vom Punkte $x_1 = x_2 = x_3 = 0$ aus bei Projection auf die Ebene $x_4 = 0$ darbietet. Ich sage nun, dass unser Γ einfach die Discriminante dieser Fläche dritter Ordnung ist, d. h., diejenige ganze rationale homogene Function 32^{ten} Grades der Coefficienten der Fläche, welche gleich Null gesetzt besagt, dass die Fläche einen Doppelpunct besitzt. In der That wird diese Discriminante so geschrieben werden können, dass sie als simultane Invariante der beiden ternären Formen

$$F = u_x \cdot \Phi - \Omega^2$$

und ϕ erscheint, vom zwölften Grade in den Coefficienten von F , vom achten Grade in den Coefficienten von Φ , genau wie es die Rücksicht auf die Homogenität in Formel (2) verlangt.

On the Reduction of a Complex Quadratic Surd to a Periodic Continued Fraction. By G. B. MATHEWS.

[Read April 11th, 1889.]

The square root of a complex integer may be reduced to a periodic chain-fraction by a process very similar to that employed for an ordinary quadratic surd. Dirichlet alludes to this, in passing, in his memoir on quadratic forms with complex coefficients (*Crelle*, t. xxiv.); but as he does not enter into any details with regard to the algorithm, and since no attempt appears to have been made to construct a complex "Canon Pellianus," the following table of results may be useful. In the column on the left will be found all the complex not-square integers $a + bi$, of which the norm does not exceed 100, and in which a, b are both positive; on the right-hand, opposite to $a + bi$, are entered the partial quotients belonging to the chain-fraction expansion of $\sqrt{a + bi}$. A semicolon has been placed before the first term of the period, so that, for instance, $2 + 2i$; $4, 4 + 4i$ means

$$2 + 2i + \frac{1}{4 + \frac{1}{(4 + 4i) + \frac{1}{4 + \frac{1}{(4 + 4i) + \&c.}}}}$$

Each stage of the process of reduction is of the type

$$z_n = \frac{E_n + \sqrt{A}}{D_n} = a_n + \frac{1}{z_{n+1}},$$

where a_n is a complex integer, determined so that

$$Nm(z_n - a_n) < \frac{1}{2}.$$

As in the ordinary theory (cf. Serret, *Algèbre Supérieure*, t. i., c. 2) all the quantities E_n , D_n are integers, and we have

$$E_n^2 + D_{n-1}D_n = A,$$

$$E_n + E_{n-1} = a_{n-1}D_{n-1}.$$

Inspection of the table suggests many curious theorems analogous to those of the ordinary theory: for instance, in all the cases examined the period begins with the second partial quotient; and, if a be the first quotient, then the last is $2a - e - e'i$, where e , e' are 0 or 1. It would be interesting to account for this independently of the theory of reduced quadratic forms; the analogy of the ordinary theory soon breaks down because the restriction that a , a_1 , a_2 ... are *positive* integers does not hold good.

It is not difficult, however to show that the expansion is really periodic: namely, it follows from the way in which a_{n-1} is determined that

$$Nm\left(\frac{E_{n-1} - a_{n-1}D_{n-1} + \sqrt{A}}{D_{n-1}}\right) < \frac{1}{2},$$

$$\text{i.e.,} \quad Nm(\sqrt{A} - E_n) < \frac{1}{2} Nm(D_{n-1}),$$

$$\text{therefore} \quad \text{mod.}(\sqrt{A} - E_n) < \frac{1}{\sqrt{2}} \text{mod.}(D_{n-1}),$$

$$\text{therefore} \quad \text{mod.}(\sqrt{A} + E_n) > \sqrt{2} \text{mod.}(D_n).$$

$$\text{Now, suppose} \quad \text{mod.}D_{n-1} < \mu \text{mod.}\sqrt{A};$$

$$\text{then} \quad \text{mod.}E_n < \text{mod.}\sqrt{A} + \text{mod.}(\sqrt{A} - E_n)$$

$$< \left(1 + \frac{\mu}{\sqrt{2}}\right) \text{mod.}\sqrt{A},$$

$$\text{and} \quad \sqrt{2} \text{mod.}D_n < \text{mod.}\sqrt{A} + \text{mod.}E_n$$

$$< \left(2 + \frac{\mu}{\sqrt{2}}\right) \text{mod.}\sqrt{A},$$

$$\text{therefore} \quad \text{mod.}D_n < \left(\frac{\mu}{2} + \sqrt{2}\right) \text{mod.}\sqrt{A}.$$

