Thursday, February 13th, 1890.

J. J. WALKER, Esq., F.R.S., President, in the Chair.

The following communications were made:—

Concerning Semi-Invariants: Mr. S. Roberts.

Ether-Squirts: Prof. K. Pearson.

On Class-Invariants: Prof. G. B. Mathews.

Note on the Imaginary Roots of an Equation: Prof. Cayley.

The following presents were received:—

"Proceedings of the Royal Society," Vol. XLVI., No. 284.

"Educational Times" for February.

"The Elements of Astronomy, with a Uranography," by Dr. C. A. Young; 8vo | Boston and London, 1890.

"The Lightning Calculator," by C. H. B. de Botwor; 8vo pamphlet; London.

"Proceedings of the Canadian Institute," No. 152 (whole Vol. XXV.); Toronto.

"Bulletin des Sciences Mathématiques," Tome XIV., Jan. 1890.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. V., Fasc. 7-10 | Roma, 1889.

"Journal für die reine und angewandte Mathematik," Band CVI., Heft 1.

"Beiblätter zu den Annalen der Physik und Chemie," Band XIII., Stück 12 | Band XIV., Stück 1.

"Memorias de la Sociedad Científica 'Antonio Alzate,'" Tomo II., No. 12 | Tomo III., Nos. 1 and 2; Mexico.

"Bollettino delle Pubblicazioni Italiane ricevute per Diritto di Stampa," Nos. 95, 97, 98.

"Annali di Matematica," Tomo XVII^o, Fasc. 3^o.

"Rendiconti del Circolo Matematico di Palermo," Tomo III., Fasc. 6; Nov., Dic., 1889.

"Tidsskrift for Mathematik," Række V., Aargang 6; Hefte 1-7; Kjobenhavn, 1888.

"Nyt Tidsskrift for Mathematik," Aargang 1; Nr. 1, A; Aargang 1, Nr. 1, B.

"Festschrift der Mathematischen Gesellschaft in Hamburg, 1890," Teil II. and III. (Katalog).

G. Lejeune Dirichlet's Werke—Erster Band; 4to (Edited by L. Kronecker); Berlin, 1889.

"Journal of the College of Science," Imperial University, Japan; Vol. III., Part 3; Tokyo.

Concerning Semi-Invariants. By SAMUEL ROBERTS, F.R.S.

[*Read Feb. 13th, 1890.*]

1. I remarked in a former paper (*Quarterly Journal of Mathematics*, Vol. VII., p. 17, 1865), that if we expand the product

$$\begin{aligned} & \left\{ a_0 + a_1(x-y) + a_2 \frac{(x-y)^2}{1.2} + a_3 \frac{(x-y)^3}{1.2.3} + \dots \right\} \\ & \times \left\{ a_0 + a_1(y-z) + a_2 \frac{(y-z)^2}{1.2} + a_3 \frac{(y-z)^3}{1.2.3} + \dots \right\} \\ & \times \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ & \times \left\{ a_0 + a_1(u-v) + a_2 \frac{(u-v)^2}{1.2} + a_3 \frac{(u-v)^3}{1.2.3} + \dots \right\} \\ & \times \left\{ a_0 + a_1(v-x) + a_2 \frac{(v-x)^2}{1.2} + a_3 \frac{(v-x)^3}{1.2.3} + \dots \right\}, \end{aligned}$$

the coefficients of powers and products of powers of $x, y, z, \dots u, v$ are the leading coefficients of certain covariants of quantics, to a numerical factor *près*. These quantics are of the form

$$(a_0, a_1, a_2 \dots a_p, \chi(x, y))^p;$$

i.e., the literal coefficients are affected with binomial multipliers.

In the place cited, a proof of the general theorem is not explicitly given. The line of demonstration is, however, indicated in the case of the product of two indefinitely extended series,

$$\left(a_0 + a_1x + \frac{a_2x^2}{1.2} + \frac{a_3x^3}{1.2.3} + \dots\right) \left(a_0 - a_1x + \frac{a_2x^2}{1.2} - \frac{a_3x^3}{1.2.3} + \dots\right) \dots (a).$$

If Π_1, Π_2 denote the two operators

$$\left(a_1 \frac{d}{da_0} + a_2 \frac{d}{da_1} + a_3 \frac{d}{da_2} + \dots\right)$$

and

$$\left(a'_1 \frac{d}{da'_0} + a'_2 \frac{d}{da'_1} + a'_3 \frac{d}{da'_2} + \dots\right),$$

then $(\Pi_1 - \Pi_2)^p a_0 a'_0$ is a semi-invariant* annulled by the operator

$$\left(a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + 3a_2 \frac{d}{da_3} + \dots\right) + \left(a'_0 \frac{d}{da'_1} + 2a'_1 \frac{d}{da'_2} + 3a'_2 \frac{d}{da'_3} + \dots\right).$$

* When a "semi-invariant" is mentioned, it is supposed to belong to binary forms with binomial coefficients. The literal coefficients, as a_0, a_1, a_2 , &c., are often styled "elements."

