II.—*On a Process in the Differential Calculus, and its application to the Solution of certain Differential Equations.* By the Rev. P. KELLAND, M.A., F.R.S.S.L. & E., F.C.P.S., late Fellow of Queen's College, Cambridge; Professor of Mathematics, &c., in the University of Edinburgh.

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The facilities which are afforded by the introduction of the function $\sqrt{}$ into certain classes of Differential expressions are well known. This function has effected the combination and generalization of Problems, which, although found to be capable of solution in particular cases, were regarded rather as isolated and exceptional forms, than as integral parts of some comprehensive expression. But the subject is far from exhausted. Some of the most important Differential Equations have never as yet been solved by a general method. The present Memoir is intended to supply this defect. The process employed differs little from that which I have previously exhibited; but the range of Problems which it embraces is much more extensive, and the Problems themselves are of a more important character.

SECTION I. PRELIMINARY THEOREMS.

$$\begin{aligned} 1. \text{ Let } & y = e^{-\alpha x^r}, \\ \text{then } & \left(\frac{1}{x^{r-1}} \frac{d}{dx} \right) y = -\alpha r y, \end{aligned}$$

from which it follows, since the operation reproduces the function itself, that

$$\left(\frac{1}{x^{r-1}} \frac{d}{dx} \right)^\mu y = (-\alpha r)^\mu y \quad (1.), \text{ whatever } \mu \text{ may be.}$$

It is necessary to observe, that if μ be negative, the above equation takes no cognizance of functions of integration, which would be introduced by means of the added arbitrary constants; and this remark applies to all our processes.

$$\text{In the equation } \int_0^\infty e^{-\theta} \theta^{n-1} d\theta = \bar{n},$$

$$\text{let } \theta = \alpha x^r,$$

$$\text{then } \int_0^\infty e^{-\alpha x^r} \alpha^{n-1} x^{r(n-1)} x^r d\alpha = \bar{n}$$

$$\text{or } \frac{\bar{n}}{x^{rn}} = \int_0^\infty \alpha^{n-1} e^{-\alpha x^r} d\alpha,$$

or, which is the same thing,

$$\frac{\sqrt[m]{r}}{x^m} = \int_0^\infty a^{\frac{m}{r}-1} e^{-a x^r} d a; \text{ from which it follows, by equation (1.), that}$$

$$\begin{aligned} \frac{\sqrt[m]{r}}{x^m} \left(\frac{1}{x^{r-1}} \frac{d}{dx} \right)^\mu \cdot \frac{1}{x^m} &= \int_0^\infty a^{\frac{m}{r}-1} (-a r)^\mu e^{-a x^r} d a \\ &= (-r)^\mu \frac{1}{x^{m+r\mu}} \int_0^\infty \theta^{\frac{m+r\mu}{r}-1} e^{-\theta} d \theta \\ &= (-r)^\mu \frac{\sqrt[m+r\mu]{r}}{x^{m+r\mu}} \end{aligned}$$

Consequently,
$$\left(\frac{1}{x^{r-1}} \frac{d}{dx} \right)^\mu \frac{1}{x^m} = (-r)^\mu \frac{\sqrt[m+r\mu]{r}}{\sqrt[m]{r}} \frac{1}{x^{m+r\mu}} \dots (2.)$$

Let u be a function, supposed to be capable of being represented under the form of a series of functions, such as $e^{-a x^r}$; then we shall have

$$x^{\mu r} \left(\frac{1}{x^{r-1}} \frac{d}{dx} \right)^\mu u = (-r)^\mu \frac{\sqrt{-\frac{D}{r} + \mu}}{\sqrt{-\frac{D}{r}}} u \dots (3.)$$

2. Since
$$\frac{1}{x^{r-1}} \frac{d}{dx} u = \frac{d u}{x^{r-1} d x} = \frac{r d u}{d x^r}$$
 it follows that
$$\left(\frac{1}{x^{r-1}} \frac{d}{dx} \right)^\mu u = r^\mu \left(\frac{d}{d x^r} \right)^\mu u$$

$$x^{\mu r} \left(\frac{d}{d x^r} \right)^\mu u = (-1)^\mu \frac{\sqrt{-\frac{D}{r} + \mu}}{\sqrt{-\frac{D}{r}}} u \dots (4.)$$

3. If we write $d_r u$ for $\frac{1}{x^{r-1}} \frac{d}{dx} u$; then from the last Art. it will be evident, without demonstration, that

$$d_r^\mu (u v) = v d_r^\mu u + \mu d_r v d_r^{\mu-1} u + \frac{\mu(\mu-1)}{1 \cdot 2} d_r^2 v d_r^{\mu-2} u + \&c.$$

4. Let $\phi(d_r)$ be any function of d_r capable of expansion in the form $\phi(d_r) = \Sigma a d_r^\mu$; then, by the last Article

$$\begin{aligned} \phi(d_r)(u v) &= \Sigma a \{ v d_r^\mu u + \mu d_r v d_r^{\mu-1} u + \&c. \} \\ &= v (\Sigma a d_r^\mu) u + d_r v (\Sigma a \mu d_r^{\mu-1}) u + \&c. \end{aligned}$$

$$= v \phi(d_r) u + d_r v \phi'(d_r) u + \frac{1}{1 \cdot 2} d_r^2 v \phi''(d_r) u + \&c.$$

where $\phi'(d_r)$ is the differential coefficient of $\phi(d_r)$ with respect to d_r .

5. In Art. 3, let $v = e^{\beta x^r}$, then

$$\begin{aligned} d_r^\mu (e^{\beta x^r} u) &= e^{\beta x^r} \{d_r^\mu u + \mu \beta r d_r^{\mu-1} u \\ &\quad + \frac{\mu(\mu-1)}{1 \cdot 2} (\beta r)^2 d_r^{\mu-2} u + \&c.\} \\ &= e^{\beta x^r} (d_r + \beta r)^\mu u; \end{aligned}$$

and hence, generally, as in Art. 1.,

$$\phi(d_r) \{e^{\beta x^r} u\} = e^{\beta x^r} \phi(d_r + \beta r) u \dots \dots (5.)$$

