XXV.—*The Generating Function of the Reciprocal of a Determinant.*

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(1) This is a subject to which very little study has been directed. The first to enunciate any proposition regarding it was JACOBI; but the solitary result which he reached received no attention from mathematicians,—certainly no fruitful attention,—during seventy years following the publication of it.

JACOBI was concerned with a problem regarding the partition of a fraction with composite denominator $(u_1 - t_1)(u_2 - t_2) \dots$ into other fractions whose denominators are factors of the original, where $u_1, u_2, \dots$ are linear homogeneous functions of one and the same set of variables. The specific character of the partition was only definable by viewing the given fraction $(u_1 - t_1)^{-1}(u_2 - t_2)^{-1} \dots$ as expanded in series form, it being required that each partial fraction should be the aggregate of a certain set of terms in this series. Of course the question of the *order* of the terms in each factor of the original denominator had to be attended to at the outset, since the expansion for $(a_1x + b_1y + c_1z - t)^{-1}$ is not the same as for $(b_1y + c_1z + a_1x - t)^{-1}$. Now one general proposition to which JACOBI was led in the course of this investigation was that *the coefficient of $x_1^{-1}x_2^{-1}x_3^{-1} \dots$ in the expansion of $u_1^{-1}u_2^{-1}u_3^{-1} \dots$, where*

$$\begin{aligned} u_1 &= a_1x_1 + a_2x_2 + a_3x_3 + \dots \\ u_2 &= b_1x_1 + b_2x_2 + b_3x_3 + \dots \\ u_3 &= c_1x_1 + c_2x_2 + c_3x_3 + \dots \\ &\dots \dots \dots \end{aligned}$$

is $|a_1b_2c_3 \dots|^{-1}$, provided that in every case the first term of u_r is that containing x_r .

This is the proposition with which we are here concerned.

(2) JACOBI gave no proof of it. He considered first the case of $(ax + by)^{-1}(b'y + a'x)^{-1}$, passing thence to $(ax + by - t)^{-1}(b'y + a'x - t')^{-1}$; next he dealt with $(ax + by + cz)^{-1}(b'y + c'z + a'x)^{-1}(c''z + a''x + b''y)^{-1}$, from which he passed readily as before to $(ax + by + cz - t)^{-1} \dots$; and then he found the work so forbidding that he turned to other things,—“Ad quatuor pluresve variables haec extendere non lubet, cum iam pro tribus tam prolixa exstiterint. Progredimur ad alia” (p. 354).*

(3) Doubtless it is true that often in the theory of determinants the treatment of the first two cases of a theorem is not only a sufficient guide to the generalisation of the

* C. G. J. JACOBI, “Exercitatio algebraica circa discerptionem singularem fractionum, quae plures variables involvunt,” *Crelle's Journ.*, v. pp. 344–364 (year 1829).

theorem, but virtually involves the generalisation, and makes the enunciation of the latter a mere matter of form. Let us see, however, whether it be so in this particular instance.

In the first case JACOBI, abruptly inserting $|a_1 \ b_2|$ into the numerator of the given fraction, says that

$$\frac{a_1 b_2 - a_2 b_1}{(a_1 x_1 + a_2 x_2)(b_2 x_2 + b_1 x_1)} = \frac{a_1}{x_2} \cdot \frac{1}{a_1 x_1 + a_2 x_2} - \frac{b_1}{x_1} \cdot \frac{1}{b_2 x_2 + b_1 x_1},$$

also that

$$\frac{a_1}{x_2} \cdot \frac{1}{a_1 x_1 + a_2 x_2} = \frac{1}{x_1 x_2} - \frac{1}{x_1} \cdot \frac{a_2}{a_1 x_1 + a_2 x_2},$$

and that therefore

$$\frac{a_1 b_2 - a_2 b_1}{(a_1 x_1 + a_2 x_2)(b_2 x_2 + b_1 x_1)} = \frac{1}{x_1 x_2} - \frac{1}{x_1} \cdot \frac{a_2}{a_1 x_1 + a_2 x_2} - \frac{1}{x_2} \cdot \frac{b_1}{b_2 x_2 + b_1 x_1}, \dots \dots (a)$$

a glance being then all that is necessary to make clear that the expansion of the last two fractions on the right consists of terms involving negative powers of one variable and positive powers of the other, and consequently that the cofactor of $x_1^{-1} x_2^{-1}$ in $(a_1 x_1 + a_2 x_2)^{-1} (b_2 x_2 + b_1 x_1)^{-1}$ is $(a_1 b_2 - a_2 b_1)^{-1}$.

In the second case, after certain preliminaries regarding subsidiary functions which are necessary for the expression of his final result but which have nothing to do with the theorem at present before us, he with equal abruptness throws down the "easily verified" identity

$$\begin{aligned} |a_1 b_2 c_3| \cdot x_1 x_2 x_3 &= a_1 u_2 u_3 - x_1 u_1 (b_1 c_1 x_1 + c_1 b_2 x_2 + b_1 c_3 x_3) \\ &\quad - x_2 u_2 (c_2 a_2 x_2 + a_2 c_3 x_3 + c_2 a_1 x_1) \\ &\quad - x_3 u_3 (a_3 b_3 x_3 + b_3 a_1 x_1 + a_3 b_2 x_2), \end{aligned}$$

and dividing by $x_1 x_2 x_3 \cdot u_1 u_2 u_3$ obtains of course

$$\begin{aligned} \frac{|a_1 b_2 c_3|}{(a_1 x_1 + \dots)(b_2 x_2 + \dots)(c_3 x_3 + \dots)} &= \frac{1}{x_1 x_2 x_3} - \frac{b_1 c_1 x_1 + c_1 b_2 x_2 + b_1 c_3 x_3}{x_2 x_3 (b_2 x_2 + \dots)(c_3 x_3 + \dots)} \dots \dots (b) \\ &\quad - \frac{c_2 a_2 x_2 + a_2 c_3 x_3 + c_2 a_1 x_1}{x_3 x_1 (c_3 x_3 + \dots)(a_1 x_1 + \dots)} \\ &\quad - \frac{a_3 b_3 x_3 + b_3 a_1 x_1 + a_3 b_2 x_2}{x_1 x_2 (a_1 x_1 + \dots)(b_2 x_2 + \dots)} \end{aligned}$$

This is the analogue to (a) in the previous case, and reasoning with it exactly as with (a) he concludes that the cofactor of $x_1^{-1} x_2^{-1} x_3^{-1}$ in $(a_1 x_1 + a_2 x_2 + a_3 x_3)^{-1} (b_2 x_2 + \dots)^{-1} (c_3 x_3 + \dots)^{-1}$ is $|a_1 b_2 c_3|^{-1}$.