The product (a) is symbolically expressed by

$$e^{x(\Pi_1 - \Pi_k)} a_0 a'_0,$$

the accents being removed after operation. It is implied that the general result was obtained by means of the same artifice from the semi-invariant

$$[(x-y)\Pi_1 + (y-z)\Pi_2 + \dots + (u-v)\Pi_{k-1} + (v-x)\Pi_k]^p a_0 a'_0 \dots a^{(k-1)'} \dots \dots \dots (b),$$

where Π_i operates on a set of a 's with i accents, and corresponds in form to Π_1 . It is apparent, by taking a limited form, as

$$(x-y)\Pi_1 + (y-z)\Pi_2 + (z-x)\Pi_3,$$

for the general one, that (b) is annulled by the operator

$$\Sigma \left(a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + 3a_2 \frac{d}{da_3} + \dots \right),$$

the summation extending to all the sets of elements.

Mr. Hammond has informed me that a result closely connected with (b) has been given by Mr. Kempe. In fact, since (b) may be written

$$[x(\Pi_1 - \Pi_k) + y(\Pi_2 - \Pi_1) + \dots \dots \dots + u(\Pi_{k-1} - \Pi_{k-2}) + v(\Pi_k - \Pi_{k-1})]^p a_0 a'_0 \dots a_0^{(k-1)'},$$

and $x, y, z, \dots u, v$ are indeterminate, it is plain that

$$(\Pi_1 - \Pi_k)^m (\Pi_2 - \Pi_1)^n \dots (\Pi_k - \Pi_{k-1})^t a_0 a'_0 \dots a_0^{(k-1)'}$$

is a semi-invariant, and this is Mr. Kempe's form. It is a particular case of a more general one. For we may write for (b)

$$(X_1 \Pi_1 + X_2 \Pi_2 + \dots + X_k \Pi_k)^p a_0 a'_0 \dots a_0^{(k-1)'},$$

subject to $X_1 + X_2 + \dots + X_k = 0$.

Now, let $(a_{11}, a_{12}, \dots a_{1k}), (a_{21}, a_{22}, \dots a_{2k}), \dots \dots (a_{k1}, a_{k2}, \dots a_{kk})$

be k sets of numbers such that

$$\sum_{p=k}^{p-1} a_{1p} = \sum_{p=k}^{p-1} a_{2p} = \dots = \sum_{p=k}^{p-1} a_{kp} = 0;$$

$$\begin{aligned}
& \text{then } Y_1(a_{11}\Pi_1 + a_{12}\Pi_2 + \dots + a_{1k}\Pi_k) + Y_2(a_{21}\Pi_1 + a_{22}\Pi_2 + \dots + a_{2k}\Pi_k) + \dots \\
& \quad \dots + Y_k(a_{k1}\Pi_1 + a_{k2}\Pi_2 + \dots + a_{kk}\Pi_k) \\
& = (Y_1a_{11} + Y_2a_{21} + \dots + Y_ka_{k1})\Pi_1 + (Y_1a_{12} + Y_2a_{22} + \dots + Y_ka_{k2})\Pi_2 + \dots \\
& \quad \dots + (Y_1a_{1k} + Y_2a_{2k} + \dots + Y_ka_{kk})\Pi_k,
\end{aligned}$$

and the sum of the coefficients of $\Pi_1, \Pi_2, \dots, \Pi_k$ in the last expression is zero by hypothesis. Consequently,

$$\begin{aligned}
& (a_{11}\Pi_1 + a_{12}\Pi_2 + \dots + a_{1k}\Pi_k)^m (a_{21}\Pi_1 + a_{22}\Pi_2 + \dots + a_{2k}\Pi_k)^n \dots \\
& \quad \dots (a_{k1}\Pi_1 + a_{k2}\Pi_2 + \dots + a_{kk}\Pi_k)^r a_0 a_0' \dots a_0^{(k-1)r}
\end{aligned}$$

is a semi-invariant.

With reference again to the theorem first before given, it is, on the face of it, a very particular case, but I did not dwell on it at the time. I propose now to investigate some general theorems which it suggests. It will be unnecessary in the present connection to repeat the phrase "to a numerical factor *près*."

Let u, u_1, u_2 , &c. represent the indefinitely extended function

$$a_0 + a_1 y X + \frac{a_2 y^2 X^2}{1 \cdot 2} + \frac{a_3 y^3 X^3}{1 \cdot 2 \cdot 3} + \dots$$

(which I hereafter call "elementary"), and its successive derived functions with respect to X . Also, let Φ denote the semi-invariant annihilator

$$a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + 3a_2 \frac{d}{da_3} + \dots + ia_{i-1} \frac{d}{da_i} + \dots,$$

continued indefinitely. We have immediately, and by differentiation with respect to X , since Φ and $\frac{d}{dX}$ are commutative,

$$\left. \begin{aligned}
\Phi u &= yXu \\
\Phi u_1 &= yXu_1 + yu \\
\Phi u_2 &= yXu_2 + 2yu_1 \\
\dots &\dots \dots \dots \\
\Phi u_m &= yXu_m + myu_{m-1}
\end{aligned} \right\} \dots\dots\dots (A).$$

2. Consider F , an integral homogeneous function of any number k of elementary sub-functions of the type of u , and of limited series of their successive derived functions relative to X , beginning in each

case with the first. We may represent such a group of functions of yX by $(u, u_1, u_2, \dots)_k$, the literal coefficients of powers of yX being different in each set corresponding to u, u_1, u_2 , &c. The operator Φ must now be of the extended form

$$\Sigma \left(a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + 3a_2 \frac{d}{da_3} + \dots \right)_k,$$

the sign of summation covering the whole group of elements, and F , of the degree μ in the elementary sub-functions, will stand for

$$F[(u, u_1, u_2, \dots)_k].$$

It must always be borne in mind that the sub-functions are indefinitely continued.

