X.—*The Hessian of a General Determinant.*

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(1) There being n^2 independent variables in a general n -line determinant, the Hessian of the determinant with respect to the said variables must be a determinant with n^2 lines: and as a general n -line determinant has all its terms linear in the elements involved, it follows that each element of the Hessian being a second differential-quotient cannot be of a higher degree in the variables than the $(n-2)^{\text{th}}$, and that consequently the degree of the Hessian itself cannot exceed $n^2(n-2)$. The object of the present short paper is to show that this degree is attained by the $n(n-2)^{\text{th}}$ power of the given determinant being a factor of the Hessian.

(2) Beginning with the case of $n = 3$, let the originating determinant be

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

so that single differentiation with respect to the nine elements produces

$$\begin{array}{ccc|ccc} |b_2c_3|, & -|b_1c_3|, & |b_1c_2| & A_1, & A_2, & A_3 \\ -|a_2c_3|, & |a_1c_3|, & -|a_1c_2| & B_1, & B_2, & B_3 \\ |a_2b_3|, & -|a_1b_3|, & |a_1b_2| & C_1, & C_2, & C_3 \end{array} \quad \text{or} \quad \begin{array}{ccc} A_1, & A_2, & A_3 \\ B_1, & B_2, & B_3 \\ C_1, & C_2, & C_3 \end{array} \quad \text{say.}$$

These being each differentiated in the same way, *i.e.* with respect to each of the nine original elements, we obtain the eighty-one elements of the Hessian: and if we agree to take the independent variables in the order, $a_1, a_2, a_3; b_1, b_2, b_3; c_1, c_2, c_3$, the Hessian itself will be

$$\begin{vmatrix} \cdot & \cdot & \cdot & \cdot & c_3 & -c_2 & \cdot & -b_3 & b_2 \\ \cdot & \cdot & \cdot & -c_3 & \cdot & c_1 & b_3 & \cdot & -b_1 \\ \cdot & \cdot & \cdot & c_2 & -c_1 & \cdot & -b_2 & b_1 & \cdot \\ \cdot & -c_3 & c_2 & \cdot & \cdot & \cdot & \cdot & a_3 & -a_2 \\ c_3 & \cdot & -c_1 & \cdot & \cdot & \cdot & -a_3 & \cdot & a_1 \\ -c_2 & c_1 & \cdot & \cdot & \cdot & \cdot & a_2 & -a_1 & \cdot \\ \cdot & b_3 & -b_2 & \cdot & -a_3 & a_2 & \cdot & \cdot & \cdot \\ -b_3 & \cdot & b_1 & a_3 & \cdot & -a_1 & \cdot & \cdot & \cdot \\ b_2 & -b_1 & \cdot & -a_2 & a_1 & \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$

By way of remembering its constitution we should think of it as divided into three equal parts by two left-to-right lines, and at the same time into three equal parts by

two top-to-bottom lines. The nine small square arrays thus resulting may then be symbolized by

$$\begin{array}{ccc} \cdot & \gamma & -\beta \\ -\gamma & \cdot & a \\ \beta & -a & \cdot \end{array},$$

each of the three-line matrices a , β , γ being zero-axial skew, and composed of elements from only one row of the original determinant.

Now if we multiply this by $|a_1 b_2 c_3|^3$ in the form

$$\begin{vmatrix} a_1 & a_2 & a_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_1 & b_2 & b_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_1 & c_2 & c_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & a_1 & a_2 & a_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & b_1 & b_2 & b_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & c_1 & c_2 & c_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_1 & a_2 & a_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_1 & b_2 & b_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_1 & c_2 & c_3 \end{vmatrix},$$

we obtain

$$\begin{vmatrix} \cdot & \cdot & \cdot & -B_1 & A_1 & \cdot & -C_1 & \cdot & A_1 \\ \cdot & \cdot & \cdot & -B_2 & A_2 & \cdot & -C_2 & \cdot & A_2 \\ \cdot & \cdot & \cdot & -B_3 & A_3 & \cdot & -C_3 & \cdot & A_3 \\ B_1 & -A_1 & \cdot & \cdot & \cdot & \cdot & \cdot & -C_1 & B_1 \\ B_2 & -A_2 & \cdot & \cdot & \cdot & \cdot & \cdot & -C_2 & B_2 \\ B_3 & -A_3 & \cdot & \cdot & \cdot & \cdot & \cdot & -C_3 & B_3 \\ C_1 & \cdot & -A_1 & \cdot & C_1 & -B_1 & \cdot & \cdot & \cdot \\ C_2 & \cdot & -A_2 & \cdot & C_2 & -B_2 & \cdot & \cdot & \cdot \\ C_3 & \cdot & -A_3 & \cdot & C_3 & -B_3 & \cdot & \cdot & \cdot \end{vmatrix},$$

which if divided like the multiplicand into nine three-line minors takes the form

$$\begin{array}{ccc} \cdot & \gamma' & \beta' \\ -\gamma' & \cdot & a' \\ -\beta' & -a' & \cdot \end{array}.$$