(4) The natural commentary upon this is that there is a certain arbitrariness in the mode of procedure in both cases: that there is no similarity until the final identity is reached: and that the result of this similarity is to force upon us the conviction that for the establishment of the next case of the theorem there must be an identity (c) similar to (b) and (a). Now an examination of the last two fractions in (a) and the last three fractions in (b) shows that each set forms a cycle, the generating substitutions in (a) being

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix};$$

and in (b)

$$\begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

This, of course, is as it should be, because the remaining terms of the two identities are unaltered by the substitution: but the observation derives additional importance from the fact that the general application of it does away with the necessity for any stipulation as to the order of the terms in $u_1, u_2, \dots$, for, having $u_1 \equiv a_1x_1 + a_2x_2 + a_3x_3$ cyclical substitution gives $u_2 \equiv b_2x_2 + b_3x_3 + b_1x_1$ and not $\equiv b_1x_1 + b_2x_2 + b_3x_3$. The identities (a) and (b) are thus more appropriately written

$$\frac{|a_1b_2|}{u_1u_2} = \frac{1}{x_1x_2} - \sum_{x_2u_2}^{\circ} b_1,$$

$$\frac{|a_1b_2c_3|}{u_1u_2u_3} = \frac{1}{x_1x_2x_3} - \sum_{c_2u_2 \cdot c_3u_3}^{\circ} \frac{b_1c_1x_1 + c_1b_2x_2 + b_1c_3x_3}{c_2u_2 \cdot c_3u_3},$$

or if we fall back to the stage which JACOBI views in the second case as being fundamental

$$|a_1b_2| \cdot x_1x_2 = u_1u_2 - \sum x_1u_1 \cdot b_1,$$

$$|a_1b_2c_3| \cdot x_1x_2x_3 = u_1u_2u_3 - \sum x_1u_1 \cdot (b_1c_1x_1 + c_1b_2x_2 + b_1c_3x_3).$$

The simple question with which we are therefore brought face to face is as to the existence of a general identity of the latter kind.

(5) By way of answer to this an examination of the next case may be found sufficient. Is there, then, a quadric function, F_1 , of x_1, x_2, x_3, x_4 such that

$$|a_1b_2c_3d_4| \cdot x_1x_2x_3x_4 = u_1u_2u_3u_4 - \sum x_1u_1 \cdot F_1?$$

So probable is it made by analogy, by enumeration of terms and by actual experiment, that a searching examination cannot be dispensed with.

Denoting the ten possible coefficients of F_1 by $1_{11}, 1_{12}, \dots, 1_{44}$ so that

$$\sum x_1u_1 \cdot F_1 \equiv x_1x_1 \cdot \begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ 1_{11} & 1_{12} & 1_{13} & 1_{14} \\ & 1_{22} & 1_{23} & 1_{24} \\ & & 1_{33} & 1_{34} \\ & & & 1_{44} \end{vmatrix} x_1 + x_2u_2 \cdot \begin{vmatrix} x_2 & x_3 & x_4 & x_5 \\ 2_{22} & 2_{23} & 2_{24} & 2_{21} \\ & 2_{33} & 2_{34} & 2_{31} \\ & & 2_{44} & 2_{41} \\ & & & 2_{11} \end{vmatrix} x_2 + \dots$$

we have for the determination of the 40 unknowns the set of 35 equations obtained by taking the coefficients on the right-hand side of the supposed identity and equating to zero the coefficients of $x_1^4, x_2^4, \dots, x_1^3x_2, \dots$, and to $|a_1b_2c_3d_4|$ the coefficient of $x_1x_2x_3x_4$.*

$$a_1b_1c_1d_1 - a_11_{11} = 0, \quad a_2b_2c_2d_2 - a_22_{22} = 0, \quad \dots \quad (1)_1, (1)_2, \dots$$

* In view of the insufficiency in the number of equations it may be noted that in the preceding case there is a redundancy, viz., ten equations to nine unknowns, but no inconsistency.

give at once for $1_{11}, 2_{22}, 3_{33}, 4_{44}$ the values

$$b_1c_1d_1, \quad c_2d_2a_2, \quad d_3a_3b_3, \quad a_4b_4c_4.$$

Taking next the coefficient of $x_1^3x_2$, we have

$$a_1l_{12} + a_2l_{11} + b_1^2_{11} = a_1b_1c_1d_2 + a_1b_1c_2d_1 + a_1b_2c_1d_1 + a_2b_1c_1d_1,$$

which, in consequence of the previous determination of 1_{11} , is reduced by a term on both sides. Consideration, therefore, of this coefficient and the eleven others like it will be seen to lead to the twelve equations

$$a_1l_{12} + b_1^2_{11} = a_1b_1c_1d_2 + a_1b_1c_2d_1 + a_1b_2c_1d_1, \quad (2)_1$$

$$b_2^2_{22} + c_2^2_{22} = b_2c_2d_2a_2 + b_2c_2d_3a_2 + b_2c_3d_2a_2, \quad (2)_2$$

$$c_3^2_{33} + d_3^2_{33} = c_3d_3a_3b_3 + c_3d_3a_4b_3 + c_3d_4a_3b_3, \quad (2)_3$$

$$d_4^2_{44} + a_4^2_{44} = d_4a_4b_4c_4 + d_4a_4b_1c_4 + d_4a_1b_4c_4, \quad (2)_4$$

$$a_1l_{13} + c_1^2_{11} = a_1b_1c_1d_3 + a_1b_1c_2d_1 + a_1b_2c_1d_1, \quad (3)_1$$

$$b_2^2_{24} + d_2^2_{22} = b_2c_2d_2a_4 + b_2c_2d_4a_2 + b_2c_4d_2a_2, \quad (3)_2$$

$$c_3^2_{13} + a_3^2_{13} = c_3d_3a_3b_1 + c_3d_3a_1b_3 + c_3d_1a_3b_3, \quad (3)_3$$

$$d_4^2_{424} + b_4^2_{244} = d_4a_4b_4c_2 + d_4a_4b_2c_4 + d_4a_2b_4c_4, \quad (3)_4$$

$$a_1l_{14} + d_1^2_{11} = a_1b_1c_1d_4 + a_1b_1c_4d_1 + a_1b_4c_1d_1, \quad (4)_1$$

$$b_2^2_{12} + a_2^2_{122} = b_2c_2d_2a_1 + b_2c_2d_1a_2 + b_2c_1d_2a_2, \quad (4)_2$$

$$c_3^2_{23} + b_3^2_{233} = c_3d_3a_3b_2 + c_3d_3a_2b_3 + c_3d_2a_3b_3, \quad (4)_3$$

$$d_4^2_{434} + c_4^2_{344} = d_4a_4b_4c_3 + d_4a_4b_3c_4 + d_4a_3b_4c_4, \quad (4)_4$$

Dealing next with the coefficients of $x_1^2x_2^2, x_1^2x_3^2, \dots$ we obtain

$$a_1l_{22} + a_2l_{12} + b_2^2_{11} + b_1^2_{12} = a_1b_1c_2d_2 + a_1c_1b_2d_2 + a_1d_1b_2c_2 + b_1c_1a_2d_2 + b_1d_1a_2c_2 + c_1d_1a_2b_2 \quad (5)_1$$

$$b_2^2_{23} + b_3^2_{233} + c_3^2_{22} + c_2^2_{233} = b_2c_2d_2a_3 + b_2d_2c_2a_3 + b_2a_2c_2d_3 + c_2d_2b_2a_3 + c_2a_2b_2d_3 + d_2a_2b_2c_3 \quad (5)_2$$

$$c_3^2_{34} + c_4^2_{344} + d_4^2_{33} + d_3^2_{344} = c_3d_3a_4b_4 + c_3a_3d_4b_4 + c_3b_3d_4a_4 + d_3a_3c_4b_4 + d_3b_3c_4a_4 + a_3b_3c_4d_4 \quad (5)_3$$

$$d_4^2_{411} + d_1^2_{414} + a_1^2_{44} + a_4^2_{144} = d_4a_4b_1c_1 + d_4b_4a_1c_1 + d_4c_4a_1b_1 + a_4b_4d_1c_1 + a_4c_4d_1b_1 + b_4c_4d_1a_1 \quad (5)_4$$

$$a_1l_{33} + a_3^2_{133} + c_3^2_{311} + c_1^2_{313} = a_1b_1c_3d_3 + a_1c_1b_3d_3 + a_1d_1b_3c_3 + b_1c_1a_3d_3 + b_1d_1a_3c_3 + c_1d_1a_3b_3 \quad (6)_1$$

$$b_2^2_{244} + b_4^2_{244} + d_4^2_{222} + d_2^2_{422} = b_2c_2d_4a_4 + b_2d_2c_4a_4 + b_2a_2c_4d_4 + c_2d_2b_4a_4 + c_2a_2b_4d_4 + d_2a_2b_4c_4 \quad (6)_2$$