By means of (A), we obtain

$$\begin{aligned} \Phi F &= \Sigma \left(\Phi u \frac{d}{du} + \Phi u_1 \frac{d}{du_1} + \Phi u_2 \frac{d}{du_2} + \dots \right)_k F \\ &= yX \Sigma \left(u \frac{d}{du} + u_1 \frac{d}{du_1} + u_2 \frac{d}{du_2} + \dots \right)_k F \\ &\quad + y \Sigma \left(u \frac{d}{du_1} + 2u_1 \frac{d}{du_2} + 3u_2 \frac{d}{du_3} + \dots \right)_k F. \end{aligned}$$

If, therefore, F is of the form of a semi-invariant as to the u 's considered as elements, and is annulled by the operator

$$\Sigma \left(u \frac{d}{du_1} + 2u_1 \frac{d}{du_2} + 3u_2 \frac{d}{du_3} + \dots \right)_k,$$

or, what is the same thing, if the leading term of F is isobaric, and a semi-invariant annulled by Φ , the result is, immediately and by differentiation with respect to X ,

$$\left. \begin{aligned} \Phi F &= y\mu XF, \\ \Phi F_1 &= y\mu XF_1 + y\mu F, \\ \Phi F_2 &= y\mu XF_2 + 2y\mu F_1, \\ \dots &\dots \dots \dots \\ \Phi F_m &= y\mu XF_m + my\mu F_{m-1}, \end{aligned} \right\} \dots \dots \dots (B),$$

where F_1, F_2 , &c. are the successive derived functions of F with respect to X .

Let the development of F , according to ascending powers of X , be

$$A_0 + A_1 X + \frac{A_2 X^2}{1 \cdot 2} + \dots + \frac{A_r X^r}{r!} + \dots,$$

and let Π denote the operator

$$\Sigma \left(a_1 \frac{d}{da_0} + a_2 \frac{d}{da_1} + a_3 \frac{d}{da_2} + \dots \right)_k;$$

then

$$A_r = y^r \Pi^r A_0,$$

and the function F may be compendiously represented by

$$e^{\nu_{XII}} A_{0,*}$$

3. Next, suppose $\mathbf{F}$ to be an integral homogeneous function of any number l of sub-functions of the type of F , and of limited series of their successive derived functions (beginning with the first) with respect to X , but depending on different groups of elementary sub-functions.

Also, each sub-function of the type F may have a different μ and a different y . The operator Φ is to be extended to all the literal coefficients of the elementary sub-functions, or, as we may say, to all the elements.

Then we have, analogously,

$$\begin{aligned}\Phi \mathbf{F} &= \Sigma \left(\Phi F \frac{d}{dF} + \Phi F_1 \frac{d}{dF_1} + \Phi_2 F_2 \frac{d}{dF_2} + \dots \right)_i \mathbf{F} \\ &= X \Sigma \left[\gamma \mu \left(F \frac{d}{dF} + F_1 \frac{d}{dF_1} + F_2 \frac{d}{dF_2} + \dots \right)_i \right] \mathbf{F} \\ &\quad + \Sigma \left[\gamma \mu \left(F \frac{d}{dF} + 2F_1 \frac{d}{dF_1} + 3F_2 \frac{d}{dF_2} + \dots \right)_i \right] \mathbf{F}.\end{aligned}$$

If in this expression we make throughout the summations $y_\mu = 1$, &c., and assume that $\mathbf{F}$ is a semi-invariant as to $(F, F_1, \text{&c.})_n$, considered as elements, and is annulled by the operator

$$\Sigma \left(F \frac{d}{dF_1} + 2F_1 \frac{d}{dF_2} + 3F_2 \frac{d}{dF_3} + \dots \right),$$

we have immediately, and by differentiation with respect to X ,

$$\left. \begin{aligned} \Phi \mathbf{F} &= \pi X \mathbf{F}, \\ \Phi \mathbf{F}_1 &= \pi X \mathbf{F}_1 + \pi \mathbf{F}, \\ \Phi \mathbf{F}_2 &= \pi X \mathbf{F}_2 + 2\pi \mathbf{F}_1, \\ \dots &\quad \dots \quad \dots \\ \Phi \mathbf{F}_m &= \pi X \mathbf{F}_m + m\pi \mathbf{F}_{m-1}. \end{aligned} \right\} \dots\dots\dots (C),$$

* *Quarterly Journal of Mathematics*, Vol. iv., "On Arbogast's Calculus of Derivations," p. 195; communicated July 1860, printed February 1861.

where π is the degree of $\mathbf{F}$ in the sub-functions of the type F , and their derived functions with respect to X , and $\mathbf{F}_1, \mathbf{F}_2$, &c., $\mathbf{F}_m$ are the successive derived functions of $\mathbf{F}$ with respect to X .

