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On the Flexure and the Vibrations of a Curved Bar.

By Professor HORACE LAMB, M.A., F.R.S.

[*Read May 10th*, 1888.]

The flexure of a curved bar has been treated in a general manner by Kirchhoff, Clebsch, and Thomson and Tait, but the special applications which have been made of the theory are very few. In this paper I propose to discuss the flexure in its own plane of a uniform bar whose axis forms in the unstrained state an arc of a circle. After establishing the general equations and the terminal conditions, some simple statical problems are solved, and I then proceed to discuss the vibrations of a "free-free" bar, with special reference to the case where the total curvature is slight. This latter problem is interesting as bearing on some observations by Chladni, referred to by Tyndall in his book on "Sound," Chap. iv.

Taking the centre of the circle as origin, and denoting the radius by a, let the polar coordinates of any point of the bar be changed by the flexure from (a, θ) to $(a+R, \theta+\Theta)$, where R, Θ are small. If we neglect the extensibility of the bar, these quantities are not independent, but are connected by the relation

$$R = -a\frac{d\Theta}{d\theta}.*$$

The rotation experienced by any element $ad\theta$ is easily found to be

$$\Theta+\Theta'',$$

the accents denoting differentiations with respect to θ, whence, for the change of curvature, we have

$$\Delta\rho^{-1} = (\Theta'+\Theta''')/a.\dagger$$

The formula for the potential energy is therefore

$$V = \tfrac{1}{2}B\int(\Delta\rho^{-1})^2\,ad\theta$$

$$= \tfrac{1}{2}\frac{B}{a}\int(\Theta'+\Theta''')^2\,d\theta.$$

The applied forces at any point of the bar may be specified by the radial component P and the tangential component Q, both estimated per unit length. We may also include the case where finite forces are concentrated in an infinitely short element of the length; these may be denoted by P_0, Q_0. The force on either end may be analysed into a radial component $\overline{P}$, a tangential component $\overline{Q}$, and a couple $\overline{N}$.

The variational equation of motion is then

$$\int(\ddot{R}\delta R+a^2\ddot{\Theta}\delta\Theta)\,\sigma a d\theta+\delta V = \int(P\delta R+Qa\delta\Theta)\,a\,d\theta$$

$$+\Sigma\,\{P_0\delta R+Q_0 a\delta\Theta\}+\left[\overline{P}\delta R+\overline{Q}a\delta\Theta+\overline{N}\delta(\Theta+\Theta'')\right],$$

where σ is the mass per unit length, and the square brackets [ ] refer to the extremities.

If we substitute $R = -a\Theta'$, and integrate by parts in the usual

* Rayleigh, *Sound*, § 233.

† Or from the approximate formula, $\frac{1}{\rho} = \frac{1}{r} - \frac{1}{r^2}\frac{d^2r}{d\theta^2}$.

way, we find

$$\int \left\{ \sigma a^3 (\ddot{\Theta} - \ddot{\Theta}'') - \frac{B}{a} (\Theta'' + 2\Theta^{iv} + \Theta^{vi}) \right\} \delta\Theta \, d\theta$$
$$+ \Sigma \left[\sigma a^3 \ddot{\Theta}' + \frac{B}{a} (\Theta' + 2\Theta''' + \Theta^{v}) \right] \delta\Theta$$
$$- \Sigma \left[\Theta'' + \Theta^{iv} \right] \delta\Theta' + \Sigma \left[\Theta' + \Theta''' \right] \delta\Theta''$$
$$= \int \left(Q + \frac{dP}{d\theta} \right) a^2 \delta\Theta \, d\theta + \Sigma Q_0 a \delta\Theta - \Sigma P_0 a \delta\Theta'$$
$$+ \left[(-\overline{P}a^2 + \overline{Q}a + \overline{N}) \delta\Theta - \overline{P} a \delta\Theta' + \overline{N} \delta\Theta'' \right],$$

where the integrated terms on the left-hand side refer as well to the points of discontinuity (as regards the form of Θ) at which the forces P_0, Q_0 act, as to the extremities of the bar. The differential equation to be satisfied at each point of the bar is therefore

$$\sigma a^3 (\ddot{\Theta} - \ddot{\Theta}'') - \frac{B}{a} (\Theta'' + 2\Theta^{iv} + \Theta^{vi}) = a^2 \left(Q + \frac{dP}{d\theta} \right).$$