These six, it will be found, do not contain any other unknowns than those contained in the previous set of twelve; and, what is still more important, the unknowns contained in any one of the set of six are exactly those contained in two of the set of twelve. For the purposes of solution, therefore, it is better to view the eighteen as belonging to six sets of three, viz.,

$$\begin{array}{l} (5)_1, (2)_1, (4)_2 \\ (5)_2, (2)_2, (4)_3 \\ (5)_3, (2)_3, (4)_4 \\ (5)_4, (2)_4, (4)_1 \end{array} \quad \text{and} \quad \begin{array}{l} (6)_1, (3)_1, (3)_3 \\ (6)_2, (3)_2, (3)_4 \end{array}.$$

Taking the first triad, viz.,

$$\left. \begin{array}{l} a_1l_{22} + a_2l_{12} + b_2^2_{11} + b_1^2_{12} = a_1b_1c_2d_2 + \dots \\ a_1l_{12} + b_1^2_{11} = a_1b_1c_1d_2 + a_1b_1c_2d_1 + a_1b_2c_1d_1 \\ a_2l_{22} + b_2^2_{12} = b_2c_2d_2a_1 + b_2c_2d_1a_2 + b_2c_1d_2a_2 \end{array} \right\}$$

we see that there are four possible solutions of the third equation; and, as substitution

for 1_{22} and 2_{12} in the first equation produces an equation like the second, it follows that there are not more than four solutions possible for the triad. These are

$$\begin{aligned} 1_{22} &= 0, \\ 2_{12} &= c_2 d_2 a_1 + c_2 d_1 a_2 + c_1 d_2 a_3 \left\{ \begin{array}{l} b_2 c_2 d_1 \\ c_2 d_2 a_1 + c_1 d_2 a_2 \\ c_2 d_2 a_1 + c_2 d_1 a_2 \end{array} \right\} \left\{ \begin{array}{l} b_2 c_1 d_2 \\ c_2 d_2 a_1 + c_2 d_1 a_2 \\ c_2 d_2 a_1 \end{array} \right\} \left\{ \begin{array}{l} b_2 c_2 d_1 + b_2 c_1 d_2 \\ c_2 d_2 a_1 \\ c_2 d_2 a_1 \end{array} \right\} \\ 1_{12} &= b_2 c_1 d_1 \\ 2_{11} &= a_1 c_1 d_2 + a_1 c_2 d_1 \left\{ \begin{array}{l} b_1 c_2 d_1 + b_2 c_1 d_1 \\ a_1 c_1 d_2 \\ a_1 c_2 d_1 \end{array} \right\} \left\{ \begin{array}{l} b_1 c_1 d_2 + b_2 c_1 d_1 \\ a_1 c_1 d_2 \\ a_1 c_2 d_1 \end{array} \right\} \left\{ \begin{array}{l} b_1 c_1 d_2 + b_1 c_2 d_1 + b_2 c_1 d_1 \\ 0 \\ 0 \end{array} \right\}. \end{aligned}$$

The solutions of the next three triads are got by cyclic substitution.

The fifth triad in the same way gives us

$$\begin{aligned} 1_{33} &= 0 \\ 3_{13} &= a_3 b_1 d_3 + a_1 b_3 d_3 + a_3 b_3 d_1 \left\{ \begin{array}{l} b_1 c_3 d_3 \\ a_1 b_3 d_3 + a_3 b_3 d_1 \\ a_3 b_1 d_3 + a_1 b_3 d_3 \end{array} \right\} \left\{ \begin{array}{l} b_1 c_3 d_3 + b_3 c_3 d_1 \\ a_1 b_3 d_3 \\ a_1 b_3 d_3 \end{array} \right\} \left\{ \begin{array}{l} b_1 c_3 d_3 + b_3 c_3 d_1 \\ a_1 b_3 d_3 \\ a_1 b_3 d_3 \end{array} \right\} \\ 1_{13} &= b_1 c_3 d_1 \\ 3_{11} &= a_1 b_1 d_3 + a_1 b_3 d_1 \left\{ \begin{array}{l} b_1 c_1 d_3 + b_1 c_1 d_3 \\ a_1 d_1 c_3 + c_1 d_1 b_3 \\ a_1 d_1 c_3 \end{array} \right\} \left\{ \begin{array}{l} b_1 c_1 d_3 + b_1 d_1 c_3 + b_3 c_1 d_1 \\ a_1 d_1 c_3 \\ a_1 d_1 c_3 \end{array} \right\} \left\{ \begin{array}{l} b_1 c_1 d_3 + b_1 d_1 c_3 + b_3 c_1 d_1 \\ 0 \\ 0 \end{array} \right\}. \end{aligned}$$

and the solutions of the sixth are got therefrom by cyclical substitution.

Here, however, a hitch occurs, for if we proceed with the cyclical substitution we find that while in the matter of the equations we are led back, as we ought to be, from the sixth triad to the fifth, the same is not true of the solutions. In fact, the solutions of the fifth triad alone are immediately convincing of the hopelessness of our quest; for pairs of the unknowns in that triad, *e.g.*, 1_{33} , 3_{11} belong to the same cycle, and the value of the second of the pair should therefore be obtainable from that of the first by cyclical substitution, which is not the case.

(6) A different mode of procedure from JACOBI'S being thus necessary for the establishment of his theorem, I would substitute the following, which, though ostensibly concerned only with the special case which we have just been examining, is of perfectly general application.

Since the sum of the elements of every one of its columns is zero, the determinant

$$\begin{vmatrix} u_1 - a_2 x_2 & -b_2 c_2 & -c_2 x_2 & -d_2 x_2 \\ -a_2 c_3 & u_2 - b_3 c_3 & -c_3 x_3 & -d_3 x_3 \\ -a_4 c_4 & -b_4 c_4 & u_3 - c_4 x_4 & -d_4 x_4 \\ -a_1 x_1 & -b_1 x_1 & -c_1 x_1 & u_4 - d_1 x_1 \end{vmatrix}$$

vanishes identically. Expanding it in a series arranged according to products of the u 's we have therefore

$$\begin{aligned} 0 &= u_1 u_2 u_3 u_4 - \sum u_1 u_2 u_3 \cdot d_1 x_1 + \sum u_1 u_2 \cdot |c_4 d_1| \cdot x_4 x_1 \\ &\quad - \sum u_1 \cdot |b_3 c_4 d_1| \cdot x_3 x_4 x_1 + |a_2 b_3 c_4 d_1| \cdot x_2 x_3 x_4 x_1 \end{aligned}$$

and consequently

$$\begin{aligned} |a_1 b_2 c_3 d_4| \cdot x_1 x_2 x_3 x_4 &= u_1 u_2 u_3 u_4 - \sum u_1 u_2 u_3 \cdot d_1 x_1 + \sum u_1 u_2 \cdot |c_4 d_1| \cdot x_4 x_1 \\ &\quad + \sum u_1 u_3 \cdot |b_3 d_1| \cdot x_3 x_1 - \sum u_1 |b_3 c_4 d_1| \cdot x_3 x_4 x_1. \end{aligned}$$