6. Let $\phi(x) = \Sigma B e^{\beta x^r}$, then

$$\begin{aligned} \phi(x) \psi(d_r) u &= \Sigma B e^{\beta x^r} \psi(d_r) u \\ &= \Sigma B \psi(d_r - \beta r) e^{\beta x^r} u \text{ by (5)} \\ &= \Sigma B \{ \psi(d_r) - \psi'(d_r) \beta r + \psi''(d_r) \frac{(\beta r)^2}{1 \cdot 2} - \&c. \} e^{\beta x^r} u \\ &= \psi(d_r) \{ \Sigma B e^{\beta x^r} u \} - \psi'(d_r) \{ d_r (\Sigma B e^{\beta x^r}) u \} + \&c. \\ &= \psi(d_r) \{ \phi(x) u \} - \psi'(d_r) \{ d_r \phi(x) \cdot u \} \\ &\quad + \frac{1}{1 \cdot 2} \psi''(d_r) \{ d_r^2 \phi(x) \cdot u \} - \&c. \end{aligned}$$

7. It is easily seen that, if m be a whole number,

$$d_r^{-m} \log x = \frac{r^{-m} x^{rm}}{1+m} \left\{ \log x - \frac{1}{r} \left(\frac{1}{1} + \frac{1}{2} + \&c + \frac{1}{m} \right) \right\};$$

and that in other cases,

$$d_r^\mu \log x = (-r)^{\mu-1} \mu x^{-r\mu}$$

8. The operation denoted by d_r^{-m} must not be confounded with $\left(x^{r-1} \int dx\right)^m$: it is, in reality, $\left(\int x^{r-1} dx\right)^m$. The one expression can be deduced from the other, thus:—

$$\begin{aligned} \left(x^{r-1} \int dx\right)^m \log x &= x^{r-1} d_r^{-m} x^{-(r-1)} \log x = x^{r-1} d_r^{-m} \frac{1}{p} (x^{p-r+1} - x^{q-r+1}) \\ &= \frac{(-r)^{-m}}{p} \left\{ \frac{-\left(\frac{p-r+1}{r} + m\right)}{\sqrt{-\frac{p-r+1}{r}}} x^p - \frac{-\left(\frac{q-r+1}{r} + m\right)}{\sqrt{-\frac{q-r+1}{r}}} x^q \right\} x^{rm} \end{aligned}$$

where p , q , and $\frac{q}{p}$ are indefinitely small.

$$\begin{aligned} \text{Hence} \quad & \left(x^{r-1} \int dx \right)^m \log x = \frac{x^{rm}}{p r^m} \left\{ \frac{\sqrt{\frac{1+p}{r}}}{\sqrt{\frac{1+p}{r}} + m} x^p - \frac{\sqrt{\frac{1+q}{r}}}{\sqrt{\frac{1+q}{r}} + m} x^q \right\} \\ & = x^{rm} r^{-m} \frac{\sqrt{\frac{1}{r}}}{\sqrt{\frac{1}{r}} + m} \left\{ \log x - \left(\frac{1}{1} + \frac{1}{1+r} + \frac{1}{1+2r} + \&c. + \frac{1}{1+(m-1)r} \right) \right\} \end{aligned}$$

9. To find the differential equation, on the solution of which depends the value of $d_r^\mu e^{ax}$.

Since $e^{ax} = 1 + ax + \frac{a^2 x^2}{1 \cdot 2} + \&c.$

$$d_r^\mu e^{ax} = (-r)^\mu x^{-r\mu} \left\{ ax \frac{\sqrt{\frac{-1+r\mu}{r}}}{\sqrt{-\frac{1}{r}}} + \frac{a^2 x^2}{1 \cdot 2} \frac{\sqrt{\frac{-2+r\mu}{r}}}{\sqrt{-\frac{2}{r}}} + \&c. \right\}$$

This equation can be made to depend on the solution of a differential equation, when r is a whole number; for, in that case, the terms will recur after the r th. To reduce the equation, we observe that

$$\frac{\sqrt{\frac{-(r+1)+r\mu}{r}}}{\sqrt{-\frac{r+1}{r}}} = \frac{r+1}{r+1-r\mu} \frac{\sqrt{\frac{-1+r\mu}{r}}}{\sqrt{-\frac{1}{r}}}, \text{ and so on.}$$

$$\begin{aligned} \text{Hence} \quad d_r^\mu e^{ax} &= (-r)^\mu x^{-r\mu} \left[\frac{\sqrt{\frac{-1+r\mu}{r}}}{\sqrt{-\frac{1}{r}}} \left(ax + \frac{r+1}{r+1-r\mu} \frac{a^{r+1} r^{r+1}}{\underline{r+1}} + \&c. \right) \right. \\ &+ \left. \frac{\sqrt{\frac{-2+r\mu}{r}}}{\sqrt{-\frac{2}{r}}} \left(\frac{a^2 x^2}{1 \cdot 2} + \frac{r+2}{r+2-r\mu} \frac{a^{r+2} x^{r+2}}{\underline{r+2}} + \&c. \right) + \&c., \&c., \right] \dots \dots \dots (A) \end{aligned}$$

$$\begin{aligned} \text{Let} \quad y &= x^{1-r\mu} + \frac{a^r x^{r+1-r\mu}}{\underline{r+1}} \frac{r+1}{r+1-r\mu} \\ &+ \frac{a^{2r} x^{2r+1-r\mu}}{\underline{2r+1}} \frac{(r+1)(2r+1)}{(r+1-r\mu)(2r+1-r\mu)} + \&c. \dots \dots \dots; \text{ then} \\ \frac{d^{r-1}}{dx^{r-1}} \left(x^{r\mu} \frac{dy}{dx} \right) &= a^r \left\{ \frac{x}{1} + \frac{a^r x^{r+1}(r+1)}{\underline{r+1}(r+1-r\mu)} + \&c. \right\} \\ &= a^r x^{r\mu} y \end{aligned}$$

$$\text{Again, if } y = \frac{1}{1 \cdot 2} x^{2-r\mu} \frac{\sqrt{\frac{-2+r\mu}{r}}}{\sqrt{-\frac{2}{r}}} + \frac{a^r x^{r+2-r\mu}}{\underline{r+2}} \frac{\sqrt{\frac{-(r+2)+r\mu}{r}}}{\sqrt{-\frac{r+2}{r}}} + \&c.$$

$$\begin{aligned}
 &= \frac{\sqrt{\frac{-2+r\mu}{r}}}{\sqrt{-\frac{2}{r}}} \left\{ \frac{1}{1 \cdot 2} x^{2-r\mu} + \frac{r+2}{r+2-r\mu} \frac{a^r x^{r+2-r\mu}}{\sqrt{r+2}} \right. \\
 &\quad \left. + \frac{a^{2r} x^{2r+2-r\mu}}{\sqrt{2r+2}} \frac{(r+2)(2r+2)}{(r+2-r\mu)(2r+2-r\mu)} + \&c. \right\} \\
 \therefore \frac{d^{r-1}}{dx^{r-1}} \left(x^{r\mu} \frac{dy}{dx} \right) &= \frac{\sqrt{\frac{-2+r\mu}{r}}}{\sqrt{-\frac{2}{r}}} a^r \left\{ \frac{x^2}{1 \cdot 2} + \frac{a^r x^{r+2}}{\sqrt{r+2}} \frac{r+2}{r+2-r\mu} + \&c. \right\} \\
 &= a^r x^{r\mu} y \dots (B).
 \end{aligned}$$

The same process will give the same result for all the terms of this expansion except the last. But as $\frac{\sqrt{-1+\mu}}{\sqrt{-1}}$ is a factor of that term, it is evidently 0.