We might proceed to treat in a similar manner, and with similar result, a function of sub-functions of the type of $\mathbf{F}$, and of limited series of their successive derived functions with respect to X , and so on indefinitely.

4. The relation which a function of the type $\mathbf{F}$ bears to one of the type F , when $y = 1$, may be exhibited in the following manner. Since, as we have seen, an elementary sub-function of the type u is represented, when $y = 1$, by $e^{X\Pi} a_0$, and a sub-function of the type F is represented, when $y = \mu^{-1}$, by $e^{X\Pi/\mu} A_0$, where A_0 is a semi-invariant relative to k groups of a -elements, and is annulled by Φ , we may write, for a function F ,

$$\psi \left[(e^{X\Pi} a_0, e^{X\Pi} a_1, e^{X\Pi} a_2, \dots)_k \right] \dots\dots\dots (c);$$

and for a function $\mathbf{F}$,

$$\psi \left[\left(e^{X\Pi/\mu} A_0, e^{X\Pi/\mu} \frac{A_1}{\mu}, e^{X\Pi/\mu} \frac{A_2}{\mu^2}, \dots \right)_k \right] \dots\dots\dots (d);$$

that is to say, if in (c) we change a_0 into A_0 , a_1 into $\frac{A_1}{\mu}$, a_2 into $\frac{A_2}{\mu^2}$, and so forth, we obtain (d).

Further, let $\mathbf{F}$ become $\mathbf{F}_0$ when X is made equal to zero. From the first equation of (C) we get

$$\Phi \mathbf{F}_0 = 0 \dots\dots\dots (D),$$

and $\mathbf{F}_0$ stands for

$$\mathbf{F} \left[\left(A_0, \frac{\Pi A_0}{\mu}, \frac{\Pi^2 A_0}{\mu^2}, \dots \frac{\Pi^i A_0}{\mu^i}, \dots \right)_i \right],$$

the corresponding substitutions being made for all the leading terms of the sub-functions of the type F , and for those of their successive derived functions. But in the condition

$$\Sigma \left(F \frac{d}{dF_1} + 2F_1 \frac{d}{dF_2} + 3F_2 \frac{d}{dF_3} + \dots \right), \mathbf{F} = 0,$$

we may write any letters b, b_1, b_2, b_3 , &c. for F, F_1, F_2, F_3 , &c.; for these F 's are here regarded in the light of elements of a binary form. The result (D) may therefore be interpreted as follows:—If in any semi-invariant of any system of independent binary forms involving

the elements $b_0, b_1, b_2, \&c., b'_0, b'_1, b'_2, \&c.,$ and so forth, we substitute, for $b_0, b_1, b_2, \&c., A_0, \frac{\Pi A_0}{\mu}, \frac{\Pi^2 A_0}{\mu^2}, \&c.,$ and for $b'_0, b'_1, b'_2, \&c., A'_0, \frac{\Pi A'_0}{\mu_1}, \frac{\Pi^2 A'_0}{\mu_1^2}, \&c.,$ and so forth, $A_0, A'_0, \&c.$ being semi-invariants of the degrees $\mu, \mu_1, \&c.$ relative to the elements (represented by a 's) of any systems of independent binary forms, and Π being the operator

$$\Sigma \left(a_1 \frac{d}{da_0} + a_2 \frac{d}{da_1} + a_3 \frac{d}{da_2} + \dots \right),$$

where the summation extends to all the systems of elements; the result will be a semi-invariant of all the substituted systems of forms.*

As a simple example, take the semi-invariant $b_0 b'_2 - 2b_1 b'_1 + b_2 b'_0,$ and substitute therein (a_0, a_1, a_2) for $(b_0, b_1, b_2);$ and

$$\left(a_0 a_2 - a_1^2, \frac{a_0 a_3 - a_1 a_2}{2}, \frac{a_0 a_4 - a_1^2}{4} \right) \text{ for } (b'_0, b'_1, b'_2).$$

We have as the result,

$$\frac{1}{4} a_0 (a_0 a_4 - 4a_1 a_3 + 3a_2^2),$$

which, multiplied by 4, is the source of the covariant found by the product of a quintic, and the S of its quartic emanant; or, considering a_0 as the source of a quartic, the other factor is the S of the quartic. Of course, a semi-invariant becomes an invariant if the order of the covariant is zero.

5. I proceed now to theorems, directly analogous to the one quoted at the beginning of § 1.

Take any number l of functions $v, w, \dots z$ of the pattern of $u,$ only writing $X_1, X_2, \dots X_l$ for X therein, and making $y = 1.$ Then, by (A),

$$\Phi u . v . w \dots z = (X + X_1 + \dots + X_l) u . v . w \dots z = 0,$$

if $X + X_1 + \dots + X_l = 0.$

This is the theorem quoted in § 1, since we may write X for $x - y,$ X_1 for $y - z, \dots X_l$ for $v - x.$

We obtain a more general result by applying similar considerations to (B), putting $y = 1$ and xX for $X.$ We have, subject to the conditions prescribed,

$$\Phi F = x\mu XF.$$

* This is equivalent to "Théorème V." enunciated by M. Ferrin (*Comptes Rendus*, 1887, pp. 1258-1260).