The terminal conditions are

$$\sigma a^3 \ddot{\Theta}' + \frac{B}{a} (\Theta' + 2\Theta''' + \Theta^{v}) = -\overline{P}a^2 + \overline{Q}a + \overline{N},$$
$$\frac{B}{a} (\Theta'' + \Theta^{iv}) = \overline{P}a,$$
$$\frac{B}{a} (\Theta' + \Theta''') = \overline{N};$$

whilst at a point of discontinuity we have

$$\left[\sigma a^3 \ddot{\Theta}' + \frac{B}{a} (\Theta' + 2\Theta''' + \Theta^{v}) \right] = Q_0 a,$$
$$\frac{B}{a} \left[\Theta'' + \Theta^{iv} \right] = P_0 a,$$
$$\left[\Theta' + \Theta''' \right] = 0,$$

the square brackets indicating that the differences of the values of the enclosed quantities on the two sides of the point in question are to be taken. These latter conditions may be simplified with the help of the obvious geometrical condition that the values of O, Θ', Θ' must be continuous.

As a first example, consider the equilibrium of the bar subject to applied force at its extremities only. The general equation becomes

$$\Theta'' + 2\Theta^{iv} + \Theta^{vi} = 0,$$

while the terminal conditions reduce to

$$\frac{B}{a^3}(\Theta'''+\Theta^{\text{v}}) = \overline{Q},$$

$$\frac{B}{a^3}(\Theta''+\Theta^{\text{iv}}) = \overline{P},$$

$$\frac{B}{a}(\Theta'+\Theta''') = \overline{N}.$$

The differential equation gives

$$\Theta = C+D\theta+(E+F\theta)\cos\theta+(G+H\theta)\sin\theta,$$

from which the terms in C, E, and G may, for the present purpose, be discarded as expressing a mere displacement of the bar as a whole. There remain three simple types of solution, from which the most general case can be derived by superposition. In the first place, if the applied forces reduce to two equal and opposite couples $\pm\overline{N}$ at the extremities, we find

$$\Theta = \frac{\overline{N}a}{B}\theta, \quad R = -\frac{\overline{N}a^3}{B},$$

i.e., the bar remains circular in form, but its radius is altered by the fraction $\overline{N}a/B$. Next, taking the origin of θ at the middle of the bar, consider the case where Θ is an odd function, viz.,

$$\Theta = D\theta+F\theta\cos\theta.$$

If there be no couples at the ends ($\theta = \pm\alpha$), we have

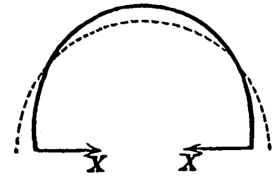
$$D = 2F\cos\alpha,$$

whence $$\pm\overline{Q} = 2F\frac{B}{a^3}\cos\alpha, \quad \overline{P} = 2F\frac{B}{a^3}\sin\alpha.$$

The resultant force at either extremity is along the chord. Denoting it by X, we have

$$\Theta = -\frac{a^2X}{B}(\cos\alpha+\tfrac{1}{2}\cos\theta)\,\theta,$$

$$R = \frac{a^3X}{B}(\cos\alpha+\tfrac{1}{2}\cos\theta-\tfrac{1}{2}\theta\sin\theta).$$