This agrees with the inequality supposed satisfied by $\text{mod.}D_{n-1}$, if

$$\mu = \frac{\mu}{2} + \sqrt{2},$$

or $\mu = 2\sqrt{2}.$

Hence, if $\text{mod. } D_{n-1} < 2\sqrt{2} \text{ mod. } \sqrt{A},$

it follows that $\text{mod. } D_n < 2\sqrt{2} \text{ mod. } \sqrt{A},$

$$\text{mod. } E_n < 3 \text{ mod. } \sqrt{A},$$

and $\text{mod. } a_n < \text{mod. } E_n + \text{mod. } E_{n-1} < 6 \text{ mod. } \sqrt{A}.$

We conclude, therefore, by induction, that the moduli of the quantities E_n, D_n, a_n cannot exceed certain finite limits, and hence the expansion must be periodic.

$1+i$	$1; -2i, 2$
$1+2i$	$1+i; 2+2i$
$1+3i$	$1+i; 2, 2+2i$
$1+4i$	$2+i; -2-i, 4+2i$
$1+5i$	$2+i; -1-2i, -4i, -1-2i, 4+2i$
$1+6i$	$2+2i; -1+2i, 2-i, -1+2i, 4+4i$
$1+7i$	$2+2i; 4i, 4+4i$
$1+8i$	$2+2i; 4+4i$
$1+9i$	$2+2i; 4, 4+4i$
$1+10i$	$2+2i; 3-i, -2, 3i, -2+3i, -1+3i, 2, 2-i, 4+4i$
$2+i$	$1; 1-i, 2$
$2+2i$	$2+i; -1+i, -3-i, 1-i, 3+i$
$2+3i$	$2+i; -3+i, 4+2i$
$2+4i$	$2+i; -4-2i, 4+2i$
$2+5i$	$2+i; -1-3i, 4+2i$
$2+6i$	$2+i; -2i, 4+2i$
$2+7i$	$2+2i; 1+2i, -1-3i, 1+2i, 4+4i$
$2+8i$	$2+2i; 2+2i, 4+4i$
$2+9i$	$2+2i; 2+i, 1+i, 2+i, 4+4i$
$3+i$	$2; -2-2i, 4$
$3+2i$	$2+i; -1+2i, 4+2i$
$3+3i$	$2+i; -2+4i, 4+2i$
$3+5i$	$2+i; 2-4i, 4+2i$
$3+6i$	$2+i; 1-2i, 4+2i$
$3+7i$	$2+2i; 1+i, -2i, -4, 1-2i, -1+i, -4-3i, -1-i, 2i, 4,$ $-1+2i, 1-i, 4+3i$