On dividing by $x_1 x_2 x_3 x_4 \cdot u_1 u_2 u_3 u_4$ this is changed into

$$\frac{|a_1 b_2 c_3 d_4|}{u_1 u_2 u_3 u_4} = \frac{1}{x_1 x_2 x_3 x_4} - \sum \frac{d_1}{u_4 \cdot c_2 c_3 x_4} + \sum \frac{|c_4 d_1|}{u_3 u_4 \cdot x_2 x_3} + \sum \frac{|b_3 d_1|}{u_2 u_4 \cdot x_2 x_4} - \sum \frac{|b_3 c_4 d_1|}{u_2 u_3 u_4 \cdot x_1}$$

and as the expansion of each of the four fractions under the sign of summation contains in every term a positive power of x_1 , it follows that the cofactor of $x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1}$ in the expansion of $|a_1b_2c_3d_4|1u_1u_2u_3u_4$ is 1, as was to be proved.

(7) The construction of the vanishing determinant on which the proof rests is easily completed when the diagonal elements are given; in every case, however, there is more than one set of diagonal elements suitable for the purpose. The requirements are (1) that the elements chosen from $|a_1b_2c_3 \dots|$ to appear in the diagonal of the vanishing determinant shall form a positive or negative term of $|a_1b_2c_3 \dots|$ according as the number of these elements is odd or even: and (2) that no suffix shall be in the place which it occupied in the original diagonal.

The first requirement is due to the need for having the sign of $|a_1b_2c_3 \dots| \cdot x_1x_2x_3 \dots$ negative at the end of the expansion of the vanishing determinant: and the second to the need for having a positive power of one of the x 's in every term, except one, of the expansion of the subsequent fractions.

Both requirements are seen to be satisfied by taking the suffixes 1, 2, 3, $\dots$, n and moving either the last to the first place or the first to the last place: for then the number of inverted-pairs in the suffixes is $n-1$ and no element is in its original place. In the case where $n=3$, these are the only possibilities: in the case where $n=4$ there are, however, four others, viz., a_2, b_4, c_1, d_3 ; a_3, b_1, c_4, d_2 ; a_3, b_4, c_2, d_1 ; a_4, b_3, c_1, d_2 .

(8) For the purpose of clearing up JACOBI'S mode of procedure, let us examine the new proof as applied to his second case. The two vanishing determinants are

$$\begin{vmatrix} u_1 - a_2x_2 & -b_2x_2 & -c_2x_2 \\ -a_3x_3 & u_2 - b_3x_3 & -c_3x_3 \\ -a_1x_1 & -b_1x_1 & u_3 - c_1x_1 \end{vmatrix}, \quad \begin{vmatrix} u_1 - a_3x_3 & -b_3x_3 & -c_3x_3 \\ -a_1x_1 & u_2 - b_1x_1 & -c_1x_1 \\ -a_2x_2 & -b_2x_2 & u_3 - c_2x_2 \end{vmatrix},$$

their expansions being

$$\begin{aligned} u_1u_2u_3 &= \sum u_1u_2 \cdot c_1x_1 + \sum u_1 \cdot |b_3c_1| \cdot x_3x_1 - |a_2b_3c_1| \cdot x_2x_3x_1, \\ u_1u_2u_3 &= \sum u_1u_2 \cdot c_2x_2 + \sum u_1 \cdot |b_1c_2| \cdot x_1x_2 - |a_3b_1c_2| \cdot x_3x_1x_2; \end{aligned}$$

and the resulting identities

$$\begin{aligned} \frac{|a_1b_2c_3|}{u_1u_2u_3} &= \frac{1}{x_1x_2x_3} - \sum \frac{c_1}{u_3 \cdot x_2x_3} + \sum \frac{|b_3c_1|}{u_2u_3 \cdot x_2}, \\ \frac{|a_1b_2c_3|}{u_1u_2u_3} &= \frac{1}{x_1x_2x_3} - \sum \frac{c_2}{u_3 \cdot x_1x_3} + \sum \frac{|b_1c_2|}{u_2u_3 \cdot x_3}, \end{aligned}$$

the one being as suitable as the other for establishing the theorem regarding the coefficient of $x_1^{-1}x_2^{-1}x_3^{-1}$. If now, however, we combine the last two terms of each, there results

$$\begin{aligned} \frac{|a_1b_2c_3|}{u_1u_2u_3} &= \frac{1}{x_1x_2x_3} - \sum \frac{b_1c_1x_1 + b_3c_1x_2 + b_1c_3x_3}{x_2x_3 \cdot u_2u_3}, \\ \frac{|a_1b_2c_3|}{u_1u_2u_3} &= \frac{1}{x_1x_2x_3} - \sum \frac{b_2c_1x_1 + b_2c_2x_2 + b_3c_2x_3}{x_1x_3 \cdot u_2u_3}, \end{aligned}$$

one of which, as might have been expected, is identical with JACOBI'S, and therefore not less suitable than in its previous form. In the other case combination has proved harmful, the reason being that then it is not the same x which occurs with positive index in the expansions of the two fractions concerned.*

An examination of the corresponding fractions in § 6 will attest the accuracy of the result reached in § 5.

(9) JACOBI'S theorem that

the coefficient of $x_1^{-1}x_2^{-1}x_3^{-1}\dots$ in the expansion of $u_1^{-1}u_2^{-1}u_3^{-1}\dots$ is $|a_1b_2c_3\dots|^{-1}$,

has a curious analogue in the theorem that

the coefficient of $x_1x_2x_3\dots$ in the expansion of $u_1u_2u_3\dots$ is $|^+a_1b_2c_3\dots|^+$,

where the permanent $|^+a_1b_2c_3\dots|^+$ is a function differing from the determinant $|a_1b_2c_3\dots|$ in having all its terms positive.† I now propose to show that this is not a solitary instance of such dualism,—a dualism which surely warrants the two theorems being called *per-reciprocals* of one another: but before doing so it is desirable to change the point of view somewhat, so as to have the theorems in a wider field.