Take any number l of functions $U, V, W, \dots Z$ of the pattern of F , and involving elementary sub-functions of xX, x_1X, x_2X , &c., with different coefficients, and limited series of their successive derived functions (beginning in each case with the first) with respect to xX, x_1X, x_2X , &c.). Let the degrees of homogeneity of U, V, W , &c. Z be, respectively, $\mu, \mu_1, \mu_2, \dots \mu_{l-1}$; then

$$\Phi U \cdot V \cdot W \dots Z = 0 \dots\dots\dots (E),$$

provided that $x\mu + x_1\mu_1 + \dots + x_{l-1}\mu_{l-1} = 0$,

and Φ be adapted to all the systems of elementary sub-functions involved.

$$\text{Since} \quad \Phi U^a = a U^{a-1} \Phi U,$$

we may write for (E),

$$\Phi U^a \cdot V^{a_1} \cdot W^{a_2} \dots Z^{a_{l-1}} = 0 \dots\dots\dots (F),$$

provided that $x\mu a + x_1\mu_1 a_1 + x_2\mu_2 a_2 + \dots + x_{l-1}\mu_{l-1} a_{l-1} = 0$.

This is really only another rendering of (E); because, if U is of the degree of homogeneity μ , U^a will be of the degree of homogeneity $a\mu$. But in the last form we can have $x, x_1, \dots x_{l-1}$, $a, a_1, a_2, \dots a_{l-1}$ negative or fractional.

When such is the case, however, we have still to develop according to ascending powers of X , and the coefficients of the expansion will consequently be affected with denominators consisting of powers of the leading terms of the negative powers expanded, and, if the powers expanded are fractional, with factors consisting of corresponding fractional powers of the leading terms. In the case of negative indices, the numerators of the coefficients, after reduction to a common denominator, will be integral semi-invariants. In the case of fractional indices, the irrational factor or factors may be removed. An easy example will, perhaps, make this clearer than description in words.

Consider the product of the indefinitely extended series

$$\left(a_0 + a_1 xX + a_2 \frac{x^2 X^2}{1 \cdot 2} + \dots \right) \left(b_0 + b_1 x_1 X + b_2 \frac{x_1^2 X^2}{1 \cdot 2} + \dots \right)^a,$$

where $b_0 = a_0 a_2 - a_1^2$, $b_1 = a_0 a_3 - a_1 a_2$, $b_2 = a_0 a_4 - a_2^2$, &c.

Here, since b_0 is of the second degree in the a 's, we must have

$$ax + 2/3x_1 = 0,$$

or, say, $x = 2\beta, \quad x_1 = -\alpha.$

The coefficient of X in the product is

$$a_0^* b_0^e \left\{ 2\alpha\beta \frac{a_1}{a_0} - \alpha\beta \frac{b_0}{b_0} \right\},$$

or $- \alpha\beta a_0^{e-1} (a_0 a_2 - a_1^2)^{e-1} \{ a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^2 \}.$

The coefficient of X^2 is

$$a_0^* b_0^e \left\{ -2\alpha^2 \beta^2 \frac{a_1 b_1}{a_0 b_0} + \frac{4\alpha\beta^2}{1 \cdot 2} \frac{a_2}{a_0} + \frac{\alpha^2 \beta}{1 \cdot 2} \frac{b_2}{b_0} + \frac{\beta(\beta-1)}{1 \cdot 2} \alpha^2 \frac{b_1^2}{b_0^2} \right. \\ \left. + \frac{\alpha(\alpha-1)}{1 \cdot 2} 4\beta^2 \frac{a_1^2}{a_0^2} \right\},$$

giving

$$\alpha\beta \frac{a_0^{e-2} (a_0 a_2 - a_1^2)^{e-2}}{2} \{ \alpha\beta (a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^2)^2 + 4\beta (a_0 a_2 - a_1^2)^2 \\ + \alpha a_0^3 [a_0 a_2 a_4 + 2a_1 a_2 a_3 - a_0 a_3^2 - a_1^2 a_4 - a_2^2] \}.$$

In these two cases, then, we can remove the factors with indices depending on α and β . The remaining factors, when, if necessary, a proper numerical factor has been applied, become rational and integer semi-invariants.

6. To illustrate (E), let $w_0, w'_0, w''_0, \dots w_0^{(k)}$ be semi-invariants of any systems of binary forms whose elements are $a_0, a_1, a_2, \&c.$; $a'_0, a'_1, a'_2, \&c.$; and so on.

Let Π , as before, be the developing operator

$$\Sigma \left(a_1 \frac{d}{da_0} + a_2 \frac{d}{da_1} + a_3 \frac{d}{da_2} + \dots \right).$$

The equation (E) shows that

$$e^{x_0 X \Pi} w_0, \quad e^{x_1 X \Pi} w'_0, \quad e^{x_2 X \Pi} w''_0, \dots e^{x_k X \Pi} w_0^{(k)}, \dots \dots \dots (G)$$

is a semi-invariant, if

$$\mu_0 x_0 + \mu_1 x_1 + \mu_2 x_2 + \dots + \mu_k x_k = 0,$$

where $\mu_0, \mu_1, \dots \mu_k$ are the degrees of $w_0, w'_0, \dots w_0^{(k)}$, respectively, in the elements.

