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439. The Definition of a Complex Number

Author(s): G. H. Hardy

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437. [K. 2. d. (vol. vi. p. 153, No. 349.)] Through Q , the centre of the inscribed ellipse, draw $Mm \parallel$ the $\perp^r AK$ to meet in m a tangent $blm \parallel$ the base BC , and also to meet in N a line bisecting the sides. Then $KA = 2MN$; and as Q is midway between the tangents $Kk = 2MQ$, $\therefore Ak = 2QN$.

Now the $\triangle^s Abs$ and ASC are similar (see No. 2 of the *Gazette*, or p. 311 of No. 73).

$$\therefore As \cdot AS = Ab \cdot AC = Ak \cdot 2R \text{ from similar right-angled triangles}$$

$$= 4R \cdot QN. \quad \text{Q.E.D.}$$

H. P. ROUSE.

438. [J. 2. c.] *The St. Petersburg Problem.*

A man is to throw a coin until he throws head. If he throws head at the n th throw, and not before, he is to receive $\pounds 2^n$.

What is the value of his expectation? The paradox that the value of his expectation is infinite has caused an extensive literature, but I have not seen the following very obvious remark:

He is charged $\pounds 1$ for the (even) chance of getting $\pounds 2$ at his first throw, $\pounds 1$ for his chance of getting $\pounds 4$ at his second throw, and $\pounds 1$ for each possibility. Consequently it is not worth his while to pay more than n pounds to any person who could not pay him $\pounds 2^n$. The value of the chance is determined by the capital of the opponent.

C. S. JACKSON.

439. [I. 5. a.] *The Definition of a Complex Number.*

It is generally recognised by now that a "complex number" $a + ib$ is merely a symbolic equivalent for a pair of real numbers (a, b) .

Writers of elementary text-books have sometimes attempted to introduce complex numbers somewhat as follows.

Let $O(a)$ denote the result of an operation performed on a real number a , and such that

$$O\{O(a)\} = -a. \quad \dots\dots\dots(1)$$

Then ia is identified with $O(a)$.

This procedure is open to a multitude of objections. For one thing, $O(a)$ is *ex hypothesi* an operation on a real number. If then $O(a)$ is not itself real, what is $O\{O(a)\}$? But there is an even more fatal objection, and the revival of the procedure in question in a recent book (a book too of some merit) leads me to think that it may be worth while to develop the objection in detail.

The objection is that the property (1) does not suffice to define any unique

operation. Let us suppose, for example, that O , operating on any complex number $a+ib$ or (a, b) , produces

$$O(a, b) = (\lambda a + \mu b, \lambda' a + \mu' b),$$

where $\lambda, \mu, \lambda', \mu'$ are real.

Then

$$O\{O(a, b)\} = (A, B),$$

where

$$A = \lambda(\lambda a + \mu b) + \mu(\lambda' a + \mu' b),$$

$$B = \lambda'(\lambda a + \mu b) + \mu'(\lambda' a + \mu' b).$$

It will easily be verified that we shall have $A = -a, B = -b$, if the two conditions

$$\lambda + \mu' = 0, \quad \lambda^2 + \lambda'\mu = \lambda'\mu + \mu'^2 = -1$$

are satisfied. We can satisfy these equations by taking

$$\lambda = \sinh \theta, \quad \mu = -\rho \cosh \theta, \quad \lambda' = \frac{\cosh \theta}{\rho}, \quad \mu' = -\sinh \theta,$$

θ and ρ being any real numbers; so that

$$O(a, b) = \left(a \sinh \theta - b \rho \cosh \theta, \frac{a \cosh \theta}{\rho} - b \sinh \theta \right).$$

In particular the operation O , when performed on a real number a , produces the complex number

$$a \sinh \theta + \frac{ia \cosh \theta}{\rho}.$$

The special case required is of course that in which $\theta = 0, \rho = 1$; when

$$O(a, b) = (-b, a), \quad O(a) = ia. \quad \text{G. H. HARDY.}$$

440. [K¹. 7. a.] To find the condition that the four straight lines represented by

$$ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4 = 0$$

shall form a harmonic pencil.

The lines joining the origin to the intersections of the straight lines $l_1x + m_1y - 1 = 0$ and $l_2x + m_2y - 1 = 0$ with the parabola $y^2 = x$ have for their equation

$$\{y^2 - x(l_1x + m_1y)\} \cdot \{y^2 - x(l_2x + m_2y)\} = 0.$$

These lines form a harmonic pencil if $m_1m_2 + 2(l_1 + l_2) = 0$; that is to say, if the coefficient of y^2 in $(l_1x + m_1y - 1)(l_2x + m_2y - 1) = 0$ is twice the coefficient of x .

The given equation may be written

$$ax^2 + 4bxy + 2\lambda y^2 + 2(3c - \lambda)x(y^2/x) + 4dy(y^2/x) + e(y^4/x^2) = 0,$$

and therefore represents the straight lines joining the origin to the points of intersection of the parabola $y^2 = x$ with the conic

$$ax^2 + 4bxy + 2\lambda y^2 + 2(3c - \lambda)x + 4dy + e = 0.$$

This conic will represent two straight lines if

$$\begin{vmatrix} a & 2b & 3c - \lambda \\ 2b & 2\lambda & 2d \\ 3c - \lambda & 2d & e \end{vmatrix} = 0,$$

and moreover, the joins of the origin to its intersections with $y^2 = x$ will form a harmonic pencil if (as above) $2\lambda = 12c - 4\lambda$ or $\lambda = 2c$.

Thus

$$\begin{vmatrix} a & 2b & c \\ 2b & 4c & 2d \\ c & 2d & e \end{vmatrix} = 4 \begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix} = 0,$$

which is the required condition.

R. F. DAVIS.

441. [A.] (Vol. vii. p. 48, No. 391.) It may be worth noting that the symbol $\simeq$, which is suggested by C. F. Boon (*Gazette*, No. 103, January 1913, p. 48) as an abbreviation for "approximately equal to," has been already so used. It occurs in Ernst Gottfried Fischer's *Lehrbuch der Elementar-Mathematik, vierter Theil:—Anfangsgründe der Algebra*. Berlin and Leipzig, 1829, p. 147 ff.

R. C. ARCHIBALD.