It is better however to write the three zero minors differently, viz.—

$$\begin{array}{ccc} A_1 - A_1 & \cdot & \cdot & B_1 - B_1 & \cdot & \cdot & C_1 - C_1 \\ A_2 - A_2 & \cdot & \cdot & B_2 - B_2 & \cdot & \cdot & C_2 - C_2 \\ A_3 - A_3 & \cdot & \cdot & B_3 - B_3 & \cdot & \cdot & C_3 - C_3 \end{array},$$

for then we are able to state more easily the law of formation of the nine. Thus, we may then say that the three

$$\cdot \quad \gamma' \quad \beta'$$

have $-A_1, -A_2, -A_3; -B_1, -B_2, -B_3; -C_1, -C_2, -C_3$ respectively in their first columns, and A_1, A_2, A_3 in a column of each, the particular column being the first of the first, the second of the second, and the third of the third. Similarly, the next three

$$-\gamma' \quad \cdot \quad a'$$

have $-A_1, -A_2, -A_3; -B_1, -B_2, -B_3; -C_1, -C_2, -C_3$ respectively in their second columns, and B_1, B_2, B_3 , in a column of each; and so in the remaining case. The product-determinant has thus two kinds of columns, viz., those with three non-zero elements, and those with six. Of the former kind there are six, and of the latter three. Further, when there are only three non-zero elements in a column they are all negative, and when there are six they are all positive. Adding together the three columns which have six non-zero elements, we can remove the factor 2 from the resulting column: and if we then diminish the two other columns by this altered column, we shall change them into columns with only three non-zero elements like the rest. Our result then will take the form

$$2 \begin{vmatrix} A_1 & . & . & -B_1 & . & . & -C_1 & . & . \\ A_2 & . & . & -B_2 & . & . & -C_2 & . & . \\ A_3 & . & . & -B_3 & . & . & -C_3 & . & . \\ B_1 & -A_1 & . & . & -B_1 & . & . & -C_1 & . \\ B_2 & -A_2 & . & . & -B_2 & . & . & -C_2 & . \\ B_3 & -A_3 & . & . & -B_3 & . & . & -C_3 & . \\ C_1 & . & -A_1 & . & . & -B_1 & . & . & -C_1 \\ C_2 & . & -A_2 & . & . & -B_2 & . & . & -C_2 \\ C_3 & . & -A_3 & . & . & -B_3 & . & . & -C_3 \end{vmatrix}$$

which is easily changed further into

$$(-)^3 2 \begin{vmatrix} A_1 & B_1 & C_1 & . & . & . & . & . & . \\ A_2 & B_2 & C_2 & . & . & . & . & . & . \\ A_3 & B_3 & C_3 & . & . & . & . & . & . \\ B_1 & . & . & A_1 & -B_1 & C_1 & . & . & . \\ B_2 & . & . & A_2 & -B_2 & C_2 & . & . & . \\ B_3 & . & . & A_3 & -B_3 & C_3 & . & . & . \\ C_1 & . & . & . & . & . & A_1 & B_1 & -C_1 \\ C_2 & . & . & . & . & . & A_2 & B_2 & -C_2 \\ C_3 & . & . & . & . & . & A_3 & B_3 & -C_3 \end{vmatrix}$$

by interchanging the 2nd, 3rd, 6th columns with the 4th, 7th, 8th respectively. We are thus led to the equation

$$H(|a_1 b_2 c_3|) \cdot |a_1 b_2 c_3|^3 = (-)^5 2 |A_1 B_2 C_3|^3,$$

and therefore finally to

$$H(|a_1 b_2 c_3|) = -2 |a_1 b_2 c_3|^3.$$

(3) Taking next the case of $n = 4$, the originating determinant being

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix},$$

we obtain, from single differentiation with respect to the sixteen elements,

$$\begin{array}{cccc}
|b_2c_3d_4|, & -|b_1c_3d_4|, & |b_1c_2d_4|, & -|b_1c_2d_3|, \\
-|a_2c_3d_4|, & |a_1c_3d_4|, & -|a_1c_2d_4|, & |a_1c_2d_3|, \\
|a_2b_3d_4|, & -|a_1b_3d_4|, & |a_1b_2d_4|, & -|a_1b_2d_3|, \\
-|a_2b_3c_4|, & |a_1b_3c_4|, & -|a_1b_2c_4|, & |a_1b_2c_3|,
\end{array}$$

or say

$$\begin{array}{cccc}
A_1 & A_2 & A_3 & A_4 \\
B_1 & B_2 & B_3 & B_4 \\
C_1 & C_2 & C_3 & C_4 \\
D_1 & D_2 & D_3 & D_4.
\end{array}$$

