for  $p$ . These curves are called the integral curves of the equation. This method owes its development chiefly to Lie.

An instructive example of a differential equation from this point of view is furnished by a well-known experiment in magnetism. A magnet exerts on another magnet, placed in its neighbourhood, a force whose direction and magnitude depend, in a given medium, solely on the strength of the two magnets and on their relative position, and, if one of the magnets is very small, the force on it due to the other is merely directive. We have here, then, a physical representation of a differential equation. If now we cover a magnet with a sheet of paper and sprinkle iron filings on it, each filing becomes a magnet by induction, and therefore sets itself longitudinally in the direction of the force at the point where it falls, and, if the paper is gently tapped, the filings arrange them-

selves in curves, namely the lines of force. These lines of force are the integral curves of the differential equation.

A differential equation of the first order  $\phi(xyp)=0$  besides defining the integral family, also defines a family of curves obtained by regarding  $p$  as an arbitrary constant in  $\phi=0$ , and these curves have the property that all the integral curves, that intersect any particular curve of the second family, have the same direction (*i.e.*, the same  $p$ ) at the points of intersection. The latter family has been called the "Loci of Contacts of Parallel Tangents," by Hill (*Proc. Lond. Math. Soc.*, xix., 1888, p. 561), but, at the suggestion of Professor Chrystal, I propose to use in this paper the more convenient term "Isoclinal Family." This family gives a method of describing any integral curve; for if, beginning at any arbitrarily chosen point, we draw an infinitesimal line in the direction specified by one of the isoclinal curves passing through that point, we in general come to another isoclinal giving a new direction differing infinitesimally from the original direction, and so on. Of course, as the starting-point is arbitrary, the integral curve must be developed on both sides of it; also it must be noticed that the direction specified by the isoclinal is not in general the direction of the isoclinal itself. Now it is evident that there is in general one and only one isoclinal that passes through the point  $(x+dx, y+pdx)$  and is also contiguous to the isoclinal  $p$  through  $(x, y)$  and similarly for the point  $(x-dx, y-pdx)$ ; also in general the isoclinal  $(p+dp)$  lies wholly on one side of the isoclinal  $p$  in the neighbourhood of  $(x, y)$ , and at such a point the  $y$  of the integral curve and its first differential co-efficients are in general synectic functions of  $x$ . This is, however, not in general the case in the neighbourhood of the envelope. For, in general, one of a family of curves does not cross the envelope of the family in the neighbourhood of the point of contact. Thus the three contiguous curves  $p-dp$ ,  $p$  and  $p+dp$  all lie on the same side of the envelope, and all touch it. Moreover the direction  $p$  will not in general be that of the envelope, but will cross it. But as there is no contiguous isoclinal across the envelope to indicate a new direction, the integral curve cannot cross the envelope, and must therefore have either a cusp or a stop point. If the envelope is the curve indicated in figure 1 by E and the isoclinal  $p$  touches

it in  $P$ , and if  $PQ$  be an infinitesimal element of the integral curve at  $P$ , we can in general find two and only two isoclinals contiguous to  $p$  which pass through  $Q$ , and they in general touch the envelope on opposite sides of  $P$ . Therefore if we start from  $P$  to draw the integral curve, we have a choice of two directions at  $Q$ , each differing infinitesimally from  $p$ ; and these in general give two distinct branches of the integral curve as the sign of the variation of  $p$  is different in the two. The envelope is therefore a locus of cusps for the integral curves. Now Cayley has shown that if  $\phi(xyp) = 0$  is an integral algebraic function of  $p$ , the isoclinical family has in general an envelope which is given by the  $p$ -discriminant of  $\phi$ . The  $p$ -discriminant is therefore in general a locus of cusps on the integral family.

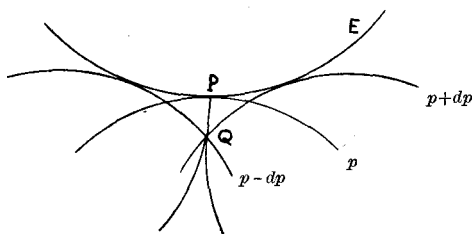


Fig. 1.

