On Noether's fundamental theorem.

By

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In the Math. Annalen Bd. 6. Noether has stated and proved that if three algebraic plane curves F, Φ, Ψ be such that at every intersection $x = a, y = b$ of Φ and Ψ it be possible to find two infinite power series in $x - a, y - b$: A', B' so as to make $F = A'\Phi + B'\Psi$, up to any dimension in $x - a, y - b$, then there exist two algebraic curves K, M such that F is identically equal to $K\Phi + M\Psi$. In the Annalen Bd. 34, p. 447, Bertini has stated that such series A', B' can be found provided that for the intersection (say $x = 0 = y$) — where Φ has its lowest terms of order k and Ψ has its lowest terms of order l and β is a positive integer more precisely defined below — there exist two curves A, B (respectively of orders $l - 2 + \beta, k - 2 + \beta$) such that the lowest terms in $F_1 = F - (A\Phi + B\Psi)$ are of order $k + l - 1 + \beta$. In the Annalen Bd. 40, p. 143, at the conclusion of a new proof of his theorem, Noether has restated this result — referring to Bertini's note for the proof. The following proof of Bertini's result differs from Bertini's and is a development of Brill's note in the Annalen Bd. 39.

The proof of Lemma III is added in § 2 (an extension of Voss' proof of the 'simple case', Math. Annalen Bd. 27, p. 532).

Lemma 1. At the intersection $x = 0 = y$ of the curves Φ, Ψ , where Φ has its lowest terms of order k , the term y^k occurring and Ψ has its lowest terms of order l , the term y^l occurring, it is possible to find two infinite power series κ, λ , convergent in sufficiently near neighbourhood of $x = 0 = y$ and not vanishing for $x = 0 = y$, such that $\kappa\Phi (= \varphi$, say) is equal to $y^k + y^{k-1}x \cdot a_1 + y^{k-2}x^2 \cdot a_2 + \dots + x^k \cdot a_k$

$$= (1, xa_1, x^2a_2, \dots, x^ka_k)(y, 1)^k,$$

 $\lambda\Psi (= \psi$, say) is equal to $y^l + y^{l-1}x \cdot b_1 + y^{l-2}x^2 \cdot b_2 + \dots + x^l \cdot b_l$

$$= (1, xb_1, x^2b_2, \dots, x^lb_l)(y, 1)^l,$$

where a_i, b_j denote ordinary (convergent) power series in x , not becoming infinite for $x = 0$ (Weierstrass, Abh. a. d. Functionentheorie p. 112).

Lemma II. If (for $r > p - 1$)

be divided by

$$y^r + e_1y^{r-1} + e_2y^{r-2} + \dots + e_r$$

$$y^p + g_1y^{p-1} + g_2y^{p-2} + \dots + g_p,$$

where e_i, g_j are ordinary power series in positive powers of x which vanish to the respective orders i, j , then the remainder will be of the form

$$x^{r-p+1}(c_0 y^{p-1} + c_2 y^{p-2} x + \dots + c_{p-1} x^{p-1}),$$

where $c_0, \dots, c_{p-1}$ are power series in x not becoming infinite for $x=0$. This is readily proved by 'induction'.

Lemma III. If

$$f = y^{k+i-1} c_0 + y^{k+i-2} x c_1 + \dots + x^{k+i-1} c_{k+i-1},$$

where $c_0, c_1, \dots$ are ordinary power series finite for $x=0$, then the Sylvester y -resultant of φ, ψ [defined here as that determinant of $k+l$ rows whose first two rows are $1, xa_1, x^2 a_2, \dots, x^k a_k, 0, \dots$ and $0, 1, xa_1, x^2 a_2, x^3 a_3, \dots$ respectively] divides identically by x^{k+i} , and denoting it by $x^{k+i} \Delta$ we have $\Delta f = P\varphi + Q\psi$ where P, Q are polynomials in y of orders $l-1, k-1$ respectively, the coefficients being functions of x not becoming infinite for $x=0$ and Δ is an integral expression in the functions $a_1, a_2, \dots, b_1, b_2, \dots$.

§ 1.