In particular, if $\alpha = \tfrac{1}{2}\pi$,

$$R_0 = -\tfrac{1}{4}\pi\, a^3X/B;$$

whilst, if $\alpha = \pi$, $$a\Theta_0 = \tfrac{3}{2}\pi a^3X/B.$$

Finally, we have the solution

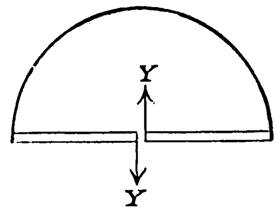
$$\Theta = H\theta \sin\theta,$$

which gives

$$\overline{Q} = 2H\frac{B}{a^2}\sin a, \quad \pm\overline{P} = 2H\frac{B}{a^2}\cos a, \quad \overline{N} = 2H\frac{B}{a}\sin a.$$

If we write

$$Y = 2H.B/a^2,$$



we have the case of a bar bent by equal and opposite forces Y applied at the extremities of rigid pieces attached to the ends, in the manner shown in the figure. The case of a nearly complete circle ($a = \pi$) is worth notice.

As an example of points of discontinuity, take the case of a circular hoop deformed by a pair of equal and opposite forces at the extremities of a diameter. The differential equation is, as before,

$$\Theta'' + 2\Theta^{iv} + \Theta^{vi} = 0,$$

whilst the dynamical conditions to be satisfied at the points of discontinuity are

$$\left[\Theta' + 2\Theta''' + \Theta^{v}\right] = 0,$$

$$\frac{B}{a}\left[\Theta'' + \Theta^{iv}\right] = P_0 a,$$

$$\left[\Theta' + \Theta'''\right] = 0.$$

Combined with the geometrical conditions, these show that Θ, Θ', Θ'', Θ''', Θ^{v} are to be continuous, whilst

$$\frac{B}{a}\left[\Theta^{iv}\right] = P_0 a.$$

Taking the diameter in question as initial line, we may assume, from $\theta = 0$ to $\theta = \pi$,

$$\Theta = C + D\theta + (E + F\theta)\cos\theta + (G + H\theta\sin\theta),$$

and from $\theta = 0$ to $\theta = -\pi$,

$$\Theta = C_1 + D_1\theta + (E_1 + F_1\theta)\cos\theta + (G_1 + H_1\theta\sin\theta).$$

The foregoing conditions then lead to*

$$C_1 - C = 4H, \quad E_1 - E = -4H,$$

* [*Oct.* 1888.—A numerical error has been corrected here.]

$$D_1 = D = \frac{4}{\pi} H, \quad F = F_1 = 0, \quad G_1 = G,$$

where $$H = P_0 a^2/4B.$$

Hence we may write, from $\theta = 0$ to $\theta = \pi$,

$$\Theta = \frac{P_0 a^2}{2B}\left(-1 + \frac{2\theta}{\pi} + \cos\theta + \tfrac{1}{2}\theta\sin\theta\right),$$

whilst, from $\theta = 0$ to $\theta = -\pi$,

$$\Theta = \frac{P_0 a^2}{2B}\left(1 + \frac{2\theta}{\pi} - \cos\theta - \tfrac{1}{2}\theta\sin\theta\right).$$

We have here omitted certain terms of the form

$$K + L\cos\theta + M\sin\theta,$$

which are common to both expressions, and represent a mere displacement without deformation. The corresponding values of R are

$$R = -\frac{P_0 a^3}{2B}\left(\frac{2}{\pi} - \tfrac{1}{2}\sin\theta + \tfrac{1}{2}\theta\cos\theta\right),$$

and $$R = -\frac{P_0 a^3}{2B}\left(\frac{2}{\pi} + \tfrac{1}{2}\sin\theta - \tfrac{1}{2}\theta\cos\theta\right).$$

We thence ascertain that the diameter $\theta = 0$ is increased by the amount $(\pi^2 - 8)/4\pi \,.\, P_0 a^3/B$, whilst the perpendicular diameter is shortened by $(4-\pi)/2\pi \,.\, P_0 a^3/B$.*

As a final statical example, we may calculate the deformation, due to its own weight, of a hoop suspended from a point of the circumference. Taking the radius through this point as initial line, we have

$$P = -g\sigma\cos\theta, \quad Q = g\sigma\sin\theta,$$

so that the differential equation is

$$\Theta'' + 2\Theta^{iv} + \Theta^{vi} = -\frac{2g\sigma a^3}{B}\sin\theta.$$

We therefore write, omitting unnecessary terms,

$$\Theta = D\theta + F\theta\cos\theta + H\theta\sin\theta + K\theta^2\sin\theta,$$

where $$K = -\frac{g\sigma a^3}{4B}.$$

* [*Oct.* 1888.—This agrees with the result quoted by Pearson from some unpublished lectures of Saint Venant, *History of Elasticity*, § 1575.]