If the general operator Π be broken up into portions, viz., Π_0 operating exclusively on w_0 , Π_1 operating on w'_0 exclusively, and

generally Π_p operating on $w^{(p)'} exclusively, the coefficient of $\frac{X^m}{m!}$, viz.,$

$$(x_0 \Pi_0 + x_1 \Pi_1 + x_2 \Pi_2 + \dots + x_k \Pi_k)^m w_0 w'_0 \dots w_0^{(k)'},$$

is a semi-invariant.

If $k = 1$, we may put $x_0 = \mu_1$, $x_1 = -\mu_0$,*

and the coefficient becomes

$$(\mu_1 \Pi_0 - \mu_0 \Pi_1)^m w_0 w'.$$

This is the extension of M. D'Ocagne's theorem given by M. Perrin (*Comptes Rendus*, 1887, pp. 1097-1099, 1258-1260; see also "Développements sur la théorie des formes binaires," par M. J. Deruyts, *Bull. de l'Académie royale de Belgique*, 3me série, t. xiv., p. 65).

If $w_0, w'_0, \dots w_0^{(k)'}$ be identically the same, and w_i be written for $\Pi_i w_0$, the coefficient of X^m is

$$\begin{aligned} w_0^k \frac{w_m}{m!} \Sigma x_0^m + w_0^{k-1} w_1 \frac{w_{m-1}}{m-1!} \Sigma x_0^{m-1} + w_0^{k-1} \frac{w_2}{2!} \frac{w_{m-2}}{m-2!} \Sigma x_0^2 x_1^{m-2} \\ + w_0^{k-2} w_1^2 \frac{w_{m-2}}{m-2!} \Sigma x_0 x_1 x_2^{m-1} + \&c., \end{aligned}$$

each term containing a symmetric function of $x_0 x_1 \dots x_k$ as a factor. If we denote $\Sigma x_0^\alpha x_1^\beta \dots x_r^\gamma$, where $\alpha, \beta, \dots \rho$ are different integers, by $[a\beta\gamma \dots \rho]$, and if any, as α, β, γ , be equal, by $[\alpha^s \dots \rho]$, the coefficient may be symbolically expressed by

$$\Sigma \left[\left(\frac{w_\alpha}{\alpha!} \right)^p \left(\frac{w_\beta}{\beta!} \right)^q \left(\frac{w_\gamma}{\gamma!} \right)^r \dots \right] [\alpha^p \beta^q \gamma^r \dots],$$

with $ap + \beta q + \gamma r + \dots = m$, $p + q + r + \dots = k + 1$,

and one of the numbers $\alpha, \beta, \gamma, \dots$ may be zero, and $0! = 1$, as usual in such cases.

Since $x_0 + x_1 + \dots + x_k = 0$,

* Like the expression (c) in § 1,

$$\begin{aligned} \left(\frac{\alpha_{00}}{\mu} \Pi_0 + \frac{\alpha_{01}}{\mu_1} \Pi_1 + \dots + \frac{\alpha_{0k}}{\mu_k} \Pi_k \right)^m \left(\frac{\alpha_{10}}{\mu} \Pi_0 + \frac{\alpha_{11}}{\mu_1} \Pi_1 + \dots + \frac{\alpha_{1k}}{\mu_k} \Pi_k \right)^n \dots \\ \dots \left(\frac{\alpha_{k0}}{\mu} \Pi_0 + \frac{\alpha_{k1}}{\mu_1} \Pi_1 + \dots + \frac{\alpha_{kk}}{\mu_k} \Pi_k \right)^s w_0 w'_0 \dots w_0^{(k)}, \end{aligned}$$

is a semi-invariant, if $\sum_{p=k}^{p=0} \alpha_{0p} = \sum_{p=k}^{p=0} \alpha_{1p} = \dots = \sum_{p=k}^{p=0} \alpha_{kp} = 0$.

we may substitute for the symmetric functions their values in terms of the coefficients of

$$y^{k+1} + K_2 y^{k-1} - K_3 y^{k-2} + \dots + (-)^{k+1} K_{k+1} = 0,$$

having as roots $x_0, x_1, \dots x_k$.

If, still supposing $w_0, w'_0, \dots w_0^{(k)}$ identically the same, we divide each factor of the product (G) by w_0 , and put generally

$$\left(1 + \frac{w_1}{w_0} x_p X + \frac{w_2}{w_0} \frac{x_p^2 X^2}{1 \cdot 2} + \dots\right) = (1 - \lambda x_p X)(1 - \lambda' x_p X)(1 - \lambda'' x_p X) \dots,$$

we may select, from each of the expressions corresponding to the factors of (G), the factor corresponding to $1 - \lambda x_p X$.

Multiplying these factors together, we get

$$1 + L_2 \lambda^2 X^2 - L_3 \lambda^3 X^3 + L_4 \lambda^4 X^4 - \&c. + (-)^{k+1} L_{k+1} \lambda^{k+1} X^{k+1},$$

and (G) becomes, after division by w_0^{k+1} , an infinite product of factors of the above form. The coefficients of the powers of X in the expansion will therefore be non-unitary symmetric functions of the roots $\lambda, \lambda', \lambda'', \&c$.