Supposing then that $\Delta = x^\beta \Delta_1$, where Δ_1 does not become infinite for $x=0$ (so that the intersection of Φ, Ψ at $x=0=y$ counts for $kl+\beta$ intersections) and supposing that in the curve

$$F_1 = F - (A\Phi + B\Psi)$$

the term $y^{k+i-1+\beta}$ actually occurs in the lowest terms (which as explained, are now supposed to be of order $k+l-1+\beta$) and 'reducing' F_1 as above, viz. writing

$$\mu F_1 = f_1 = (1, x d_1, x^2 d_2, \dots, \chi y, 1)^{k+i-1+\beta}$$

and dividing f_1 (in case $k+l-1+\beta > k+l-1$ or $\beta > 0$) by $\varphi\psi$, we obtain by Lemma II

$$\begin{aligned} \mu F_1 = f_1 &= \varrho \varphi \psi + x^\beta (y^{k+i-1} c_0 + y^{k+i-2} x c_1 + \dots + x^{k+i-1} c_{k+i-1}) \\ &= \varrho \varphi \psi + x^\beta f \text{ say} \end{aligned}$$

and thence by Lemma III

$$\mu F_1 = \varrho \varphi \psi + \frac{x^\beta}{\Delta} (P\varphi + Q\psi) = \varrho \varphi \psi + \frac{1}{\Delta_1} (P\varphi + Q\psi)$$

therefore

$$F = A\Phi + B\Psi + \frac{\varrho x^\lambda}{\mu} \Phi \Psi + \frac{x P}{\mu \Delta_1} \Phi + \frac{\lambda Q}{\mu \Delta_1} \Psi$$

and this is of the form $A'\Phi + B'\Psi$: in fact we may take

$$A' = \frac{x P}{\mu \Delta_1} + A + \frac{\varrho x^\lambda}{\mu} \Psi, \quad B' = \frac{\lambda Q}{\mu \Delta_1} + B.$$

(But these functions are only defined for sufficiently small values of x, y . The series x, λ can clearly become infinite for large values of x, y).

§ 2.

Proposition I. The determinant

$$\begin{vmatrix} p_0 & a_0 & a_1 e & a_2 e^2 & \dots & a_m e^m \\ p_1 & & a_0^{(1)} & a_1^{(1)} e & a_2^{(1)} e^2 & \dots & a_m^{(1)} e^m \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ p_{n-1} & & & a_0^{(n-1)} & & & a_m^{(n-1)} e^m \\ q_0 & b_0 & b_1 e & b_2 e^2 & \dots & b_n e^n \\ q_1 & & b_0^{(1)} & b_1^{(1)} e & \dots & & b_n^{(1)} e^n \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ q_{m-1} & & & b_0^{(m-1)} & \dots & & b_n^{(m-1)} e^n \\ k & c_0 & c_1 e & c_2 e^2 & \dots & & c_{m+n-1} e^{m+n-1} \end{vmatrix}$$

is of the form

$$e^{mn} [k \nabla + p_0 P_0 + p_1 P_1 e + \dots + p_{n-1} P_{n-1} e^{n-1} + q_0 Q_0 + q_1 Q_1 e + \dots + q_{m-1} Q_{m-1} e^{m-1}]$$

where

$$\nabla, P_0, \dots, P_{n-1}, Q_0, \dots, Q_{m-1}$$

are rational integral expressions in

$$a_0 \dots a_m, a_0^{(1)} \dots a_m^{(1)} \dots b_0 b_1 \dots c_0 \dots c_{m+n-1}$$

and do not contain e . — To see this, write it in the form

$$T e^{-\frac{1}{2} n(n-1) - \frac{1}{2} m(m-1)}$$

where T differs from the original determinant only in the fact that its second row is multiplied by e , its third row by $e^2 \dots$, its n th row by e^{n-1} , its $(n+2)$ th row by e , its $(n+3)$ th row by $e^2 \dots$, its $(n+m)$ th row by e^{m-1} . And denote the $(j+1)$ th element of the i th row of T for $i < n+1$, by $A_{i,j}$, the $(j+1)$ th element of the $(n+i)$ th row, for $i < m+1$, by $A_{n+i,j}$ and the $(j+1)$ th element of the last row by $A_{n+m+1,j}$.

