XXXIX. *On the General Extension of Fourier's Theorem.*

By THOMAS PRESTON, M.A., F.R.U.I.*

IN a recent number of the Philosophical Magazine Dr. G. J. Stoney† has announced the following important optical theorem in relation to microscopic vision:—“*However complex the contents of the objective field, and whether it or parts of it be self-luminous or illuminated in any way, however special, the light which emanates from it may be resolved into undulations each of which consists of uniform plane waves.*”

In this theorem it is explicitly stated as a general principle that any disturbance however complex within a given region of space may be resolved into a system of plane wave components, or that a system of trains of plane waves can be determined such that when compounded they will reproduce any arbitrarily specified disturbance throughout a given region.

When the disturbance is a function of a single variable this statement forms the verbal expression of Fourier's Theorem, viz. that any arbitrary function of a variable can be expressed as a series of sines or cosines of multiples of that variable, between assigned limits ; and when the disturbance is a function of two or more variables the theorem in its most general form implies that any function of any number of variables can be expanded in a series, each term of which can be expressed in terms of a linear function of the variables.

Since the time of Fourier it has been customary to represent a general function of any number of variables by a series each term of which is a continued product of cosines of multiples of the variables ; thus

$$f(x, y, z) = \Sigma A \cos lx \cos my \cos nz. \quad . \quad . \quad . \quad (1)$$

This form in fact was used by Fourier in his investigation of “the movement of heat in a solid cube.” Now if the exponential values of $\cos lx$ &c. be substituted for them, it is seen at once that the product of the cosines $\cos lx \cos my \cos nz$ resolves itself into terms of the type $\cos (lx \pm my \pm nz)$, and consequently the form of expansion (1) in the case of any number of variables is equivalent to

$$f(x, y, z, \dots) = \Sigma A \cos (lx + my + nz + \dots) ;$$

* Communicated by the Author.

† Phil. Mag. vol. xlii. p. 332 &c. (1896), and vol. xliii. p. 139 (1897).

which is the analytical expression of the foregoing general statement when the variables are taken to be x, y, z, t , namely the coordinates of a point in space and the time t .

In order to investigate this expansion we return to the case of a single variable and write Fourier's expansion in the form

$$\begin{aligned}\pi f(x) &= \Sigma \cos lx \int_0^{2\pi} f(\alpha) \cos \alpha \, d\alpha + \Sigma \sin lx \int_0^{2\pi} f(\alpha) \sin \alpha \, d\alpha \\ &= \Sigma \int_0^{2\pi} f(\alpha) \cos l(x-\alpha) \, d\alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)\end{aligned}$$

where l is any whole number positive or negative.

Now, if in (2) we replace $f(x)$ by a function of two variables $f(x, y)$, we have

$$\pi f(x, y) = \Sigma \int_0^{2\pi} f(\alpha, y) \cos l(x-\alpha) \, d\alpha; \quad . \quad . \quad (3)$$

and if $f(\alpha, y)$ in the right-hand member of (3) be replaced by its expansion in a Fourier series of cosines and sines of multiples of y , as denoted in equation (2), we have

$$\pi^2 f(x, y) = \Sigma \Sigma \int_0^{2\pi} \int_0^{2\pi} f(\alpha, \beta) \cos l(x-\alpha) \cos m(y-\beta) \, d\alpha \, d\beta \quad . \quad . \quad . \quad (4)$$

where l and m are any whole numbers. Similarly the equation for any number of variables may be written down at once. Since the general term of (4) is of the form $Ae^{i(lx+my)}$, it is clear that any function of any number of variables can be expanded in the form

$$f(x, y, \dots) = \Sigma A \cos (lx + my + \dots) + \Sigma B \sin (lx + my + \dots); \quad . \quad . \quad . \quad (5)$$

where l, m , &c. are any whole numbers positive or negative, and the coefficients A, B , &c. are given by (4).

Thus, in the case of two variables the coefficients of $\cos (lx + my)$ and $\sin (lx + my)$ are respectively

$$A_{lm} = \frac{2}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} f(\alpha, \beta) \cos (l\alpha + m\beta) \, d\alpha \, d\beta, \quad . \quad . \quad (6)$$

$$B_{lm} = \frac{2}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} f(\alpha, \beta) \sin (l\alpha + m\beta) \, d\alpha \, d\beta; \quad . \quad . \quad (7)$$

and similarly in the case of n variables we have

$$A_{lm\dots} = \frac{2}{(2\pi)^n} \int_0^{2\pi} f(\alpha, \beta, \gamma \dots) \cos(l\alpha + m\beta + n\gamma + \dots) d\alpha d\beta d\gamma \dots \quad (8)$$

$$B_{lm\dots} = \frac{2}{(2\pi)^n} \int_0^{2\pi} f(\alpha, \beta, \gamma \dots) \sin(l\alpha + m\beta + n\gamma + \dots) d\alpha d\beta d\gamma \dots \quad (9)$$

These values of the coefficients may also be obtained directly if we assume the expansion to be expressed in the form (5). For if we multiply both sides of (5) by $\cos(lx + my + \dots)$, and integrate between the limits zero and 2π with respect to each variable, we obtain at once

$$\int_0^{2\pi} f(x, y, \dots) \cos(lx + my + \dots) dx dy \dots = \frac{1}{2} A (2\pi)^n,$$

which agrees with (8).