Consequently, the equation $\frac{d^{r-1}}{dx^{r-1}} \left(x^{r\mu} \frac{dy}{dx} \right) = a^r x^{r\mu} y$ gives the complete value of $y = d_r^\mu e^{ax}$, it being observed that one of the constants of integration is 0. The other constants must be determined by means of equation (A).

10. To find $d_r^\mu \cos ax$.

$$\text{Since} \quad \cos ax = 1 - \frac{a^2}{2} x^2 + \frac{a^4}{4} x^4 - \&c. \}$$

if $-a^2 = b$, we shall have

$$\begin{aligned}
 \left(\frac{1}{x^{r-1}} \frac{d}{dx} \right)^\mu \cos ax &= (-r)^\mu \left\{ \frac{b}{2} \frac{\sqrt{\frac{-2+r\mu}{r}}}{\sqrt{-\frac{2}{r}}} x^{2-r\mu} \right. \\
 &\quad \left. + \frac{b^2}{4} \frac{\sqrt{\frac{-4+r\mu}{r}}}{\sqrt{-\frac{4}{r}}} x^{4-r\mu} + \frac{b^3}{6} \frac{\sqrt{\frac{-6+r\mu}{r}}}{\sqrt{-\frac{6}{r}}} x^{6-r\mu} + \&c. \right\}
 \end{aligned}$$

It remains to express this series in the form of a differential equation when r is a whole number.

Let s be any whole number less than $r+1$, $S = (-r)^\mu b^s \frac{\sqrt{\frac{-2s+r\mu}{r}}}{\sqrt{-\frac{2s}{r}}}$; then the

series may be written

$$\begin{aligned}
 y &= S \left\{ \frac{x^{2s-r\mu}}{\sqrt{2s}} + \frac{b^r}{\sqrt{2r+2s}} \frac{(r+2s)(2r+2s)}{(r+2s-r\mu)(2r+2s-r\mu)} x^{2r+2s-r\mu} + \&c. \right\} \\
 \therefore \frac{d^{r-1}}{dx^{r-1}} \left(x^{r\mu} \frac{dy}{dx} \right) &= S \left\{ \frac{2s-r\mu}{2s} \frac{x^{2s-r}}{\sqrt{2s-r}} + \frac{b^r}{\sqrt{2s+r-1}} \frac{x^{2s+r}}{r+2s-r\mu} \right.
 \end{aligned}$$

$$\begin{aligned}
& + \frac{b^{2r}}{\sqrt{2s+3r-1}} \frac{(r+2s)(2r+2s)x^{2s+3r}}{(r+2s-r\mu)(2r+2s-r\mu)(3r+2s-r\mu)} + \&c. \} \\
\frac{d}{dx} x^{-r\mu} \frac{d^{r-1}}{dx^{r-1}} x^{r\mu} \frac{dy}{dx} &= S \left\{ \frac{(2s-r\mu)(2s-r-r\mu)}{2s(2s-r)} \frac{x^{2s-r-1-r\mu}}{\sqrt{2s-r-1}} \right. \\
& + \frac{b^r}{\sqrt{2s+r-1}} x^{2s+r-1-r\mu} + \frac{b^{2r}}{\sqrt{2s+3r-1}} \frac{(r+2s)(2r+2s)x^{2s+3r-1-r\mu}}{(r+2s-r\mu)(2r+2s-r\mu)} + \&c. \} \\
\frac{d^{r-1}}{dx^{r-1}} x^{r\mu} \frac{d}{dx} x^{-r\mu} \frac{d^{r-1}}{dx^{r-1}} x^{r\mu} \frac{dy}{dx} &= S \left\{ \frac{(2s-r\mu)(2s-r-r\mu)}{2s(2s-r)} \frac{x^{2s-2r}}{\sqrt{2s-2r}} \right. \\
& + \frac{b^r}{\sqrt{2s}} x^{2s} + \frac{b^{2r}}{\sqrt{2r+2s}} \frac{(r+2s)(2r+2s)}{(r+2s-r\mu)(2r+2s-r\mu)} x^{2s+2r} + \&c. \} \\
&= S \frac{(2s-r\mu)(2s-r-r\mu)}{2s(2s-r)} \frac{x^{2s-2r}}{\sqrt{2s-2r}} + b^r x^{r\mu} y.
\end{aligned}$$

If $s < r$, $\frac{1}{\sqrt{2s-2r}} = 0$

If $s = r$, $S \frac{(2s-r\mu)(2s-r-r\mu)}{2s(2s-r)} \frac{x^{2s-2r}}{\sqrt{2s-2r}} = (-r)^\mu \frac{\sqrt{-\frac{2s}{r} + \mu + 2}}{\sqrt{-\frac{2s}{r} + 2}}$

$$= (-r)^\mu \frac{\sqrt{\mu}}{\sqrt{0}} = 0;$$

Therefore, generally, if $y = d_r^\mu \cos ax$, we have

$$\frac{d^{r-1}}{dx^{r-1}} x^{r\mu} \frac{d}{dx} x^{-r\mu} \frac{d^{r-1}}{dx^{r-1}} x^{r\mu} \frac{dy}{dx} = (-a^2)^r x^{r\mu} y;$$

or, if $x^{r\mu} y = z$, the equation may be written

$$\left(\frac{d^{r-1}}{dx^{r-1}} x^{r\mu} \frac{d}{dx} x^{-r\mu} \right)^2 z = (-a^2)^r z, \dots (C)$$

a form which exhibits its relation with the differential equation resulting from $e^{\alpha x}$.