Conversely, if  $E$  is a locus of cusps for the integral family, it is part at least of the envelope of the isoclinical family; for two contiguous isoclinals intersect at  $Q$ .

If, however, the direction  $p$  is also the direction of the isoclinical at  $P$ , and therefore of the envelope, the direction  $p$  does not cross the envelope and there is no discontinuity in the variation of  $p$  for the integral curve. The direction of the isoclinical family is given by

$$\phi_x + \phi_y \frac{dy}{dx} = 0 \quad . \quad . \quad . \quad . \quad (1)$$

therefore the condition for the  $p$ -discriminant being an envelope locus for the integral family is

$$\phi_x + p\phi_y = 0 \quad . \quad . \quad . \quad . \quad (2)$$

and this condition is in general both necessary and sufficient.

Several other cases now present themselves. The  $p$ -discriminant

may be a locus of double points on the isoclinal family—in general a node locus. In this case the isoclinals, and therefore the integral curves, in general cross the  $p$ -discriminant. For, if  $P$  and  $Q$  have the same meaning as formerly, we find the same phenomena appearing on the integral curve as in the case of an envelope, with this difference, that we also have a point  $Q'$  on the other side of the  $p$ -discriminant from  $Q$  at which also two integral curves or two branches of the same integral curve diverge.

As the latter is a higher order of singularity it is less general. (It must be noticed that the two isoclinals contiguous to  $p$  which pass through  $Q'$  are not in general the same as the two which pass

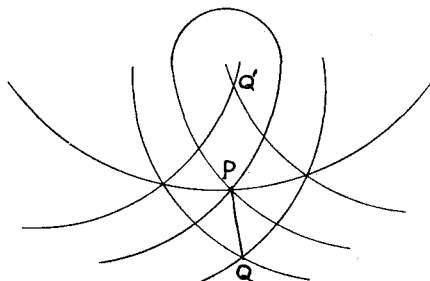


Fig. 2.

through  $Q$ .) The conditions for a node locus on the isoclinal family, and therefore for a tac-locus on the integral family, are

$$\phi_x = 0 \quad \phi_y = 0 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in addition  $\phi = 0$  and  $\phi_p = 0$ , and these are in general sufficient. Now, in general, if an isoclinal pass through a point  $P$ , there is one and only one curve contiguous to it which passes through a given point which is contiguous to  $P$ , but if  $P$  is a tac-point of the integral family, and  $Q$  is a point contiguous to  $P$  on both the integral curves that touch at  $P$ , since the rate of variation of  $p$  is in general different for the two integral curves, two contiguous isoclinals must pass through  $Q$ ; and similarly for  $Q'$  on the other side of  $P$  from  $Q$ , and therefore two isoclinals must pass through  $P$ ; and as the  $p$  of both integral curves is the same at  $P$ , these two isoclinals must be branches of the same curve, therefore the conditions (3) are also in general necessary.

The branches of the isoclinal curve at a node divide the plane

into two regions, one of which contains the  $p$ -discriminant and one which does not. There are three species of tac-loci according as the direction of the integral curve lies in the former or the latter region or along the isoclinal: (i) if it lies in the region not containing the  $p$ -discriminant the two integral curves have opposite curvature; (ii) if it lies in the region containing the  $p$ -discriminant the curvature is the same for both in direction but not in general in magnitude; (iii) if the direction of the integral curve lie along the isoclinal, there is, as will be shown

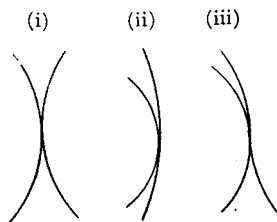


Fig. 3.

later, an inflexion on one of the branches of the integral curve. Figure 3 gives a geometrical representation of the three cases. If the double point is a point of the first order, the directions of the tangents to the isoclinal are given by the quadratic

$$\varpi^2\phi_{yy} + 2'\varpi\phi_{xy} + \phi_{xx} = 0 \quad . \quad . \quad . \quad (4)$$

therefore in any particular case it is easy to decide to which species the tac-locus belongs. (See examples (1), (2), (3) and (7).)