(10) If to each linear homogeneous function u_r of $x_1, x_2, \dots$ there be prefixed as a factor another letter, τ say, with the same suffix, the terms in $\tau_r u_r$ consist of one in which the suffix of τ is the same as the suffix of the x with which it is associated, and others where this similarity does not exist: in other words, $\tau_r u_r$ is partitionable into $a_r \tau_r x_r$ and $\tau_r(u_r - a_r x_r)$. Terms in which each x is accompanied by a τ with the same suffix may be conveniently spoken of as *conjugal* terms, and the operation of taking the aggregate of the conjugal terms existing in the expansion of any function of $\tau_1 u_1, \tau_2 u_2, \dots$ may be denoted by T. It is of course clear that

$$\begin{aligned} T(\phi_1 + \phi_2 + \dots) &= T(\phi_1) + T(\phi_2) + \dots, \\ T(C\phi) &= CT(\phi), \end{aligned}$$

where C is a constant with respect to the τ 's and x 's: and it is convenient to consider

$$T(C) = C.$$

Now, returning to the first of the two identities in § 8, and dividing by $\tau_1 \tau_2 \tau_3$, we have

$$\frac{|a_1b_2c_3|}{\tau_1 u_1 \cdot \tau_2 u_2 \cdot \tau_3 u_3} = \frac{1}{\tau_1 x_1 \cdot \tau_2 x_2 \cdot \tau_3 x_3} - \sum \frac{c_1}{u_3 \cdot \tau_1 \cdot \tau_2 x_2 \cdot \tau_3 x_3} + \sum \frac{|b_3c_1|}{\tau_1 \tau_3 \cdot u_2 u_3 \cdot \tau_2 x_2};$$

and, as in the expansion of the fractions under the sign of summation τ_1 is negative and x_1 positive, it follows that the aggregate of the conjugal terms in the expansions on the right is $1/\tau_1 x_1 \cdot \tau_2 x_2 \cdot \tau_3 x_3$, and therefore that the aggregate of the conjugal

* Had we taken as our vanishing determinant that corresponding to the only other positive term of $|a_1b_2c_3|$ we should have found the resulting identity objectionable in both forms.

† *Proceedings Roy. Soc. Edin.*, xi. pp. 409-418 (Sess. 1881-2).

terms in the expansion of the reciprocal of $\tau_1 u_1 \cdot \tau_2 u_2 \cdot \tau_3 u_3$ is the reciprocal of $\tau_1 x_1 \cdot \tau_2 x_2 \cdot \tau_3 x_3 \cdot |a_1 b_2 c_3|$

$$\text{i.e., } T(\tau_1 u_1 \cdot \tau_2 u_2 \cdot \tau_3 u_3)^{-1} = |a_1 b_2 c_3|^{-1} \cdot (\tau_1 x_1 \cdot \tau_2 x_2 \cdot \tau_3 x_3)^{-1}.$$

The similar variant of the other theorem of § 9 is

$$T(\tau_1 u_1 \cdot \tau_2 u_2 \cdot \tau_3 u_3) = |a_1 b_2 c_3| \cdot (\tau_1 x_1 \cdot \tau_2 x_2 \cdot \tau_3 x_3).$$

(11) Knowing the truth of this second theorem in its general form, viz.:

$$T(\tau_h u_h \cdot \tau_k u_k \cdot \tau_l u_l \cdots) = |h_k l_l \cdots| \cdot (\tau_h x_h \cdot \tau_k x_k \cdot \tau_l x_l \cdots)$$

we can readily find the aggregate of the conjugal terms in any series

$$C_1 \phi_1 + C_2 \phi_2 + C_3 \phi_3 + \cdots$$

where the ϕ 's are of the form $\tau_h u_h \cdot \tau_k u_k \cdot \tau_l u_l \cdots$. For example:

$$\begin{aligned} & T(C\tau_1 u_1 \cdot \tau_2 u_2 + D\tau_2 u_2 \cdot \tau_3 u_3 + E\tau_3 u_3 \cdot \tau_1 u_1) \\ &= C |a_1 b_2| \cdot \tau_1 x_1 \cdot \tau_2 x_2 + D |b_2 c_3| \cdot \tau_2 x_2 \cdot \tau_3 x_3 + E |c_3 a_1| \cdot \tau_3 x_3 \cdot \tau_1 x_1. \end{aligned}$$

(12) A conspicuous instance of such a series is the expansion of $(m_1 - \tau_1 u_1)(m_2 - \tau_2 u_2)(m_3 - \tau_3 u_3) \cdots$. Restricting the number of factors to three, merely for simplicity's sake in writing, we have the aggregate of the conjugal terms of this product

$$\begin{aligned} &= T[m_1 m_2 m_3 - (m_1 m_2 \cdot \tau_3 u_3 + \cdots) + (m_1 \cdot \tau_2 u_2 \cdot \tau_3 u_3 + \cdots) - \tau_1 u_1 \cdot \tau_2 u_2 \cdot \tau_3 u_3] \\ &= m_1 m_2 m_3 - (m_1 m_2 \cdot c_3 \tau_3 x_3 + \cdots) + (m_1 \cdot |b_2 c_3| \cdot \tau_2 x_2 \cdot \tau_3 x_3 + \cdots) - |a_1 b_2 c_3| \tau_1 x_1 \cdot \tau_2 x_2 \cdot \tau_3 x_3 \\ &= \begin{vmatrix} m_1 - a_1 \cdot \tau_1 x_1 & -b_1 \cdot \tau_1 x_1 & -c_1 \cdot \tau_1 x_1 \\ -a_2 \cdot \tau_2 x_2 & m_2 - b_2 \cdot \tau_2 x_2 & -c_2 \cdot \tau_2 x_2 \\ -a_3 \cdot \tau_3 x_3 & -b_3 \cdot \tau_3 x_3 & m_3 - c_3 \cdot \tau_3 x_3 \end{vmatrix}, \end{aligned}$$

the reason for the last step being the fact that the law for the expression of a determinant with binomial elements in terms of determinants with monomial elements applies to the case of permanents also.

(13) This theorem has corresponding to it a per-reciprocal theorem, viz.,

$$T\{(m_1 - \tau_1 u_1)(m_2 - \tau_2 u_2)(m_3 - \tau_3 u_3)\}^{-1} = \begin{vmatrix} m_1 - a_1 \cdot \tau_1 x_1 & -b_1 \cdot \tau_1 x_1 & -c_1 \cdot \tau_1 x_1 \\ -a_2 \cdot \tau_2 x_2 & m_2 - b_2 \cdot \tau_2 x_2 & -c_2 \cdot \tau_2 x_2 \\ -a_3 \cdot \tau_3 x_3 & -b_3 \cdot \tau_3 x_3 & m_3 - c_3 \cdot \tau_3 x_3 \end{vmatrix}^{-1} \cdot (\tau_1 x_1 \cdot \tau_2 x_2 \cdot \tau_3 x_3)^{-1}$$

By way of proof let it be noted that the determinant, Δ say, whose reciprocal appears on the right may be written

$$\begin{vmatrix} (m_1 - \tau_1 u_1) + \tau_1(u_1 - a_1 x_1) & -\tau_1 \cdot b_1 x_1 & -\tau_1 \cdot c_1 x_1 \\ -\tau_2 \cdot a_2 x_2 & (m_2 - \tau_2 u_2) + \tau_2(u_2 - b_2 x_2) & -\tau_2 \cdot c_2 x_2 \\ -\tau_3 \cdot a_3 x_3 & -\tau_3 \cdot b_3 x_3 & (m_3 - \tau_3 u_3) + \tau_3(u_3 - c_3 x_3) \end{vmatrix},$$

and that this when expanded becomes

$$\begin{aligned}
(m_1 - \tau_1 u_1)(m_2 - \tau_2 u_2)(m_3 - \tau_3 u_3) &+ \sum (m_1 - \tau_1 u_1)(m_2 - \tau_2 u_2) \cdot \tau_3(u_3 - c_3 x_3) \\
&+ \sum (m_1 - \tau_1 u_1) \cdot \tau_2 \tau_3 \cdot \begin{vmatrix} u_2 - b_2 x_2 & -c_2 x_2 \\ -b_3 x_3 & u_3 - c_3 x_3 \end{vmatrix} \\
&+ \tau_1 \tau_2 \tau_3 \begin{vmatrix} u_1 - a_1 x_1 & -b_1 x_1 & -c_1 x_1 \\ -a_2 x_2 & u_2 - b_2 x_2 & -c_2 x_2 \\ -a_3 x_3 & -b_3 x_3 & u_3 - c_3 x_3 \end{vmatrix},
\end{aligned}$$