11. There exist numerous relations between the differential coefficients of a function; such as the following:—

Since $x^{\mu r} \left(\frac{1}{x^{r-1}} \frac{d}{dx} \right)^\mu u = (-r)^\mu \frac{\sqrt{-\frac{D}{r} + \mu}}{\sqrt{-\frac{D}{r}}} u;$

let $\phi\left(\frac{D}{r}\right) = \frac{\sqrt{-\frac{D}{r} + \mu}}{\sqrt{-\frac{D}{r}}}$; then

$$\phi\left(\frac{D}{r} + 1\right) = \frac{\sqrt{-\frac{D}{r} - 1 + \mu}}{\sqrt{-\frac{D}{r} - 1}} = \frac{\frac{D}{r} + 1}{\frac{D}{r} - \mu + 1} \phi\left(\frac{D}{r}\right)$$

Hence

$$\begin{aligned} \phi\left(\frac{D}{r}\right) &= \frac{\sqrt{\frac{D}{r} + 1}}{\sqrt{\frac{D}{r} - \mu + 1}} \\ &= x^{(\mu-1)r} \frac{\sqrt{\frac{D}{r} + \mu}}{\sqrt{\frac{D}{r}}} x^{-(\mu-1)r} \\ &= x^{(\mu-1)r} \frac{\sqrt{-\frac{D}{r} + \mu}}{\sqrt{-\frac{D}{r}}} x^{-(\mu-1)r} \end{aligned}$$

Hence

$$x^{\mu r} \left(\frac{1}{x^{r-1}} \frac{d}{dx} \right)^{\mu} u = (-1)^{\mu} x^{-r} \left(x^{r+1} \frac{d}{dx} \right)^{\mu} x^{-(\mu-1)r} u,$$

a relation between differentials with positive and negative values of r .

SECTION II. APPLICATION OF THE PRECEDING THEORY TO THE SOLUTION OF DIFFERENTIAL EQUATIONS.

12. The first example which I propose to give is the solution of equation (B), Art. 9.

Ex. 1. $\frac{d^{r-1}}{dx^{r-1}} \left(x^{r\mu} \frac{dy}{dx} \right) = a^r x^{r\mu} y$, where μ may be anything whatever.

Let $x^{r\mu} y = z$,

$$\therefore \frac{d^{r-1}}{dx^{r-1}} x^{r\mu} \frac{d}{dx} x^{-r\mu} z = a^r z$$

or $x^{r-1} \frac{d^{r-1}}{dx^{r-1}} x^{r\mu-1} D x^{-r\mu} z = a^r x^{r-1} z$

or $D(D-1) \dots (D-r+2) \times (D-r\mu+1) e^{-\theta} z = a^r e^{r\theta} e^{-\theta} z$

Let $\frac{z}{x}$ or $e^{-\theta} z = f\left(-\frac{D}{r}\right) v$; then

$$D(D-1) \dots (D-r+2) \times (D-r\mu+1) f\left(-\frac{D}{r}\right) v =$$

$$a^r e^{r\theta} f\left(-\frac{D}{r}\right) v = a^r f\left(-\frac{D}{r} + 1\right) e^{r\theta} v.$$

Which equation is reduced to

$$D(D-1) \dots (D-r+1) v = a^r e^{r\theta} v \quad (1.)$$

$$\text{by the condition } f\left(-\frac{D}{r} + 1\right) = \frac{D-r\mu+1}{D-r+1} f\left(-\frac{D}{r}\right) \quad (2.)$$

$$= \frac{-\frac{D}{r} - \frac{1}{r} + \mu}{-\frac{D}{r} - \frac{1}{r} + 1} f\left(-\frac{D}{r}\right)$$

the solution of which equation is

$$f\left(-\frac{D}{r}\right) = \frac{\sqrt{-\frac{D}{r} - \frac{1}{r} + \mu}}{\sqrt{-\frac{D}{r} - \frac{1}{r} + 1}}$$

whence

$$\begin{aligned} \frac{z}{x} \text{ or } x^{r\mu-1} y &= \frac{\sqrt{-\frac{D}{r} - \frac{1}{r} + \mu}}{\sqrt{-\frac{D}{r} - \frac{1}{r} + 1}} v \\ &= x^{r-1} \frac{\sqrt{-\frac{D}{r} + \mu - 1}}{\sqrt{-\frac{D}{r}}} x^{-r+1} v \\ &= (-r)^{-\mu+1} x^{\mu r-1} d_r^{\mu-1} x^{-r+1} v \end{aligned}$$

Now v is the solution of equation (1); or, which is the same thing, of

$$\frac{d^r v}{d x^r} = a^r v$$

$\therefore v = A e^{a x}$ is a value of v ; and hence, finally,

$$\begin{aligned} y &= (-r)^{-\mu+1} A d_r^{\mu-1} \frac{e^{a x}}{x^{r-1}} \\ &= B d_r^{\mu} e^{a x}, \text{ where } B \text{ is an arbitrary constant.} \end{aligned}$$

It is evident that any other of the r roots of the equation $\frac{d^r v}{d x^r} = a^r v$ would satisfy the conditions, or that any one of the roots of a^r will produce the same result as a itself.

This example affords an excellent illustration of the process.

Let us take, in the next place, the equation which occurs in the theory of the Figure of the Earth; an equation which has been solved by Professor BOOLE, in his admirable Memoir in the *Philosophical Transactions* for 1844, p. 251, but whose process is necessarily restricted to integral values of i . The equation is—

$$\text{Ex. 2.} \quad \frac{d^2 u}{d x^2} - q^2 u - \frac{i(i+1)u}{x^2} = 0.$$

By the usual process, this equation becomes

$$D(D-1)u - q^2 e^{2\theta} u - i(i+1)u = 0$$

or

$$(D+i)(D-i-1)u - q^2 e^{2\theta} u = 0.$$

Let

$$u = e^{-i\theta} f\left(-\frac{D}{2}\right)v, \text{ then}$$

$$(D+i)(D-i-1)e^{-i\theta} f\left(-\frac{D}{2}\right)v - q^2 e^{-i\theta} f\left(-\frac{D}{2}+1\right)e^{2\theta}v = 0$$

or

$$D(D-2i-1)f\left(-\frac{D}{2}\right)v - q^2 f\left(-\frac{D}{2}+1\right)e^{2\theta}v = 0$$

an equation which, by omitting the function of 0, is reduced to

$$D(D-1)v - q^2 e^{2\theta}v = 0 \quad (1.)$$

by the condition

$$f\left(-\frac{D}{2}+1\right) = \frac{D-2i-1}{D-1} f\left(-\frac{D}{2}\right) \quad (2.)$$

$$= \frac{-\frac{D}{2}+i+\frac{1}{2}}{-\frac{D}{2}+\frac{1}{2}} f\left(-\frac{D}{2}\right)$$

$$\therefore f\left(-\frac{D}{2}\right) = \frac{\sqrt{-\frac{D}{2}+i+\frac{1}{2}}}{\sqrt{-\frac{D}{2}+\frac{1}{2}}}$$

and

$$\begin{aligned} u &= e^{-i\theta} \frac{\sqrt{-\frac{D}{2}+i+\frac{1}{2}}}{\sqrt{-\frac{D}{2}+\frac{1}{2}}} v \\ &= e^{-(i-1)\theta} \frac{\sqrt{-\frac{D}{2}+i}}{\sqrt{-\frac{D}{2}}} e^{-\theta} v \\ &= (-2)^{-i} x^{i+1} \left(\frac{1}{x} \frac{d}{dx}\right)^i \frac{v}{x} \end{aligned}$$