If, however, the roots of (4) are equal, *i.e.*, if

$$\phi_{xy}^2 - \phi_{xx}\phi_{yy} = 0 \quad . \quad . \quad . \quad (5)$$

the  $p$ -discriminant is a cusp locus for the isoclinal family.

Similar reasoning to the above shows that it is also a cusp locus for the integral family; and in every case, except when the direction of both families is the same, the curves contiguous to  $p$  passing through  $Q$  both lie on the same side of  $P$ , and therefore the curvature of both branches is in the same direction, *i.e.*, the cusp is a ramphoid cusp. (See fig. 4.)

If, however, at any point the direction of the integral curve is the same as that of the cusp locus, there is in general a tac-point on the integral family, the contact being of higher order than the first. (See examples (4) and (5).)

If the roots of (4) are imaginary, the  $p$ -discriminant is a locus of conjugate points for both families.

If at any point  $P$ , which is not on the envelope locus of the

isoclinical family, the direction of the integral curve is the same as that of the isoclinical through P, the contiguous points Q and Q' are in general both on the same side of the isoclinical, but on opposite sides of the point P; therefore the sign of the variation of  $p$  changes on passing through P, *i.e.*, there is in general an inflexion on the integral curve. The condition for this is

$$\phi_x + p\phi_y = 0.$$

This is equivalent to the usual condition for an inflexion; for

$$\phi_x dx + \phi_y dy + \phi_p dp = 0$$

but along an integral curve  $dy = p dx$ , hence

$$\frac{dp}{dx} = - \frac{\phi_x + p\phi_y}{\phi_p} = 0$$

if  $\phi_p \neq 0$ , which, along with  $\frac{d^2p}{dx^2} \neq 0$ , is the usual condition for an

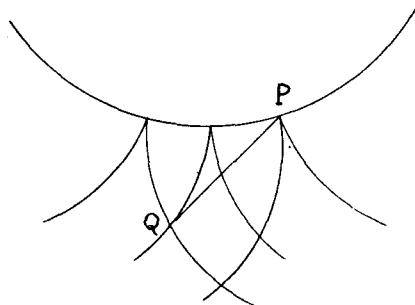


Fig. 4.

inflexion. Even if  $\phi_p = 0$  the geometrical reasoning shows there is still an inflexion, unless the  $p$ -discriminant is an envelope locus for the isoclinical family. (See example (3).)

This may obviously be generalised as follows: if the direction of the integral curve lie along the isoclinical at any point, and if the tangent at that point meet the isoclinical in  $n$  contiguous points, then it will also meet the integral curve in  $(n + 1)$  points. (See example (6).)

The following are a few examples in illustration of the above.

Example (1):

$$p^2 + (3x + 2y)p - \frac{7}{4}x^2 + 3xy + y^2 = 0.$$

The  $p$ -discriminant is easily found to be

$$x^2 = 0$$

and the directions of the isoclinal curve at the origin are given by

$$y^2 + 3xy - \frac{7}{4}x^2 = 0,$$

hence

$$y = -\frac{7}{2}x \text{ or } \frac{1}{2}x.$$

The direction of the integral curve at the same point is along the  $x$ -axis, *i.e.*, in the region not containing the  $p$ -discriminant, hence the integral curves should have opposite curvatures.

A first approximation at the origin gives

$$p^2 + 3px - \frac{7}{4}x^2 = 0,$$

hence

$$y = \frac{1}{4}(-3 \pm 4)x^2.$$

The two branches have opposite curvatures as predicted.