Then the general term in the expansion of T is

$$A_{1,j_1} A_{2,j_2} A_{3,j_3} \dots A_{n+m+1,j_{n+m+1}}$$

where $j_1, j_2, \dots, j_{n+m+1}$ are in some order the numbers $0, 1, 2, \dots, (n+m)$. And $A_{\lambda,j}$ has the factor e^{j-1} unless $j=0$, while for $j=0$

$$A_{\lambda,j} = p_{\lambda-1} e^{\lambda-1}, \text{ or } q_{i-1} e^{i-1}, \text{ or } k$$

according as

$$\lambda < n+1, \lambda = n+i \ (i < m+1) \text{ or } \lambda = n+m+1.$$

Thus the general term of the expansion has the factor e^μ where

$$\mu = \sum (j-1) + \lambda, \quad \sum (j-1) + i, \quad \sum (j-1) + 1$$

in these three cases respectively. Hence

$$\mu = \frac{1}{2} (n+m)(n+m+1) - (n+m+1) + \nu$$

where $\nu = \lambda$ or i or 1 in these three cases,

$$= \frac{1}{2} n(n-1) + \frac{1}{2} m(m-1) + mn + \nu - 1.$$

Hence the general term in

$$T e^{-\frac{1}{2} n(n-1) - \frac{1}{2} m(m-1)}$$

is $p_{\lambda-1} P_{\lambda-1} e^{mn+\lambda-1}$ or $q_{i-1} Q_{i-1} e^{mn+i-1}$ or $k \nabla e^{mn}$

proving our theorem. [And we may say, $P_{\lambda-1}$ is isobaric in a, b, c together and of weight $mn + \lambda - 1$, etc.]

Proposition II. Returning to the notation of lemma III and writing for shortness $x = ey$, we have if Ω denote the y -resultant of φ, ψ

$$\Omega f = \begin{vmatrix} 0 & 1 & xa_1 & x^2a_2 & \dots & x^ka_k \\ 0 & 0 & 1 & xa_1 & & x^ka_k \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & xb_1 & x^2b_2 & \dots & x^ib_i \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & & & & & x^ib_i \\ f & c_0 & c_1x & c_2x^2 & \dots & c_{k+i-1}x^{k+i-1} \end{vmatrix} \neq 1$$

by multiplying second column by y^{k+i-1} , third column by $y^{k+i-2}, \dots$ last column by y^0 , and subtracting from first column

$$\begin{aligned} \Omega f &= \begin{vmatrix} -\varphi \cdot y^{i-1} & 1 & xa_1 & x^2a_2 & \dots \\ -\varphi \cdot y^{i-2} & 0 & 1 & xa_1 & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -\psi \cdot y^{k-1} & 1 & xb_1 & x^2b_2 & \dots \\ -\psi \cdot y^{k-2} & 0 & 1 & xb_1 & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & c_0 & c_1x & c_2x^2 & \dots \end{vmatrix} \\ &= x^{ki} \{ -\varphi [y^{i-1}P_0 + y^{i-2}xP_1 + \dots + x^{i-1}P_{i-1}] \\ &\quad - \psi [y^{k-1}Q_0 + y^{k-2}xQ_1 + \dots + x^{k-1}Q_{k-1}] \} \end{aligned}$$

by Proposition I (viz putting therein $m = k, n = i, p_0 = -\varphi y^{i-1}, p_1 = -\varphi y^{i-2}, \dots, a_0 = a_0^{(1)} = 1, a_2^{(1)} = a_2$ etc.).

Thus $\Omega = x^{ki} \Delta$ where Δ is not infinite for $x = 0$ and

$$\Delta f = P\varphi + Q\psi,$$

which is lemma III and here

$$P = -(y^{i-1}P_0 + y^{i-2}xP_1 + \dots + x^{i-1}P_{i-1}),$$

$$Q = -(y^{k-1}Q_0 + y^{k-2}xQ_1 + \dots + x^{k-1}Q_{k-1})$$

where $P_0, P_1, \dots$ are algebraic rational integral expressions in

$$a_1, a_2, \dots, b_1, b_2, \dots, c_0, c_1, c_2, \dots$$