$3+8i$	$2+2i; 1+i, 2-i, -3-2i, 3-i, -1+i, -4-3i, -1-i,$ $-2+i, 3+2i, -3+i, 1-i, 4+3i$
$3+9i$	$2+2i; 2+i, -2+2i, 2+i, 4+4i$
$4+i$	$2; -4i, 4$
$4+2i$	$2+i; 2i, 4+2i$
$4+3i$	$2+i; 1+3i, 4+2i$
$4+4i$	$2+i; 4+2i$
$4+5i$	$2+i; 3-i, 4+2i$
$4+6i$	$2+i; 2-i, -2+2i, 2-i, 4+2i$
$4+7i$	$2+i; 1-i, 4+2i$
$4+8i$	$3+2i; -1+i, -5-3i, 1-i, 5+3i$
$4+9i$	$3+2i; -2+i, 2-2i, -1-i, -3+5i, 1+2i, 2+2i, -1+i,$ $-5-3i, 2-i, -2+2i, 1+i, 3-5i, -1-2i, -2-2i,$ $1-i, 5+3i$
$5+i$	$2; 2-2i, 4$
$5+2i$	$2; 1-2i, -3i, 1-2i, 4$
$5+3i$	$3+i; -1+i, -1+2i, 2-5i, -1-i, 2+i, 5+2i$
$5+4i$	$2+i; 2+i, 4+2i$
$5+5i$	$2+i; 2, 4+2i$
$5+6i$	$3+i; -2-i, 2-3i, -2-i, -6-2i, 2+i, -2+3i, 2+i,$ $6+2i$
$5+7i$	$2+i; 1-i, 2-2i, -3, 2+4i, -3, 2-2i, 1-i, 4+2i$
$5+8i$	$3+i; -1-i, 2i, 3-i, -1, -1-i, -2+i, -3i, 1-i, 5+2i$
$6+i$	$2; 2-i, -3-3i, 2-i, 4$
$6+2i$	$2; 1-i, 4$
$6+3i$	$3+i; -1+i, -5-i, 1-i, 5+i$
$6+4i$	$3+i; -2+i, 6+2i$
$6+5i$	$3+i; -3, 1-i, 1+i, 3-i, 1+2i, 2+i, -2, -5-2i,$ $3, -1+i, -1-i, -3+i, -1-2i, -2-i, 2, 5+2i$
$6+6i$	$3+i; -3-i, 6+2i$
$6+7i$	$3+i; -2-2i, 6+2i$
$7+i$	$3; -2-i, -2+2i, -2-i, 6$
$7+2i$	$3; -1-2i, -1-i, 2i, 1-2i, 1-i, 5$
$7+3i$	$3+i; -1+2i, 2i, -2-3i, -1-3i, 1-i, 5+i$
$7+4i$	$3+i; -2+2i, 6+2i$
$7+5i$	$3+i; -4+2i, 6+2i$
$7+6i$	$3+i; -6-2i, 6+2i$
$7+7i$	$3+i; -2-4i, 6+2i$

$8+i$	$3; -3-3i, 6$
$8+2i$	$3; -1-2i, 2i, -1-2i, 6$
$8+3i$	$3+i; -1+2i, 2, 1-6i, -2-i, -2i, 6+i$
$8+4i$	$3+i; -1+3i, 6+2i$
$8+5i$	$3+i; -2+6i, 6+2i$
$9+i$	$3; -6i, 6$
$9+2i$	$3; -3i, 6$
$9+3i$	$3; -2i, 6$
$9+4i$	$3+i; 1+3i, -2+i, 4+i, -2+i, 1+3i, 6+2i$
$10+i$	$3; 3-3i, 6$

Construction du centre de courbure de la développée de la courbe de contour apparent d'une surface que l'on projette orthogonalement sur un plan. By Prof. A. MANNHEIM.

[Read April 11th, 1889.]

Dans une communication sur les surfaces parallèles, que j'ai eu l'honneur de faire à la Société Mathématique de Londres,* je suis arrivé à une expression du rayon de courbure de la développée de la courbe de contour apparent d'une surface. La recherche de cette expression est un de ces problèmes dont la solution dépend des infiniment petits du 3^e ordre et que, le premier, j'ai traités géométriquement.

Dans la même communication, j'ai fait connaître aussi une autre expression de ce même rayon de courbure, et j'ai annoncé que je montrerais comment on peut l'obtenir directement.

C'est cette démonstration directe que je vais exposer aujourd'hui, en faisant usage de deux droites dont j'ai déjà eu plusieurs fois l'occasion de prouver l'utilité.†

Il s'agit donc moins ici du problème particulier que je traite dans

* Voir les *Proceedings*, Vol. XII., No. 177.

† Voir *Comptes rendus de l'Académie des Sciences*, Séances des 22 Mars 1875 et 6 Mars 1876.