The dynamical conditions to be satisfied at the point of suspension are

$$\frac{B}{a}\Big[\Theta'+2\Theta'''+\Theta^{\mathrm{v}}\Big]_0^{2\pi}=0,$$

$$\frac{B}{a}\Big[\Theta''+\Theta^{\mathrm{iv}}\Big]_0^{2\pi}=P_0a,$$

$$\Big[\Theta'+\Theta'''\Big]_0^{2\pi}=0.$$

To these we must add the geometrical conditions that Θ, Θ', Θ'' are to be continuous. We thence find

$$D=-4K,\quad F=4K,\quad H=-2\pi K,$$

$$P_0=-8\pi\frac{B}{a^2}K=2\pi a\sigma g.$$

In order that R', $(=-a\Theta'')$, may be zero at the point of suspension, we must add to Θ the terms $4\pi K(1-\cos\theta)$. We thus obtain

$$\Theta=K\left\{(\theta-\pi)^2\sin\theta+4(\theta-\pi)\cos\theta-4(\theta-\pi)-\pi^2\sin\theta\right\}.$$

It easily follows that the vertical diameter is increased by $(\pi^2-8)\,g\sigma a^4/4B$, whilst the horizontal diameter is shortened by $(4-\pi)\,g\sigma a^4/2B$.*

Let us next examine the flexural vibrations of the bar, supposed free from external force except at the extremities. If we assume that $\Theta\propto e^{ipt}$, and write

$$\sigma p^2/B=k^4,$$

the differential equation is

$$\Theta^{\mathrm{vi}}+2\Theta^{\mathrm{iv}}+(1-k^4a^4)\,\Theta''+k^4a^4\Theta=0.$$

At a *free* end we have the conditions

$$\Theta^{\mathrm{v}}+2\Theta'''+(1-k^4a^4)\,\Theta'=0,$$

$$\Theta^{\mathrm{iv}}+\Theta''=0,$$

$$\Theta'''+\Theta'=0,$$

* [*Oct.* 1888.—Comparing with the solution of the preceding problem, we learn that the elongation of the vertical, and the shortening of the horizontal, diameter are each half what they would have been if the weight of the hoop had been concentrated in its lowest point.]

the first of which may, in virtue of the differential equation, be replaced by

$$\int \Theta \, d\theta = 0.$$

The conditions to be satisfied at a *clamped* end are, of course,

$$\Theta = 0, \quad \Theta' = 0, \quad \Theta'' = 0.$$

Assuming that $\Theta = \Sigma A e^{\lambda\theta}$, we find

$$\lambda^6 + 2\lambda^4 + \lambda^2 + k^4a^4(1-\lambda^2) = 0.$$

The roots of this are functions of ka, and the elimination of the arbitrary constants from the six boundary conditions (three for each end) gives an equation to determine ka, and thence the frequency $p/2\pi$.

The interpretation of the solution in the general case would be difficult, but we may obtain some results of interest by proceeding to a first approximation in the case where the total curvature of the bar is slight. Fixing our attention more particularly on the case of a "free-free" bar, we see, by comparison with the known theory for the case where the bar is straight, that $\sigma p^2/B = m^4/l^4$ nearly, where l, $= 2a\alpha$, is the length, and m is a root of a certain transcendental equation. Hence $k^4a^4 = (m/2a\alpha)^4$, nearly, and is therefore large. One root of the cubic in λ^2 is therefore nearly equal to unity; and the remaining roots are given by