When the variables are taken to be the coordinates x, y, z of a point in space and the time t , we have the form

$$f(x, y, z, t) = \Sigma A \cos(px + qy + rz + st) \\ + \Sigma B \sin(px + qy + rz + st),$$

or the equivalent form

$$f(x, y, z, t) = \Sigma a \cos(px + qy + rz + st + \alpha),$$

where a and α are constants expressible in terms of the definite integrals A and B by the equations

$$a = (A^2 + B^2)^{\frac{1}{2}}, \quad \tan \alpha = -B/A.$$

This, then, is the analytical expression of the general theorem enunciated by Dr. Stoney, and means that any disturbance whatsoever in a medium either uniform or heterogeneous can be resolved into systems of trains of plane waves.

That some such resolution should be possible is not surprising when the matter is regarded from the physical standpoint. For two trains of plane waves traversing space in different directions will produce illumination of a simple grating type on a screen placed perpendicular to the plane of the wave normals; that is, the pattern on the screen consists of parallel bars, and the amplitude as we pass across a bar varies according to the cosine law. If therefore any pattern, however complex, can be built up by superposing various simple grating patterns, it follows that any disturbance within a given region can be resolved into plane waves, and that any function of x, y, z, t can be resolved into a sum of cosines or sines of

linear functions of the variables. It is not difficult to admit that a very complex pattern may be built up by superposing simple-grating patterns, but that any arbitrary pattern whatsoever may be so constituted requires demonstration.

When the disturbance is periodic throughout space the expansion in cosines or sines will represent the function completely throughout all space, and when the function is not periodic the expansion can be made to agree with the function at all points within assigned limits.

In the case of a periodic function the expansion may be derived from the condition of periodicity. Thus if $\lambda, \mu, \nu, \dots$ be the periodic intervals and $l, m, n, \dots$ any whole numbers, we have by the condition of periodicity,

$$f(x, y, z, \dots) = f(x + l\lambda, y + m\mu, z + n\nu, \dots). \quad (10)$$

If we denote the operator $\frac{d}{dx}$ by D_1 , and $\frac{d}{dy}$ by D_2 , &c., we have by applying Taylor's theorem to (10),

$$f(x, y, z, \dots) = e^{l\lambda D_1 + m\mu D_2 + n\nu D_3} f(x, y, z, \dots),$$

that is*,

$$(e^{l\lambda D_1 + m\mu D_2 + \dots} - 1)f(x, y, z, \dots) = 0. \quad (11)$$

Now any differential equation of the form

$$F(D_1, D_2, D_3, \dots)f(x, y, z, \dots) = 0 \quad (12)$$

will be satisfied by $f = e^{ax + by + cz + \dots}$ provided a, b, c , &c. satisfy the equation

$$F(a, b, c, \dots) = 0. \quad (13)$$

Hence equation (11) will be satisfied by a sum of terms of the form

$$f(x, y, z, \dots) = \sum A e^{ax + by + cz + \dots},$$

provided a, b, c satisfy the equation

$$e^{l\lambda a + m\mu b + n\nu c + \dots} - 1 = 0.$$

That is $a, b, c, \dots$ must satisfy the equation

$$l\lambda a + m\mu b + n\nu c + \dots = 2i\pi r,$$

where r is any whole number. Thus in the case of three

* This method was employed by Mr. J. O'Kinealy, for the case of a single variable (Phil. Mag. August 1874, pp. 95, 96.)

variables, the quantities ia, ib, ic may be the coordinates of any point on any one of the system of planes

$$l\xi + m\mu\eta + n\nu\zeta = 2r\pi,$$

in which l, m, n , and r are any whole numbers, positive or negative. The general expansion is consequently of the form

$$f(x, y, z, \dots) = \sum A \cos (ax + by + cz + \dots).$$

In the particular, or rather *extreme*, case in which the function f has a given value at each of a system of equally-spaced points and has zero value at all other points of space, we have the disturbance contemplated in the lemma used by Dr. Stoney as the foundation of the demonstration sketched in his first paper*.

It is interesting to notice that any function of an odd number of variables can be expanded in a series of sines or a series of cosines of linear functions of multiples of the variables, as in the case of a single variable, the limits of integration being zero and π . On the other hand, when there is an even number of variables the limits of integration require to be zero and 2π .

In the case in which l, m, n , &c. are not whole numbers the function may still be expanded in the form of a series of sines, or a series of cosines ; thus,

$$f(x, y, \dots) = \sum A \sin (lx + my + \dots),$$

where l, m , &c. are chosen so as to satisfy an equation of the form

$$\frac{\cos \{ (l+l')x + (m+m')y \dots \}}{\sin \{ (l+l')x + (m+m')y \dots \}} = \frac{\cos \{ (l-l')x + (m-m')y \dots \}}{\sin \{ (l-l')x + (m-m')y \dots \}},$$

and the coefficients A &c. are determined by definite integrals of the form (8) or (9). In the case of a single variable this equation reduces to

$$\frac{\tan lx}{lx} = \text{constant},$$

the form obtained by Fourier.

* *Loc. cit.*