where the addition of the elements of each column shows that the three-line determinant vanishes and that the two-line determinant has x_1 for a factor. As a consequence we have

$$\frac{\Delta}{(m_1 - \tau_1 u_1)(m_2 - \tau_2 u_2)(m_3 - \tau_3 u_3)} = 1 + \sum \frac{\tau_3(u_3 - c_3 x_3)}{m_3 - \tau_3 u_3} + \sum \frac{\tau_2 \tau_3(b_1 c_1 x_1 + b_3 c_1 x_3 + b_1 c_2 x_2) \cdot x_1}{(m_2 - \tau_2 u_2)(m_3 - \tau_3 u_3)}$$

and thus see that, since the first fraction on the right has for a factor τ_3 without x_3 , and the second has x_1 without τ_1 , there results

$$\mathbf{T} \left\{ \frac{\Delta}{(m_1 - \tau_1 u_1)(m_2 - \tau_2 u_2)(m_3 - \tau_3 u_3)} \right\} = 1,$$

as was to be proved.

(14) Putting $m_1 = m_2 = m_3 = \dots = 0$ in the theorem of the preceding paragraph we obtain JACOBI's case of 1829 with which we started; and putting $m_1 = m_2 = m_3 = \dots = 1$ we obtain the case which is the subject of a memoir by Major MACMAHON, printed in the *Transactions* of the Royal Society of London for 1893.*

The new per-reciprocal theorem of § 12 of course degenerates in like fashion.

(15) A still wider generalisation, however, is possible. All that is requisite is to make use of a simple fact employed by JACOBI in the memoir above noted, viz., that if $\omega_1, \omega_2, \dots$ be what $u_1, u_2, \dots$ become when $x_1, x_2, \dots$ are changed into $\xi_1, \xi_2, \dots$,—in other words, if $\xi_1, \xi_2, \dots$ be the values of $x_1, x_2, \dots$ which satisfy the equations $u_1 = \omega_1, u_2 = \omega_2, \dots$,—then any identity connecting $u_1, u_2, \dots$ and the quantities of which $u_1, u_2, \dots$ are functions will still remain an identity if in every instance x_r be changed into $x_r - \xi_r$ and u_r into $u_r - \omega_r$. As a consequence of this action we obtain from the result of § 12 the theorem,—

In the expansion of

$$\{m_1 - \tau_1(u_1 - \omega_1)\} \cdot \{m_2 - \tau_2(u_2 - \omega_2)\} \dots$$

the aggregate of the terms which are such that every τ occurring in them is accompanied by the corresponding $x - \xi$, and vice-versa, is

$$\begin{vmatrix}
m_1 - a_1 \tau_1(x_1 - \xi_1) & -b_1 \tau_1(x_1 - \xi_1) & -c_1 \tau_1(x_1 - \xi_1) & \dots & + \\
-a_2 \tau_2(x_2 - \xi_2) & m_2 - b_2 \tau_2(x_2 - \xi_2) & -c_2 \tau_2(x_2 - \xi_2) & \dots & \\
-a_3 \tau_3(x_3 - \xi_3) & -b_3 \tau_3(x_3 - \xi_3) & m_3 - c_3 \tau_3(x_3 - \xi_3) & \dots & \\
\vdots & \vdots & \vdots & \ddots & \vdots
\end{vmatrix}.$$

* MACMAHON, P. A., "A certain class of generating functions in the theory of numbers," *Trans. Roy. Soc.*, clxxxv. pp. 111-160.

Similarly from § 13 there is obtained the theorem,

In the expansion of

$$\{m_1 - \tau_1(u_1 - \omega_1)\}^{-1} \cdot \{m_2 - \tau_2(u_2 - \omega_2)\}^{-1} \dots$$

the aggregate of the terms which are such that every τ occurring in them is accompanied by the corresponding $x - \xi$, and vice-versa, is

$$\begin{vmatrix} m_1 - a_1\tau_1(x_1 - \xi_1) & -b_1\tau_1(x_1 - \xi_1) & -c_1\tau_1(x_1 - \xi_1) & \dots \\ -a_2\tau_2(x_2 - \xi_2) & m_2 - b_2\tau_2(x_2 - \xi_2) & -c_2\tau_2(x_2 - \xi_2) & \dots \\ -a_3\tau_3(x_3 - \xi_3) & -b_3\tau_3(x_3 - \xi_3) & m_3 - c_3\tau_3(x_3 - \xi_3) & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}^{-1}.$$

(16) Leaving the subject of conjugal aggregates at this point, let us return for a moment to the question of § 5. It will be remembered that JACOBI's first two cases of a supposed general theorem were

$$\begin{aligned} |a_1b_2| \cdot x_1x_2 &= u_1u_2 - \sum u_1 \cdot b_1x_1, \\ |a_1b_2c_3| \cdot x_1x_2x_3 &= u_1u_2u_3 - \sum u_1 \cdot (b_1c_1x_1 + b_2c_1x_2 + b_1c_3x_3) \cdot x_1; \end{aligned}$$

and that for three different reasons it was difficult to avoid a strong presumption that the form of the general theorem must be

$$|a_1b_2c_3 \dots| \cdot x_1x_2 \dots x_n = u_1u_2 \dots u_n - \sum u_1 \cdot F_1 \cdot x_1$$

where F_1 is an integral homogeneous function of the $(n-2)^{\text{th}}$ degree in $x_1, x_2, \dots, x_n$. A fourth reason will now incidentally appear, it being inseparable from certain curious results in determinants which were discovered in consequence of it, and which deserve on their own account to be chronicled.

Early in the investigation it was observed that the expression following $\sum u_1$ in the second case above is equal to the determinant

$$\begin{vmatrix} u_3 - c_3x_2 & -c_3x_3 \\ -b_2x_2 & u_2 - b_2x_3 \end{vmatrix},$$

and on turning to the first case it was seen that the corresponding expression there, viz., b_1x_1 , being equal to $u_2 - b_2x_2$, could be considered to be of the same form. The question thus arose whether or not there might be a similar determinant of the third order having (1) x_1 for a factor, (2) a quadratic function of x_1, x_2, x_3, x_4 for the cofactor, (3) the terms of this function sixteen in number and all positive, and (4) these terms such as to ensure an identity; and, strange to say, determinants possessing the first three of these properties were found to be realisable.