Example (2):

$$p^2 + 2(x+y)p + \frac{3}{4}x^2 + 2xy + y^2 = 0.$$

The direction of the integral curves at the origin is along the  $x$ -axis. The  $p$ -discriminant is  $x^2 = 0$ , and the directions of the isoclinals are given by

$$y = -\frac{3}{2}x \text{ or } -\frac{1}{2}x.$$

Hence the integral curves have a tac-point of the second species.

A first approximation gives

$$p^2 + 2px + \frac{3}{4}x^2,$$

hence

$$y = \frac{-2 \pm 1}{4}x^2.$$

Example (3):

$$p^2 + 2(x+y)p + 2xy + y^2 + x^3 = 0.$$

The  $p$ -discriminant is

$$x^2(x-1) = 0$$

where  $x^2$  corresponds to the tac-locus.

The directions of the isoclinals are given at the origin by

$$y(y+2x) = 0,$$

hence, as the direction of the integral curve is along the  $x$ -axis, there is an inflexion on one branch.

To find the integral curve we have

$$\begin{aligned} p &= -(x+y) \pm \sqrt{x^2(x-1)} \\ &= -x-y \pm (x-\frac{1}{2}x^2) \end{aligned}$$

to a first approximation. Hence to same order of approximation

$$y = -\frac{1}{6}x^3, \quad y = -x^2.$$

Example (4):

$$(y-p)^2 = x^3.$$

The  $p$ -discriminant is a locus of cusps for the isoclinical family, and is therefore a locus of ramphoid cusps for the integral family. To find the integral family we have

$$p-y = \pm x^{\frac{3}{2}}$$

hence

$$y = ae^x \pm e^x \int e^{-x} x^{\frac{3}{2}}$$

which to a first approximation is

$$y = a + ax + \frac{a}{2}x^2 \pm \frac{2}{5}x^{\frac{5}{2}},$$

a ramphoid cusp, except when  $a=0$ , i.e., when the direction of the integral curve is the same as that of the isoclinical.

Example (5):

$$p^2 - 2xp + x^2 - y^3 = 0.$$

The  $p$ -discriminant  $y=0$  is a locus of cusps for the isoclinical family and at the origin the direction of the integral curve lies along it, hence the origin is a tac-point. A first approximation gives

$$p = x \quad y = \frac{1}{2}x^2$$

putting

$$p = x + w \quad y = \frac{1}{2}x^2 + v;$$

and neglecting terms not required for the second approximation we get

$$w = \pm \frac{1}{2\sqrt{2}}x^3 \quad v = \pm \frac{1}{8\sqrt{2}}x^4$$

hence

$$y = \frac{1}{2}x^2 \pm \frac{1}{8\sqrt{2}}x^4.$$



Example (6):

$$y - p = x^n.$$

The  $x$ -axis has contact of  $(n-1)^{\text{th}}$  order, hence, as the direction of the integral curve at the origin is also along the  $x$ -axis, the tangent there has contact of the  $n^{\text{th}}$  order. Integrating we get

$$\begin{aligned} y &= -e^x \int x^n e^{-x} dx \\ &= -\frac{1}{n+1} x^{n+1} \end{aligned}$$

to a first approximation.

Example (7):

$$y^2 = (p-x)^2 + (p-x)^n.$$

The  $p$ -discriminant is  $y^2(y^2 - \frac{4}{27}) = 0$ .

The directions of the integral curves at the origin are

$$p=0 \quad \text{and} \quad p=-1.$$

The direction of the isoclinal curve corresponding to  $p=0$  is

$$y = \pm x;$$

the corresponding part of the  $p$ -discriminant is therefore  $y^2=0$ , and the integral family has a tac-point of the second species. The integral curves at the origin are

$$y = -x + \frac{1}{2}x^2 + \frac{1}{3}x^3, \quad y = \frac{1}{2}x^2 \pm \frac{1}{6}x^3.$$

It must be noticed that, although the direction of the integral curve lies along one branch of the isoclinal family, it is not the branch to which that direction belongs, and so there is no inflexion.

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