$$\lambda^2 + 1 = \pm k^2a^2 \text{ nearly.}$$

Continuing the approximations by ordinary methods, we find

$$\lambda^2 = \mu^2, \quad -\nu^2, \quad \varpi^2,$$

where

$$\mu^2 = k^2a^2 - \tfrac{3}{2} - \tfrac{7}{8}\frac{1}{k^2a^2} - \frac{2}{k^4a^4},$$

$$\nu^2 = k^2a^2 + \tfrac{3}{2} - \tfrac{7}{8}\frac{1}{k^2a^2} + \frac{2}{k^4a^4},$$

$$\varpi^2 = 1 + \frac{4}{k^4a^4}.$$

If we take the origin of θ at the middle of the bar, the fundamental modes will fall into two classes, according as Θ is an odd or an even function of θ. The former class is the more important as including the gravest mode, and is therefore taken first. We therefore assume

$$\Theta = F \sinh \mu\theta + G \sin \nu\theta + H \sinh \varpi\theta.$$

The conditions at the extremities ($\theta = \pm\alpha$) then give

$$\left.\begin{aligned} \frac{F}{\mu}\cosh\mu\alpha - \frac{G}{\nu}\cos\nu\alpha + \frac{H}{\varpi}\cosh\varpi\alpha &= 0 \\ \mu^3(\mu^2+1)F\sinh\mu\alpha + \nu^3(\nu^2-1)G\sin\nu\alpha + \varpi^3(\varpi^2+1)H\sinh\varpi\alpha &= 0 \\ \mu(\mu^2+1)F\cosh\mu\alpha - \nu(\nu^2-1)G\cos\nu\alpha + \varpi(\varpi^2+1)H\cosh\varpi\alpha &= 0 \end{aligned}\right\}$$

whence, by elimination of F, G, H,

$$\begin{vmatrix} \frac{1}{\mu}\cosh\mu\alpha, & -\frac{1}{\nu}\cos\nu\alpha, & \frac{1}{\varpi}\cosh\varpi\alpha \\ \mu^3(\mu^2+1)\sinh\mu\alpha, & \nu^3(\nu^2-1)\sin\nu\alpha, & \varpi^3(\varpi^2+1)\sinh\varpi\alpha \\ \mu(\mu^2+1)\cosh\mu\alpha, & -\nu(\nu^2-1)\cos\nu\alpha, & \varpi(\varpi^2+1)\cosh\varpi\alpha \end{vmatrix} = 0.$$

If we expand this according to the constituents of the last column, we find that the parts corresponding to the three constituents are of the orders k^7a^7, k^3a^3, k^8a^8, respectively. Hence, subject to an error of order $1/k^4a^4$, we may retain only the first of these, which gives

$$\mu\alpha\tanh\mu\alpha = -\nu\alpha\tan\nu\alpha.$$

To solve this by approximation, put

$$ka\alpha = \tfrac{1}{2}m + x,$$

where m is a root of $\qquad \tanh\tfrac{1}{2}m = -\tan\tfrac{1}{2}m,$

and x is small. This makes

$$\mu\alpha = ka\alpha - \tfrac{3}{4}\,\frac{\alpha^2}{ka\alpha}$$

$$= \tfrac{1}{2}m + x - \tfrac{3}{2}\,\frac{\alpha^2}{m},$$

and similarly $\qquad \nu\alpha = \tfrac{1}{2}m + x + \tfrac{3}{2}\,\frac{\alpha^2}{m}.$

Substituting in the equation, and retaining only the first powers of x and α^2, we find, after a little reduction,

$$x = -3\,\frac{\alpha^2}{m^2}\left(1 + \tfrac{1}{2}m\tan\tfrac{1}{2}m\right)\tan\tfrac{1}{2}m.$$

The values of m are approximately $3\pi/2$, $7\pi/2$, $11\pi/2$, &c., so that x is always negative. For the lowest root we have $m = 4{\cdot}7300$, whence

$$x = -{\cdot}17438\,\alpha^2.$$

To find the alteration of pitch due to the curvature, we have

$$\sigma p^2/B = k^4 = \left(\frac{m}{l}\right)^4 \left(1+\frac{2x}{m}\right)^4.$$