(17) *If $u_1 \equiv a_1x_1 + \dots + a_4x_4$, $u_2 \equiv b_1x_1 + \dots + b_4x_4$, each of the six deter-*

$$\begin{vmatrix} u_2 - b_2x_2 & -b_3x_3 & -b_4x_4 \\ -c_2x_2 & u_3 - c_3x_3 & -c_4x_4 \\ -d_2x_2 & -d_3x_3 & u_4 - d_4x_4 \end{vmatrix}, \quad \begin{vmatrix} u_2 - b_2x_2 & -b_1x_1 & -b_3x_3 \\ -c_2x_2 & u_3 - c_1x_1 & -c_3x_3 \\ -d_2x_2 & -d_1x_1 & u_4 - d_3x_3 \end{vmatrix}, \quad \begin{vmatrix} u_2 - b_3x_3 & -b_2x_2 & -b_4x_4 \\ -c_3x_3 & u_3 - c_2x_2 & -c_4x_4 \\ -d_3x_3 & -d_2x_2 & u_4 - d_4x_4 \end{vmatrix},$$

$$\begin{vmatrix} u_2 - b_3x_3 & -b_4x_4 & -b_2x_2 \\ -c_3x_3 & u_3 - c_4x_4 & -c_2x_2 \\ -d_3x_3 & -d_4x_4 & u_4 - d_2x_2 \end{vmatrix}, \quad \begin{vmatrix} u_2 - b_1x_1 & -b_2x_2 & -b_3x_3 \\ -c_1x_1 & u_3 - c_2x_2 & -c_3x_3 \\ -d_1x_1 & -d_2x_2 & u_4 - d_3x_3 \end{vmatrix}, \quad \begin{vmatrix} u_2 - b_4x_4 & -b_3x_3 & -b_2x_2 \\ -c_4x_4 & u_3 - c_3x_3 & -c_2x_2 \\ -d_4x_4 & -d_3x_3 & u_4 - d_2x_2 \end{vmatrix},$$

is presentable in the form $x_1 \cdot F_1$, where F_1 is a complete quadric function of x_1, x_2, x_3, x_4 , consisting of sixteen positive terms, the coefficient of x_1^2 being monomial and that of x_1x_i binomial.

That x_1 is a factor is readily seen by adding the elements of each row; thus, in the case of the first determinant of the six we have at once

$$x_1 \cdot \begin{vmatrix} b_1 & -b_3x_3 & -b_4x_4 \\ c_1 & c_1x_1 + c_2x_2 + c_4x_4 & -c_4x_4 \\ d_1 & -d_3x_3 & d_1x_1 + d_2x_2 + d_3x_3 \end{vmatrix}$$

But here the cofactor of b_1 consists of eight positive terms of the kind promised, the two $+c_4d_3x_4x_3, -c_4d_3x_4x_3$ cancelling each other: the cofactor of c_1 has four of similar kind, and the cofactor d_1 also four. The result consequently can be written

$$x_1 \cdot \begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ b_1c_1d_1 & b_1c_2d_1 & b_1c_1d_3 & b_1c_4d_1 \\ b_1c_1d_2 & b_1c_2d_2 & b_1c_2d_3 & b_1c_4d_2 \\ b_3c_1d_1 & b_3c_1d_2 & b_3c_1d_3 & b_4c_1d_3 \\ b_4c_1d_1 & b_4c_2d_1 & b_3c_4d_1 & b_1c_4d_1 \end{vmatrix} x_1$$

(18) If the elements of each row in the quadrate array here obtained be divided by the diagonal element of that row, there results the more interesting form

$$x_1 \cdot \begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ 1 & \frac{c_2}{c_1} & \frac{d_3}{d_1} & \frac{c_4}{c_1} \\ \frac{c_1}{c_2} & 1 & \frac{d_3}{d_2} & \frac{c_4}{c_2} \\ \frac{d_1}{d_3} & \frac{d_2}{d_3} & 1 & \frac{b_4}{b_3} \\ \frac{c_1}{c_4} & \frac{c_2}{c_4} & \frac{b_3}{b_4} & 1 \end{vmatrix} \begin{matrix} x_1 \cdot b_1c_1d_1 \\ x_2 \cdot b_1c_2d_2 \\ x_3 \cdot b_3c_1d_3 \\ x_4 \cdot b_4c_4d_1 \end{matrix},$$

of which the quadrate array is inverso-symmetric.*

The existence of this peculiar symmetry gives us the means of deriving the twelve non-diagonal elements from those in the diagonal; and as the latter four are easily obtained we can thus write out the full expansion without any delay upon intermediate work. Let us take as an example the third of the six determinants, viz.,

$$\begin{vmatrix} u_2 - b_3x_3 & -b_2x_2 & -b_4x_4 \\ -c_3x_3 & u_3 - c_2x_2 & -c_1x_1 \\ -d_3x_3 & -d_2x_2 & u_4 - d_4x_4 \end{vmatrix}.$$

* The first to draw attention to determinants of such array was probably JOSEPH HORNER: see his "Notes on Determinants" in the *Quart. Journ. of Math.*, viii. pp. 157-162 (year 1865).

We note that here as in the five other cases (1) the letters of the coefficients are b, c, d , (2) that the suffixes of these letters are the suffixes of the associated x 's; and we note that (3) in this particular case, as a glance at the elements suffices to show, b_3, c_2, d_4 do not occur. Now the diagonal elements of the quadrate array of the equivalent bipartite form are the coefficients of

$$x_1 \cdot x_1 x_1, \quad x_1 \cdot x_2 x_2, \quad x_1 \cdot x_3 x_3, \quad x_1 \cdot x_4 x_4$$

and therefore in accordance with the conditions just noted must be

$$b_1 c_1 d_1, \quad c_1 b_2 d_2, \quad b_1 c_3 d_3, \quad d_1 b_4 c_4.$$

Next we note that because of the inverso-symmetry (4) the product of any two conjugate elements of the quadrate array is equal to the product of the two diagonal elements which occupy the same rows and columns. Using this we take the product of any two diagonal elements, say the 3rd and 4th, viz.,

$$b_1 c_3 d_3 \cdot b_4 c_4 d_1$$

and rearrange the suffixes so as to have two sets of 1, 3, 4, these being the suffixes of $x_1 \cdot x_3 x_4, x_1 \cdot x_4 x_3$. The result $b_1 c_4 d_3, b_1 c_3 d_1$ and others similarly obtained gives us

$$x_1 \cdot \begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \hline b_1 c_1 d_1 & b_1 c_1 d_2 & b_1 c_1 d_3 & b_1 c_1 d_4 & x_1 \\ b_2 c_1 d_1 & b_2 c_1 d_2 & b_2 c_1 d_3 & b_2 c_1 d_4 & x_2 \\ b_1 c_3 d_1 & b_1 c_3 d_2 & b_1 c_3 d_3 & b_1 c_3 d_4 & x_3 \\ b_4 c_1 d_1 & b_4 c_1 d_2 & b_4 c_1 d_3 & b_4 c_1 d_4 & x_4 \end{array}$$

whence we have as before

$$x_1 \cdot \begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \hline 1 & \frac{d_2}{d_1} & \frac{d_3}{d_1} & \frac{c_4}{c_1} & x_1 \cdot b_1 c_1 d_1 \\ \frac{d_1}{d_2} & 1 & \frac{d_3}{d_2} & \frac{b_4}{b_2} & x_2 \cdot b_2 c_1 d_2 \\ \frac{d_1}{d_3} & \frac{d_2}{d_3} & 1 & \frac{c_4}{c_3} & x_3 \cdot b_1 c_3 d_3 \\ \frac{c_1}{c_4} & \frac{b_2}{b_1} & \frac{c_3}{c_4} & 1 & x_4 \cdot b_4 c_4 d_1 \end{array}$$

(19) It deserves to be noted in passing that in no case is the inverso-symmetry unique, any one of the non-diagonal elements of the first row of the array being replaceable by another quotient, provided the corresponding change be made in the first column. Thus, instead of d_2/d_1 in the case just dealt with, we may write b_2/b_1 provided we alter the conjugate element into b_1/b_2 .