But the solution of Equation (1.) is

$$v = A e^{qx} + B e^{-qx};$$

Hence the complete solution of the given equation is

$$u = (-2)^{-i} x^{i+1} \left(\frac{1}{x} \frac{d}{dx}\right)^i \frac{1}{x} (A e^{qx} + B e^{-qx})$$

This solution is susceptible of another form by Art. 11; for by that Article,

$$\left(\frac{1}{x} \frac{d}{dx}\right)^i \frac{v}{x} = (-1)^i x^{-(2i+2)} \left(x^3 \frac{d}{dx}\right)^i \frac{v}{x^{2i-1}}$$

$$\therefore u = 2^{-i} \frac{1}{x^{i+1}} \left(x^3 \frac{d}{dx}\right)^i \frac{A e^{qx} + B e^{-qx}}{x^{2i-1}},$$

in which form the solution has been given by Professor BOOLE, when i is a whole number.

Ex. 3. *Solution of the Equation for Laplace's Functions.*

This equation has been reduced by Professor BOOLE, in the *Cambridge Mathematical Journal*, New Series, vol. i., p. 15, to the form

$$(1-\mu^2) \frac{d^2 v}{d\mu^2} + 2(a-1) \mu \frac{dv}{d\mu} + \{n(n+1) - a(a-1)\}v = 0 \dots (1)$$

where u , the quantity sought, is connected with v by the equation

$$u = (1-\mu^2)^{-\frac{a}{2}} v \quad (2).$$

Putting $\mu = e^\theta$ and writing D for $\frac{d}{d\theta}$, we have

$$\{(e^{-2\theta} - 1) D(D-1) + 2(a-1) D + n(n+1) - a(a-1)\}v = 0$$

$$\text{or} \quad e^{-2\theta} D(D-1) v - (D-a+n)(D-a-n-1) v = 0$$

$$\text{or} \quad D(D-1) v - (D-a-n-2)(D-a+n-1) e^{2\theta} v = 0$$

Let, therefore, $v = f\left(-\frac{D}{2}\right) w$ (3), and this equation becomes

$$D(D-1) f\left(-\frac{D}{2}\right) w - (D-a-n-2)(D-a+n-1) f\left(-\frac{D}{2} + 1\right) e^{2\theta} w = 0$$

$$\text{or} \quad D(D-1) w - (D-a-n-1)(D-a-n-2) e^{2\theta} w = 0 \quad (4), \quad \text{provided}$$

$$\begin{aligned} f\left(-\frac{D}{2} + 1\right) &= \frac{D-a-n-1}{D-a+n-1} f\left(-\frac{D}{2}\right) \\ &= \frac{-\frac{D}{2} + \frac{a}{2} + \frac{n}{2} + \frac{1}{2}}{-\frac{D}{2} + \frac{a}{2} - \frac{n}{2} + \frac{1}{2}} f\left(-\frac{D}{2}\right) \end{aligned}$$

an equation of which the solution is evidently

$$f\left(-\frac{D}{2}\right) = \frac{\sqrt{-\frac{D}{2} + \frac{a}{2} + \frac{n}{2} + \frac{1}{2}}}{\sqrt{-\frac{D}{2} + \frac{a}{2} - \frac{n}{2} + \frac{1}{2}}} \quad (5.)$$

and by (3)

$$\begin{aligned} v &= f\left(-\frac{D}{2}\right) w \\ &= \frac{\sqrt{-\frac{D}{2} + \frac{a}{2} + \frac{n}{2} + \frac{1}{2}}}{\sqrt{-\frac{D}{2} + \frac{a}{2} - \frac{n}{2} + \frac{1}{2}}} w \\ &= e^{-(n-a-1)\theta} \frac{\sqrt{-\frac{D}{2} + n}}{\sqrt{-\frac{D}{2}}} e^{(n-a-1)\theta} w \end{aligned}$$

$$= (-2)^{-n} \mu^{n+\alpha+1} \left(\frac{1}{\mu} \frac{d}{d\mu} \right)^n \mu^{n-\alpha-1} w \quad (6)$$

a result which is true, whether n be integral or fractional.

The value of w from Equation (4.) is easily found, and is given by Mr BOOLE in the form

$$w = (1 + \mu)^{n+\alpha} \psi(\phi) + (1 - \mu)^{n+\alpha} \chi(\phi)$$

whence that of u is found.

Let us take, as our next example, the equation which has been discussed by M. POISSON in the *Journal de l'Ecole Polytechnique*, cah. 17, p. 614; and by Professor BOOLE in the *Philosophical Transactions* for 1844, p. 254.

$$\text{Ex. 4. } (1-x^2) \frac{d^2 u}{dx^2} + \left\{ \frac{p+1}{x} - (4n-p+1)x \right\} \frac{du}{dx} - 2n(2n-p)u = 0. \dots (1.)$$

The symbolical form of this equation is

$$u - \frac{(D+2n-2)(D+2n-2-p)}{D(D+p)} e^{2\theta} u = 0. \dots (2.)$$

$$\text{Let } u = f\left(-\frac{D}{2}\right) v \dots (3.); \text{ then}$$

$$f\left(-\frac{D}{2}\right) v - \frac{(D+2n-2)(D+2n-2-p)}{D(D+p)} f\left(-\frac{D}{2} + 1\right) e^{2\theta} v = 0 \dots (4.)$$