These being each differentiated in the same way, *i.e.* with respect to each of the sixteen original elements, we obtain the 256 elements of the Hessian; and find the latter, when partitioned into 4×4 minor matrices of the 4th order, to be of the form

$$\begin{array}{cccc}
. & (\gamma, \delta) & -(\beta, \delta) & (\beta, \gamma) \\
-(\gamma, \delta) & . & (a, \delta) & -(a, \gamma) \\
(\beta, \delta) & -(a, \delta) & . & (a, \beta) \\
-(\beta, \gamma) & (a, \gamma) & -(a, \beta) & .
\end{array}$$

where (a, β) , (a, γ) , . . . are all exactly alike,—for example

$$(a, \beta) = \begin{pmatrix} . & |a_3b_4| & -|a_2b_4| & |a_2b_3| \\ -|a_3b_4| & . & |a_1b_4| & -|a_1b_3| \\ |a_2b_4| & -|a_1b_4| & . & |a_1b_2| \\ -|a_2b_3| & |a_1b_3| & -|a_1b_2| & . \end{pmatrix},$$

—being thus all zero-axial skew like their parent matrix.*

Multiplying the Hessian by $|a_1b_2c_3d_4|^4$ we obtain as before a product which, when partitioned like the multiplicand, is of the form

$$\begin{array}{cccc}
. & (\beta_1a_2)' & (\gamma_1a_3)' & (\delta_1a_4)' \\
-(\beta_1a_2)' & . & (\gamma_2\beta_3)' & (\delta_2\beta_4)' \\
-(\gamma_1a_3)' & -(\gamma_2\beta_3)' & . & (\delta_3\gamma_4)' \\
-(\delta_1a_4)' & -(\delta_2\beta_4)' & -(\delta_3\gamma_4)' & .
\end{array}$$

where

$$(\beta_1a_2)' = \begin{pmatrix} -B_1 & A_1 & . & . \\ -B_2 & A_2 & . & . \\ -B_3 & A_3 & . & . \\ -B_4 & A_4 & . & . \end{pmatrix}$$

the suffix of a indicating the column in which the positive A's appear, and the suffix of β the column in which the negative B's appear. Any one of the four lines of matrices, say the line

$$. \quad (\beta_1a_2)' \quad (\gamma_1a_3)' \quad (\delta_1a_4)',$$

may consequently be described in language exactly similar to that employed in the case

* Note, too, that the determinant of every matrix vanishes, being, in the case exemplified, the square of the vanishing Pfaffian

$$|a_1b_2| |a_3b_4| - |a_1b_3| |a_2b_4| + |a_1b_4| |a_2b_3|.$$

where $n = 3$; and thus the product determinant comes to be transformable into 3 times a determinant whose first four columns are

$$\begin{array}{cccc} A_1 & . & . & . \\ A_2 & . & . & . \\ A_3 & . & . & . \\ A_4 & . & . & . \\ . & -A_1 & . & . \\ . & -A_2 & . & . \\ . & -A_3 & . & . \\ . & -A_4 & . & . \\ . & . & -A_1 & . \\ . & . & -A_2 & . \\ . & . & -A_3 & . \\ . & . & -A_4 & . \\ . & . & . & -A_1 \\ . & . & . & -A_2 \\ . & . & . & -A_3 \\ . & . & . & -A_4 \end{array}$$

these being followed by four columns containing B's similarly placed but all negative, these again by four columns containing negative C's, and these by four columns containing negative D's. Interchanging columns as before, with the necessary accompaniment of $(3 + 2 + 1) + 3$ changes of sign, we obtain

$$\begin{aligned} H(|a_1 b_2 c_3 d_4|) \cdot |a_1 b_2 c_3 d_4|^4 &= (-)^{9 \cdot 3} |A_1 B_2 C_3 D_4|^4, \\ &= (-)^{9 \cdot 3} |a_1 b_2 c_3 d_4|^{12}, \end{aligned}$$

and therefore

$$H(|a_1 b_2 c_3 d_4|) = -3 |a_1 b_2 c_3 d_4|^8.$$

(4) The natural generalisation for an originating determinant of order n is thus fully legitimised, the number of requisite changes of sign then being

$$\{(n-1) + (n-2) + \dots + 2 + 1\} + (n-1),$$

which of course is the same as

$$(n-2) + (n-3) + \dots + 2 + 1,$$

i.e.,

$$\frac{1}{2}(n-1)(n-2):$$

and the resulting equation is

$$H(|a_{1n}|) \cdot |a_{1n}|^n = (-)^{\frac{1}{2}(n-1)(n-2)} \cdot (n-1) \cdot |A_{1n}|^n,$$

so that

$$\begin{aligned} H(|a_{1n}|) &= (-)^{\frac{1}{2}(n-1)(n-2)} \cdot (n-1) \cdot |a_{1n}|^{n(n-1)-n}, \\ &= (-1)^{\frac{1}{2}(n-1)(n-2)} \cdot (n-1) \cdot |a_{1n}|^{n(n-2)}. \end{aligned}$$