As a very particular case of (G), we may take the function

$$\left(a_0 + a_1 X + \frac{a_2 X^2}{1 \cdot 2} + \dots\right)^k \left(a_0 - a_1 kX + a_2 \frac{k^2 X^2}{1 \cdot 2} - \dots\right).$$

The coefficient of X^m here is

$$a_0 A_m - a_1 k A_{m-1} + a_2 \frac{k^2}{1 \cdot 2} A_{m-2} - \dots + (-)^m a_m A_0,$$

where $A_m, A_{m-1}, \dots$ are the coefficients of $X^m, X^{m-1}, \dots$ in the expansion of the k^{th} power. Since this is also a case of (F), we are at liberty to make k negative or fractional, in which case the coefficients will be affected with powers of a_0 .

If $w_0, w'_0, \dots w_0^{(k)}$ are semi-invariants as to one set of elements $a_0, a_1, a_2, \&c$, it follows that

$$e^{X\Pi} \frac{w_0 w'_0 \dots w_0^{(k)}}{a_0^{(\mu_0 + \mu_1 + \dots + \mu_k)}}$$

is a semi-invariant, and the coefficients of the different powers of X are so. Here $w_0, w'_0, \dots w_0^{(k)}$ are not assumed to be of the same degree

in the elements. This, however, is not really more general than

$$e^{X\Pi} \frac{w_0}{a_0''},$$

since the product of semi-invariants is a semi-invariant.

7. Let us now take any number l of functions $\mathbf{U}, \mathbf{V}, \mathbf{W}, \dots \mathbf{Z}$ of the type $\mathbf{F}$, and let their degrees in the F functions, and in limited series of their derived functions with respect to X , be respectively $\pi_0, \pi_1, \dots \pi_{l-1}$. Also, write for X , $x_0 X$ in $\mathbf{U}$, $x_1 X$ in $\mathbf{V}$, and so on, and let Φ be adapted to all the elementary sub-functions involved; then, by means of (C), we obtain

$$\Phi \mathbf{U}^{a_0} \cdot \mathbf{V}^{a_1} \cdot \mathbf{W}^{a_2} \dots \mathbf{Z}^{a_{l-1}} = 0 \dots \dots \dots (\text{H}),$$

subject to $x_0 a_0 \pi_0 + x_1 a_1 \pi_1 + x_2 a_2 \pi_2 + \dots + x_{l-1} a_{l-1} \pi_{l-1} = 0$.

In this result, as in the analogous case (F'), we may make $x_0, x_1, \dots x_{l-1}$, $a_0, a_1, \dots a_{l-1}$ negative or fractional, subject to considerations similar to those mentioned in regard to (F'). When the product affected by Φ is expanded according to powers of X , the coefficients of those powers are semi-invariants relative to the systems of elements involved.

It seems unnecessary to write down more general expressions analogous to (F) and (H), for the last is already so general that the simplest examples are laborious.

8. There is a special case to be noted, which does not quite fall in with the general results.

Let U be a function of the type F , and of the form

$$A_0 + A_1 X + \frac{A_2 X^2}{1 \cdot 2} + \dots \text{ad inf.},$$

where A_0 is a semi-invariant of any system of binary forms, and

$$A_r = \Pi^r A_0.$$

Then

$$\Phi \log U = \frac{\Phi U}{U} = \mu X \dots \dots \dots (\text{K}),$$

μ being the degree of U in the elements; and from this it appears that the coefficients of powers of X higher than the first in the expansion of $\log U$ are semi-invariants. Similarly,

$$\Phi \frac{U_1}{U} = \mu,$$

showing that the coefficients of the powers of X , after that of X^0 (or the constant term), in the expansion of $\frac{U_1}{U}$, are semi-invariants. But this is, of course, a direct inference from (K).

$$\text{Since} \quad \log U = \log A_0 + \log \left(1 + \frac{A_1}{A_0} X + \frac{A_2}{A_0} \frac{X^2}{1 \cdot 2} + \dots \right),$$

it follows that the sums of the powers of the reciprocal roots of $U=0$ (higher than the first) are semi-invariants, and Major Macmahon's inference that any integral symmetric function of the reciprocal of the roots of

$$1 + bx + \frac{cx^2}{1 \cdot 2} + \dots = 0,$$

in which the indices of the roots are greater than unity, is a semi-invariant, holds for $U=0$, subject to a power of A_0 appearing as a denominator. That is to say, such symmetrical functions of the reciprocal roots of $U=0$ are semi-invariants not only with regard to A_0, A_1, A_2 , &c., considered as elements, but with regard to the elements involved in the A 's.

9. If, instead of an elementary function U , we take

$$v = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots,$$

indefinitely continued, the coefficients being deprived of the exponential multipliers, the coefficient of x^{m+1} in v^{-m} is a semi-invariant, relative to the elements a_0, a_1, a_2 , &c.