This last equation may be reduced to an integrable form in various ways:

$$1. \text{ By making } f\left(-\frac{D}{2} + 1\right) = \frac{D+p}{D-1} \cdot \frac{D+2n-3}{D+2n-2-p} f\left(-\frac{D}{2}\right)$$

$$\begin{aligned} \text{or } f\left(-\frac{D}{2}\right) &= \frac{\sqrt{-\frac{D}{2}-\frac{p}{2}}}{\sqrt{-\frac{D}{2}+\frac{1}{2}}} \cdot \frac{\sqrt{-\frac{D}{2}-n+\frac{3}{2}}}{\sqrt{-\frac{D}{2}-n+1+\frac{p}{2}}} \\ &= e^{\theta} \frac{\sqrt{-\frac{D}{2}-\frac{p}{2}-\frac{1}{2}}}{\sqrt{-\frac{D}{2}}} e^{-(2n-p-1)\theta} \frac{\sqrt{-\frac{D}{2}-\frac{p}{2}+\frac{1}{2}}}{\sqrt{-\frac{D}{2}}} e^{(2n-p-2)\theta} \end{aligned}$$

$$\text{or } u = (-2)^p x^{-p} \left(\frac{1}{x} \frac{d}{dx} \right)^{-\frac{p+1}{2}} x^{-2n+2} \left(\frac{1}{x} \frac{d}{dx} \right)^{-\frac{p-1}{2}} x^{2n-p-2} v$$

and equation (4.) becomes

$$v - \frac{(D+2n-2)(D+2n-3)}{D(D-1)} e^{2\theta} v = 0 \dots (5.)$$

which is a known form, and thus the given equation is completely solved.

It is evident that our solution reduces the operation to that of ordinary differentiation or integration, when p is an odd integer, positive or negative. By varying the process, however, we can obtain other forms of the solution of this

equation, which are reduced to ordinary differentiation or integration in other cases. The other forms are as follow :

2. By making $f\left(-\frac{D}{2}+1\right)=\frac{D}{D+2n-2}f\left(-\frac{D}{2}\right)$

or
$$f\left(-\frac{D}{2}\right)=\frac{\sqrt{-\frac{D}{2}}}{\sqrt{-\frac{D}{2}-n+1}}$$

$$=e^{-(2n-2)\theta}\frac{\sqrt{-\frac{D}{2}+n-1}}{\sqrt{-\frac{D}{2}}}e^{(2n-2)\theta}$$

and, consequently, $u=(-2)^{-(n-1)}\left(\frac{1}{x}\frac{d}{dx}\right)^{n-1}x^{2n-2}v$; where v is found from the equation of the first order $v-\frac{D+2n-2-p}{D+p}e^{2\theta}v=f\left(-\frac{D}{2}\right)^{-1}0$.

This solution is of the ordinary form, whenever n is a positive or negative integer.

By making $f\left(-\frac{D}{2}+1\right)=\frac{D}{D+2n-2-p}f\left(-\frac{D}{2}\right)$

or
$$f\left(-\frac{D}{2}\right)=\frac{\sqrt{-\frac{D}{2}}}{\sqrt{-\frac{D}{2}-n+1+\frac{p}{2}}}$$

$$=e^{-(2n-2-p)\theta}\frac{\sqrt{-\frac{D}{2}+n-1-\frac{p}{2}}}{\sqrt{-\frac{D}{2}}}e^{2n-2-p\theta}$$

and $u=(-2)^{-(n-\frac{p}{2}-1)}\left(\frac{1}{x}\frac{d}{dx}\right)^{n-\frac{p}{2}-1}x^{2n-p-2}v$;

where v is found from the solution of the equation of the first order,

$$v-\frac{D+2n-2}{D+p}e^{2\theta}v=f\left(-\frac{D}{2}\right)^{-1}0.$$

This solution is of the ordinary form when $n-\frac{p}{2}$ is an integer.

4. By making $f\left(-\frac{D}{2}+1\right)=\frac{D+p}{D+2n-p-2}f\left(-\frac{D}{2}\right)$,

or
$$f\left(-\frac{D}{2}\right)=\frac{\sqrt{-\frac{D}{2}-\frac{p}{2}}}{\sqrt{-\frac{D}{2}-n+\frac{p}{2}+1}}$$

$$=e^{-(2n-p-2)\theta}\frac{\sqrt{-\frac{D}{2}+n-p-1}}{\sqrt{-\frac{D}{2}}}e^{(2n-p-2)\theta}$$

and
$$u = (-2)^{-(n-p-1)} x^{-p} \left(\frac{1}{x} \frac{d}{dx} \right)^{n-p-1} x^{2n-p-2} v,$$

where v is found from the solution of the equation of the first order,

$$v - \frac{D+2n-2}{D} e^{2\theta} v = f\left(-\frac{D}{2}\right)^{-1} 0.$$

This solution is of the ordinary form when $n-p$ is an integer.

This last form appears to have been overlooked by Professor BOOLE.

It may be remarked, that we have omitted $f\left(-\frac{D}{2}\right)^{-1} 0$ in the first solution.

The reason is, that equation (5) being of the second order, will contain two arbitrary constants, and will thus render the solution of the given equation complete, without the introduction of any other terms.

Ex. 5.
$$(1-x^2) \frac{d^2 u}{dx^2} - (2m+1)x \frac{du}{dx} - (m^2-q^2)u = 0.$$

This equation has been solved by Professor BOOLE; but it is requisite to shew that his solution is not confined to integral values of m .

The symbolical form of the equation is

$$u - \frac{(D+m-2)^2 - q^2}{D(D-1)} e^{2\theta} u = 0.$$

Let $u = e^{-m\theta} f(-D) v$; then

$$f(-D) v - \frac{D-2)^2 - q^2}{(D-m)(D-m-1)} f(-D+2) e^{2\theta} v = 0.$$

By making $f(-D) = \frac{D(D-1)}{(D-m)(D-m-1)} f(-D+2);$

or
$$f(-D) = \frac{\sqrt{-D+m}}{\sqrt{-D}}$$

this equation is reduced to

$$v - \frac{(D-2)^2 - q^2}{D(D-1)} e^{2\theta} v = 0 \quad (1)$$

and
$$u = e^{-m\theta} \frac{\sqrt{-D+m}}{\sqrt{-D}} v = (-1)^{-m} \left(\frac{d}{dx} \right)^m v$$

$$= \left(\frac{d}{dx} \right)^m \left\{ A \cos(q \sin^{-1} x) + B \sin(q \sin^{-1} x) \right\}$$

This equation is, however, susceptible of a solution in a different form from that which we have just exhibited.