Hence, if $p_0/2\pi$ be the frequency of the corresponding mode for a straight bar

$$\frac{p}{p_0} = \left(1+\frac{2x}{m}\right)^2 = 1+\frac{4x}{m}, \text{ nearly,*}$$

so that the pitch is *lowered.* For the gravest mode

$$\frac{p}{p_0} \doteq 1-{\cdot}14747\,a^2.$$

The position of the nodes ($R=0$) is determined by

$$\begin{vmatrix} \mu\cosh\mu\theta, & \nu\cos\nu\theta, & \varpi\cosh\varpi\theta \\ \frac{1}{\mu}\cosh\mu a, & -\frac{1}{\nu}\cos\nu a, & \frac{1}{\varpi}\cosh\varpi a \\ \mu(\mu^2+1)\cosh\mu a, & -\nu(\nu^2-1)\cos\nu a, & \varpi(\varpi^2+1)\cosh\varpi a \end{vmatrix} = 0,$$

or $$(\nu^2+\varpi^2)(\nu^2-\varpi^2-1)\,\mu^2\frac{\cosh\mu\theta}{\cosh\mu a}+(\mu^2-\varpi^2)(\mu^2+\varpi^2+1)\,\nu^2\frac{\cos\nu\theta}{\cos\nu a}$$
$$+(\mu^2+\nu^2)(\mu^2-\nu^2+1)\frac{\cosh\varpi\theta}{\cosh\varpi a} = 0.$$

Recalling the approximate values of μ^2, ν^2, ϖ^2, we find that, subject to an error of the same order ($1/k^4a^4$) as before, this reduces to

$$(\nu^2-1)\cos\nu a\cosh\mu\theta+(\mu^2+1)\cosh\mu a\cos\nu\theta = 0,$$

or $$\cos\nu a\cosh\mu\theta+\cosh\mu a\cos\nu\theta = \frac{1}{k^2a^2}\cos\nu\theta\cosh\mu a,$$

approximately. To solve this, write $\theta/a = z+\epsilon$, where z is a root of

$$\cos\tfrac{1}{2}m\cosh\tfrac{1}{2}mz+\cosh\tfrac{1}{2}m\cos\tfrac{1}{2}mz = 0.\dagger$$

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* This calculation may be verified by the method explained in Lord Rayleigh's *Sound*, § 89; assuming as a hypothetical type that R has the same form as for a straight bar, viz.,

$$R \propto \cos\tfrac{1}{2}m\cosh\frac{m\theta}{2a}+\cosh\tfrac{1}{2}m\cos\frac{m\theta}{2a}.$$

† In comparing with the ordinary theory for a straight bar, it must be borne in mind that the origin is now at the middle point. *Cf.* Greenhill, *Messenger of Mathematics,* Dec. 1886.

We have already found

$$\mu a = \tfrac{1}{2}m + \left(x - \tfrac{3}{2}\,\frac{a^2}{m}\right),$$

$$\nu a = \tfrac{1}{2}m + \left(x + \tfrac{3}{2}\,\frac{a^2}{m}\right),$$

whence

$$\mu\theta = \mu a\,(z+\epsilon)$$

$$= \tfrac{1}{2}mz + \left(x - \tfrac{3}{2}\,\frac{a^2}{m}\right)z + \tfrac{1}{2}m\,\epsilon,$$

$$\nu\theta = \tfrac{1}{2}mz + \left(x + \tfrac{3}{2}\,\frac{a^2}{m}\right)z + \tfrac{1}{2}m\,\epsilon.$$