For, operating with

$$\Phi = a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + 3a_2 \frac{d}{da_3} + \dots$$

on v^{-m} , we get

$$-m(a_0 x + 2a_1 x^2 + 3a_2 x^3 + \dots) v^{-m-1} = -mx v^{-m} + x^2 \frac{d}{dx} v^{-m},$$

and the coefficient of x^{m+1} in this is

$$(\text{co. of } x^m) \text{ in } -mv^{-m} + (\text{co. of } x^{m-1}) \text{ in } x^2 \frac{d}{dx} v^{-m} = 0,$$

so that Φ annuls the coefficient of x^{m+1} in v^{-m} .

Now, if we revert the series (indefinitely continued)

$$z = a_0 x + a_1 x^2 + a_2 x^3 + \dots,$$

we have, putting Ω for $a_1 \frac{d}{da_0} + 2a_2 \frac{d}{da_1} + 3a_3 \frac{d}{da_2} + \dots$,

$$x = a_0^{-1} z + \frac{1}{2} \Omega a_0^{-2} z^2 + \frac{1}{3} \frac{\Omega^2 a_0^{-3}}{1.2} z^3 + \frac{1}{4} \frac{\Omega^3 a_0^{-4}}{1.2.3} z^4 + \dots,$$

$$x^{-1} = a_0 z^{-1} + \frac{a_1}{a_0} - \frac{\Omega^2 a_0^{-1}}{1.2} z - \frac{1}{2} \frac{\Omega^3 a_0^{-2}}{1.2.3} z^2 - \frac{1}{3} \frac{\Omega^4 a_0^{-3}}{1.2.3.4} z^3 - \dots.*$$

But $\frac{\Omega^{m+1} a_0^{-m}}{m+1!}$ is the coefficient of x^{m+1} in v^{-m} , and therefore is a semi-invariant as to the elements a_0, a_1, a_2 , &c. Hence in the value of x^{-1} in terms of z , and the elements a_0, a_1, a_2 , &c., every term after the first two is a semi-invariant multiplied by a power z .

Of course, the value of $\frac{1}{x}$ in question is the negative first power of the familiar series

$$\frac{z}{a_0} - \frac{a_1}{a_0^2} z^2 + \frac{2a_1^2 - a_0 a_2}{a_0^3} z^3 - \frac{a_0^3 a_3 - 5a_0 a_1 a_2 + 5a_1^3}{a_0^4} z^4 + \dots$$

If we divide this by z/a_0 , and call the quotient S , the expansion of S^{-1} according to ascending powers of x has for the coefficient of x^p the sum of the homogeneous products with repetitions (of the p^{th} order) of the reciprocals of the roots of $S = 0$. Therefore, when a_0 is put equal to unity, and p being greater than unity, the sum of such homogeneous products is a semi-invariant relative to the elements $1, a_1, a_2$, &c. There is some analogy here to the known property of

$$1 + a_1 x + \frac{a_2 x^2}{1.2} + \&c. = 0.$$

The foregoing result may be extended by taking, instead of the function v , a function

$$V = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \&c.,$$

* West's Mathematical Treatises, edited by Leslie, 1838, pp. 161-3. We have to notice that in West's notation, if O be

$$\beta \frac{d}{da} + 2\gamma \frac{d}{d\beta} + 3\delta \frac{d}{d\gamma} + \dots,$$

$$\left(\frac{a^p}{1}\right)', \left(\frac{a^p}{1}\right)'', \left(\frac{a^p}{1}\right)''' \dots \text{ mean } Oa^p, \frac{\partial^2 a^p}{1.2}, \frac{\partial^3 a^p}{1.2.3} \dots$$

where A_0 is a semi-invariant of the degree μ with elements of any system of binary forms,

$$\Pi = \Sigma \left[\left(a_1 \frac{d}{da_0} + a_2 \frac{d}{da_1} + a_3 \frac{d}{da_2} + \dots \right) \right],$$

the summation extending to all the elements, and

$$A_p = \Pi^p A_0.$$

In fact, by the comparison of coefficients on each side of the first equation in (B), we have

$$\Phi A_p = \mu p A_{p-1},^*$$

and $\Phi V^{-m} = -m (\Phi A_0 + \Phi A_1 \cdot x + \Phi A_2 \cdot x^2 + \dots) V^{-m-1}$

$$= -m\mu (A_0 x + 2A_1 x^2 + 3A_2 x^3 + \dots) V^{-m-1}$$

$$= -m\mu x V^{-m} + \mu x^2 \frac{d}{dx} V^{-m}.$$

The coefficient of x^{m+1} in this is zero. We can now revert the series

$$Z = A_0 x + A_1 x^2 + A_2 x^3 + \dots,$$

and reason as in the simpler case, with analogous result. In like manner, we might proceed to take, instead of the function V of the type F , a function of the type $\mathbf{F}$, and so on.

* This was given and employed by M. Perrin (1887). His notes on the subject of semi-invariants appear to have been suggested by a particular theorem due to M. D'Ocagno, viz., if w_p, w_q be two semi-invariants of the binary form

$$a_0 x^n + \binom{n}{1} a_1 x^{n-1} + \dots + a_n y^n,$$

and if p and q be their respective orders, and Δ is the operator

$$a_1 \frac{d}{da_0} + a_2 \frac{d}{da_1} + \dots + a_n \frac{d}{da_{n-1}},$$

then $q w_q \Delta w_p - p w_p \Delta w_q$ is a semi-invariant.