Let $u = f\left(-\frac{D}{2}\right) v$, then

$$f\left(-\frac{D}{2}\right) v - \frac{(D+m-2-q)(D+m-2+q)}{D(D-1)} f\left(-\frac{D}{2}+1\right) e^{2\theta} v = 0$$

By making $f\left(-\frac{D}{2}+1\right) = \frac{(D-1)}{D+m-2+q} f\left(-\frac{D}{2}\right) \dots (1.)$

this equation is reduced to the equation of the first order.

$$v - \frac{D+m-2-q}{D} e^{2\theta} v = f\left(-\frac{D}{2}\right)^{-1} 0 \dots (2.)$$

Also
$$f\left(-\frac{D}{2}\right) = \frac{\sqrt{-\frac{D}{2} + \frac{1}{2}}}{\sqrt{-\frac{D}{2} - \frac{m}{2} + 1 - \frac{q}{2}}}$$

$$= e^{-(m-2+q)\theta} \frac{\sqrt{-\frac{D}{2} + \frac{m}{2} - \frac{1}{2} + \frac{q}{2}}}{\sqrt{-\frac{D}{2}}} e^{(m-2+q)\theta}$$

$$u = (-2)^{-(m+q-1)} x \left(\frac{1}{x} \frac{d}{dx}\right)^{m+q-1} x^{m+q-2} v$$

which is of the ordinary form when $m+q$ is a whole number.

The same is true if $-q$ be substituted for q .

SECTION III. COMPARISON OF PROCESSES.

We shall occupy this Section in the comparison of the different methods which may be employed in the solution of a given Differential Equation.

13. *To find the value of $\sqrt{-D} \cdot v$.*

Suppose v expressed in the form

$$v = \frac{A}{x^n} + \frac{B}{x^{n+r}} + \&c.$$

$$= A e^{-n\theta} + B e^{-(n+r)\theta} + \&c.$$

then

$$\sqrt{-D} \cdot v = A \sqrt{n} e^{-n\theta} + B \sqrt{n+r} e^{-(n+r)\theta} + \&c.$$

$$= A \frac{\sqrt{n}}{x^n} + B \frac{\sqrt{n+r}}{x^{n+r}} + \&c.$$

$$= \int_0^\infty e^{-ax} d\alpha (A a^{n-1} + B a^{n+r-1} + \dots)$$

$$= \int_0^\infty e^{-ax} \frac{d\alpha}{a} (A a^n + B a^{n+r} + \&c.)$$

Hence, if

$$v = f(x)$$

$$\sqrt{-D} f(x) = \int_0^\infty e^{-ax} \frac{d\alpha}{a} f\left(\frac{1}{a}\right)$$

14. All equations of the form $y - m x^r \frac{d^n y}{d x^n} = X$ can be converted into equations in $\frac{d^r v}{d x^r}$.

For
$$y - m x^{r-n} (-1)^n \frac{\sqrt{-D+n}}{\sqrt{-D}} y = X$$

Let $y = \sqrt{-D} \cdot v$, then

$$\sqrt{-D} \cdot v - m x^{r-n} (-1)^n \sqrt{-D+n} v = X$$

or
$$v - m (-1)^n \frac{\sqrt{-D+n}}{\sqrt{-D}} e^{(r-n)\theta} v = (\sqrt{-D})^{-1} X$$

or
$$v - m (-1)^{-(r-n)} x^r \left(\frac{d}{dx} \right)^r x^{r-n} v = (\sqrt{-D})^{-1} X$$

or, if $x^{r-n} v = z$,

$$z - m (-1)^{-(r-n)} x^{2r-n} \frac{d^r z}{d x^r} = x^{r-n} (\sqrt{-D})^{-1} X$$

wherefore the above equation is reduced to an ordinary differential equation whatever be x , provided r is an integer.

As an example, let us take the simplest case, of which the solution will be found at pp. 257, 258 of my Memoir on General Differentiation, Part III.

Ex. 1.
$$y - a \sqrt{-1} x \frac{d^{\frac{1}{2}} y}{d x^{\frac{1}{2}}} = 0$$

Suppose $y = \sqrt{-D} v$,

then
$$v \sqrt{x} + a \sqrt{x} \frac{\sqrt{-D+1}}{\sqrt{-D}} (v \sqrt{x}) = C \sqrt{x}$$

or
$$z - a x^{\frac{3}{2}} \frac{dz}{dx} = C \sqrt{x}, \text{ where } z = v \sqrt{x};$$

$$\therefore z = e^{-\frac{2}{a\sqrt{x}}} \left\{ -\frac{A}{a} - \frac{C}{a} \int \frac{dx}{x} e^{\frac{2}{a\sqrt{x}}} \right\}$$

and
$$y = \sqrt{-D} \frac{e^{-\frac{2}{a\sqrt{x}}}}{\sqrt{x}} \left\{ -\frac{A}{a} - \frac{C}{a} \int \frac{dx}{x} e^{\frac{2}{a\sqrt{x}}} \right\} \dots (1.)$$

$$= \int_0^\infty e^{-px} \frac{dp}{\sqrt{p}} e^{-\frac{2\sqrt{p}}{a}} \left\{ -\frac{A}{a} + \frac{C}{a} \int \frac{dp}{p} e^{-\frac{2\sqrt{p}}{a}} \right\} \dots (2.)$$

The constants A and C are not both of them arbitrary. In fact, it will be seen that $C = \frac{A}{\sqrt{0}}$, so that the final form of equation (1.) is

$$y = -\frac{A}{a\sqrt{-D}} \frac{e^{-\frac{2}{a\sqrt{x}}}}{\sqrt{x}} \left\{ 1 + \frac{1}{\sqrt{0}} \int \frac{dx}{x} e^{\frac{2}{a\sqrt{x}}} \right\}$$

Ex. 2. To solve the equation

$$\frac{d y}{d x} - a \frac{d^{\frac{1}{2}} y}{d x^{\frac{1}{2}}} - \frac{1}{2} \frac{y}{x} = X.$$

1. By the method employed at p. 269 of my previous Memoir, this equation becomes

$$(-D + \tfrac{1}{2}) y + a (-1)^{\frac{1}{2}} \frac{\sqrt{-D+1}}{\sqrt{-D+\frac{1}{2}}} e^{\frac{\theta}{x}} y = -X x$$

or

$$\frac{y}{x} - a \frac{d^{-\frac{1}{2}} y}{d x^{-\frac{1}{2}}} \cdot \frac{y}{x} = \frac{1}{x} \cdot \frac{X x}{D - \frac{1}{2}}$$

$$= \frac{1}{\sqrt{x}} \int \frac{X dx}{x^{\frac{1}{2}}} \\ = P, \text{ suppose ;}$$

then

$$\frac{y}{x} = \frac{1}{\frac{d}{dx} - a^2} \left(\frac{d P}{d x} + a \frac{d^{\frac{1}{2}} P}{d x^{\frac{1}{2}}} \right)$$

$$y = A x e^{\alpha^2 x} + x e^{\alpha^2 x} \int e^{-\alpha^2 x} dx \left(\frac{d P}{d x} + a \frac{d^{\frac{1}{2}} P}{d x^{\frac{1}{2}}} \right)$$