Substituting in the above equation, expanding, and retaining only the first powers of x, a^2, ϵ, we find, after effecting some reductions by means of the equation

$$\tanh\tfrac{1}{2}m = -\tan\tfrac{1}{2}m,$$

$$\tfrac{1}{2}m\,(\tanh\tfrac{1}{2}mz + \tan\tfrac{1}{2}mz)\,\epsilon$$

$$= -\left(x - \tfrac{3}{2}\,\frac{a^2}{m}\right)z\tanh\tfrac{1}{2}mz - \left(x + \tfrac{3}{2}\,\frac{a^2}{m}\right)z\tan\tfrac{1}{2}mz$$

$$+3\,\frac{a^2}{m}\tan\tfrac{1}{2}m - 4\,\frac{a^2}{m^2}$$

In the gravest mode

$$m = 4{\cdot}7300 = 271^\circ\,0'\,40'',$$

$$z = {\cdot}55164,$$

whence

$$\tfrac{1}{2}mz = 1{\cdot}3046 = 74^\circ\,45'\,20'',$$

$$\tan\tfrac{1}{2}mz = 3{\cdot}6694, \quad \tanh\tfrac{1}{2}mz = {\cdot}86295,$$

$$\tan\tfrac{1}{2}m = -{\cdot}98251.$$

Using the value already found for x, we obtain finally

$$\epsilon = -{\cdot}07959\,a^2,$$

so that the position of the nodes is given by

$$\pm\,\theta/a = {\cdot}55164 - {\cdot}07959\,a^2.$$

The conclusions that the effect of curvature is to lower the pitch, and at the same time to make the nodes approach the middle of the bar, are in agreement with the observations of Chladni.*

The asymmetrical fundamental modes may be treated more

* *Akustik*, § 99.

briefly. Assuming

$$\Theta = F\cosh\mu\theta + G\cos\nu\theta + H\cosh\varpi\theta,$$

the terminal conditions lead to

$$\begin{vmatrix} \frac{1}{\mu}\sinh\mu a, & \frac{1}{\nu}\sin\nu a, & \frac{1}{\varpi}\sinh\varpi a \\ \mu^2(\mu^2+1)\cosh\mu a, & \nu^2(\nu^2-1)\cos\nu a, & \varpi^2(\varpi^2+1)\cosh\varpi a \\ \mu(\mu^2+1)\sinh\mu a, & \nu(\nu^2-1)\sin\nu a, & \varpi(\varpi^2+1)\sinh\varpi a \end{vmatrix} = 0,$$

whence, to the same degree of approximation as before,

$$\mu a \coth\mu a = \nu a\cot\nu a.$$

Writing $\qquad ka a = \tfrac{1}{2}m + y,$

where m is a root of $\qquad \coth\tfrac{1}{2}m = \cot\tfrac{1}{2}m,$

and y is small, we find

$$y = -3\frac{a^2}{m^2}(\tfrac{1}{2}m\cot\tfrac{1}{2}m - 1)\cot\tfrac{1}{2}m.$$

The effect of the curvature is to alter the frequency in the ratio

$$\frac{p}{p_0} = 1 + \frac{4y}{m}, \text{ nearly.}$$

It is easily seen that y is always negative, so that the pitch is in all cases lowered.

[*Note added Oct.* 1888.—It has been assumed throughout that the elongation of the elements of the bar may be neglected. It may be shown that the amount of elongation which actually occurs is of no importance in the problems above considered, except in one very special case. Take, for example, the problem discussed near the foot of p. 368. The stretching (or compressing) force will be greatest at the middle of the bar, where its value is X, and the energy per unit length due to the stretching will there be $X^2/2q\omega$, where q is Young's modulus, and ω the sectional area. Again, the bending moment is also greatest at the middle, where it is $X(1-\cos\alpha)a$, so that the energy due to the bending is, per unit length,

$$= \tfrac{1}{2}\frac{X^2a^2}{B}(1-\cos\alpha)^2 = \tfrac{1}{2}\frac{X^2a^2}{q\kappa^2\omega}(1-\cos\alpha)^2,$$

where κ is the proper radius of gyration of the section. The ratio of the former energy to the latter is $\kappa^2/a^2(1-\cos\alpha)^2$, which in any practical case is small, unless indeed α be small. In the problem to which the figure near the top of p. 369 refers, the corresponding ratio is readily found to be κ^2/a^2, which is always small.]