2. By Professor BOOLE'S method, given in the *Philosophical Magazine* for February 1847. The given equation when written

$$x (d - a d^{\frac{1}{2}}) y - \tfrac{1}{2} y = X x ;$$

may be thrown into the form

$$f(d) (x F(d)) y = X x ; \text{ provided}$$

$$f(d) F(d) = d - a d^{\frac{1}{2}}, \text{ and}$$

$$f'(d) F(d) = -\tfrac{1}{2}.$$

$$\text{We have, therefore, } f(d) = (d^{\frac{1}{2}} - a)^{-1}$$

$$F(d) = (d^{\frac{1}{2}} - a)^2 d^{\frac{1}{2}}$$

and

$$y = d^{-\frac{1}{2}} (d^{\frac{1}{2}} - a)^{-2} \{x^{-1} (d^{\frac{1}{2}} - a) X x\}$$

3. By Mr HARGREAVE'S method, given in the *Philosophical Transactions* for 1848, p. 31.

By changing d into x , and x into $-d$ in the equation

$$x (d - a d^{\frac{1}{2}}) y - \tfrac{1}{2} y = X x \dots (1.)$$

it becomes

$$d (x - a x^{\frac{1}{2}}) y + \tfrac{1}{2} y = d X \dots (2.)$$

which, being an ordinary linear equation, gives, as its solution, if we write d^{-1} for $\int d x$,

$$y = x^{-\frac{1}{2}} (x^{\frac{1}{2}} - a)^{-2} d^{-1} \{ (x^{\frac{1}{2}} - a) d X \} \dots (3.)$$

Now, since equation (2.) has been derived from equation (1.) by the change of d into x , and x into $-d$, equation (1.) may be derived from equation (2.) by changing x into d and d into $-x$. Consequently (in some cases, at least, of which this form is an instance) the *solution* of equation (1.) may be derived from the *solution* of equation (2.) by the same change. Hence the solution of the given equation is

$$y = d^{-\frac{1}{2}} (d^{\frac{1}{2}} - a)^{-2} \{x^{-1} (d^{\frac{1}{2}} - a) x X\},$$

which is precisely the solution given by Professor BOOLE's process.

We shall conclude the present Memoir by comparing the methods here exhibited in two particular cases of the second example.

CASE 1. Let $X=0$; then the first method gives $y = A x e^{a^2 x}$: whilst the second and third methods give

$$y = d^{-\frac{1}{2}} (d^{\frac{1}{2}} - a)^{-2} \cdot 0.$$

Now, in a former paper (*Transactions of the Royal Society of Edinburgh*, vol. xiv., p. 252), I have shewn, in Cor. 1. to Example 3, that

$$(d^{\frac{1}{2}} - a)^{-2} \cdot 0 = A e^{a^2 x} + B x e^{a^2 x}$$

Hence

$$\begin{aligned} y &= d^{-\frac{1}{2}} \{A e^{a^2 x} + B x e^{a^2 x}\} \\ &= \frac{A}{a} e^{a^2 x} + \frac{B x}{a} e^{a^2 x} - \frac{B}{2a^3} e^{a^2 x} \end{aligned}$$

The condition $B = 2 A a^2$ reduces this solution to the former.

CASE 2. Let $X = b \left(\frac{1}{x^{\frac{3}{2}}} + \frac{a\sqrt{-1}}{\sqrt{\pi}} \frac{1}{x} \right)$;

then, by the first method,

$$\begin{aligned} P &= \frac{1}{\sqrt{x}} \int \frac{X dx}{\sqrt{x}} = \frac{b}{\sqrt{x}} \left\{ -\frac{1}{x} - \frac{2a\sqrt{-1}}{\sqrt{\pi}} \frac{1}{\sqrt{x}} \right\} \\ &= -\frac{b}{x^{\frac{3}{2}}} - \frac{2ab\sqrt{-1}}{\sqrt{\pi}} \frac{1}{x} \\ \frac{dP}{dx} + a \frac{d^{\frac{1}{2}}P}{dx^{\frac{1}{2}}} &= b \left(\frac{3}{2} \frac{1}{x^{\frac{5}{2}}} + \frac{a^2}{x^{\frac{3}{2}}} \right) \\ y &= A x e^{a^2 x} + b x e^{a^2 x} \int e^{-a^2 x} dx \left(\frac{3}{2} \frac{1}{x^{\frac{5}{2}}} + \frac{a^2}{x^{\frac{3}{2}}} \right) \\ &= A x e^{a^2 x} - \frac{b}{\sqrt{x}} \end{aligned}$$

By the second and third methods,

$$\frac{1}{x} (d^{\frac{1}{2}} - a) X x = b \left(\frac{\sqrt{-1}}{\sqrt{\pi}} \frac{1}{x^2} - \frac{a}{x^{\frac{3}{2}}} - \frac{a^2 \sqrt{-1}}{\sqrt{\pi} x} \right)$$

Also

$$d^{-\frac{1}{2}} (d^{\frac{1}{2}} - a)^{-2} = \frac{d^{\frac{1}{2}} + 2a + d^{-\frac{1}{2}}}{(d - a^2)^2}$$

$$\begin{aligned} \therefore y &= (d - a^2)^{-2} (d^{\frac{1}{2}} + 2a + d^{-\frac{1}{2}}) b \left(\frac{\sqrt{-1}}{\sqrt{\pi}} \frac{1}{x^2} - \frac{a}{x^{\frac{3}{2}}} - \frac{a^2 \sqrt{-1}}{\sqrt{\pi} x} \right) \\ &= (d - a^2)^{-2} b \left(-\frac{3}{4x^{\frac{5}{2}}} - \frac{a^2}{x^{\frac{3}{2}}} - \frac{a^4}{x^{\frac{1}{2}}} \right) \end{aligned}$$

an ordinary linear differential equation, of which the solution is

$$y = A x e^{a^2 x} + B e^{a^2 x} - \frac{b}{\sqrt{x}}.$$

This agrees with the former solution by making $B=0$.