(20) If $u_1, u_2, \dots, u_5$ be used to stand for $a_1 x_1 + \dots + a_5 x_5, b_1 x_1 + \dots + b_5 x_5, \dots, e_1 x_1 + \dots + e_5 x_5$, there are twenty-four determinants of the type

$$\begin{vmatrix} u_2 - b_2 x_2 & -b_3 x_3 & -b_4 x_4 & -b_5 x_5 \\ -c_2 x_2 & u_3 - c_3 x_3 & -c_4 x_4 & -c_5 x_5 \\ -d_2 x_2 & -d_3 x_3 & u_4 - d_4 x_4 & -d_5 x_5 \\ -e_2 x_2 & -e_3 x_3 & -e_4 x_4 & u_5 - e_5 x_5 \end{vmatrix}$$

which are all presentable in the form $x_1 \cdot F_1$, where F_1 is a complete cubic in $x_1, x_2, \dots, x_5$, consisting of 125 positive terms, viz., in the case of the determinant specified, five terms of the form $b_1 c_1 d_1 e_1 \cdot x_1^3$, twenty compound terms of the form

$$x_1^2 x_2 \cdot (b_1 c_1 d_1 e_2 + b_1 c_1 d_2 e_1 + b_1 c_2 d_1 e_1),$$

and ten compound terms of the form

$$x_1 x_2 x_3 \cdot (b_1 c_1 d_2 e_3 + b_1 c_1 d_3 e_2 + b_1 c_2 d_1 e_3 + b_1 c_2 d_3 e_1 + b_2 c_1 d_1 e_2 + b_2 c_1 d_2 e_1).$$

By adding all the other columns to the first column we see at once that the determinant is equal to

$$\begin{vmatrix} b_1 & -b_3 x_3 & -b_4 x_4 & -b_5 x_5 \\ c_1 & c_1 x_1 + c_2 x_2 + c_4 x_4 + c_5 x_5 & -c_4 x_4 & -c_5 x_5 \\ d_1 & -d_3 x_3 & d_1 x_1 + d_2 x_2 + d_3 x_3 + d_5 x_5 & -d_5 x_5 \\ e_1 & -e_3 x_3 & -e_4 x_4 & e_1 x_1 + e_2 x_2 + e_3 x_3 + e_4 x_4 \end{vmatrix},$$

and by treating the cofactor of b_1 here in the same way we can separate it into two parts of the same kind as the cofactors of b_3, b_4, b_5 . F_1 is thus seen to be the sum of five similar parts, viz.,

$$b_1 x_1 \cdot \begin{vmatrix} c_1 & -c_4 x_4 & -c_5 x_5 \\ d_1 & d_1 x_1 + d_2 x_2 + d_3 x_3 + d_5 x_5 & -d_5 x_5 \\ e_1 & -e_4 x_4 & e_1 x_1 + e_2 x_2 + e_3 x_3 + e_4 x_4 \end{vmatrix} + \dots$$

Taking the first of these and dealing with it after the manner followed in the preceding paragraph we find it expressible as the sum of twenty-five positive terms, viz.,

$$b_1 x_1 \cdot \begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \hline c_1 d_1 e_1 & c_1 d_1 e_2 & c_1 d_1 e_3 & c_1 d_1 e_4 & c_1 d_1 e_5 & x_1 \\ c_1 d_2 e_1 & c_1 d_2 e_2 & c_1 d_2 e_3 & c_1 d_2 e_4 & c_1 d_2 e_5 & x_2 \\ c_1 d_3 e_1 & c_1 d_3 e_2 & c_1 d_3 e_3 & c_1 d_3 e_4 & c_1 d_3 e_5 & x_3 \\ c_4 d_1 e_1 & c_4 d_1 e_2 & c_4 d_1 e_3 & c_4 d_1 e_4 & c_5 d_1 e_4 & x_4 \\ c_1 d_5 e_1 & c_5 d_2 e_1 & c_5 d_3 e_1 & c_4 d_5 e_1 & c_5 d_5 e_1 & x_5 \end{array},$$

and the like is true of the four others. Removing the b at the beginning of each of these five expressions and attaching it to every element of the square array to which it belongs, we obtain a variant form consisting of x_1 multiplied by a quadric in $x_1, x_2, \dots, x_5$ with coefficients of the form $b_i c_k d_l e_n$, x_2 multiplied by a similar quadric, x_3 multiplied by a similar quadric, and so on. It follows therefore that the natural way to represent the result is to use space of three dimensions, each square array being made to overlie the one before it. We should thus have x_1, x_2, x_3, x_4, x_5 placed at equal distances along each of three mutually perpendicular straight lines, and the 125 coefficients disposed within the cube of which these lines are concurrent edges. As $x_1 x_2$ would arise from $x_1 x_1 x_2$ or $x_1 x_2 x_1$ or $x_2 x_1 x_1$ to each of which a coefficient attaches, the coefficient of $x_1^2 x_2$ would clearly be three-termed: and for a similar reason the coefficient of $x_1 x_2 x_3$ would be six-termed. Our theorem is thus established.

(21) The first set of twenty-five positive terms specified in the preceding paragraph may be put in the form

$$b_1x_1 \cdot \begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline 1 & \frac{e_2}{e_1} & \frac{e_3}{e_1} & \frac{e_4}{e_1} & \frac{d_5}{d_1} & x_1 \cdot c_1d_1e_1 \\ \frac{e_1}{e_2} & 1 & \frac{e_3}{e_2} & \frac{e_4}{e_2} & \frac{d_5}{d_2} & x_2 \cdot c_1d_2e_2 \\ \frac{e_1}{e_3} & \frac{e_2}{e_3} & 1 & \frac{e_4}{e_3} & \frac{d_5}{d_3} & x_3 \cdot c_1d_3e_3 \\ \frac{e_1}{e_4} & \frac{e_2}{e_4} & \frac{e_3}{e_4} & 1 & \frac{c_5}{c_4} & x_4 \cdot c_4d_1e_4 \\ \frac{d_1}{d_5} & \frac{d_2}{d_5} & \frac{d_3}{d_5} & \frac{c_4}{c_5} & 1 & x_5 \cdot c_5d_5e_1 \end{array}$$

where the square array is inverso-symmetric. Each of the four other sets can be similarly represented: but it is curious to note that the five arrays so obtained are neither all alike nor all different. In fact, if we use E to stand for that just written, the second and third sets of twenty-five terms are respectively

$$b_1x_2 \cdot \begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{array} \begin{array}{l} x_1 \cdot c_2d_1e_1 \\ x_2 \cdot c_2d_2e_2 \\ x_3 \cdot c_2d_3e_3 \\ x_4 \cdot c_4d_2e_4 \\ x_5 \cdot c_5d_3e_2, \end{array} \quad b_3x_3 \cdot \begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{array} \begin{array}{l} x_1 \cdot c_1d_1e_1 \\ x_2 \cdot c_1d_2e_2 \\ x_3 \cdot c_1d_3e_3 \\ x_4 \cdot c_4d_1e_4 \\ x_5 \cdot c_5d_5e_1, \end{array}$$

while the fourth and fifth are

$$b_4x_4 \cdot \begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline 1 & \frac{e_2}{e_1} & \frac{e_3}{e_1} & \frac{e_4}{e_1} & \frac{d_5}{d_1} & x_1 \cdot c_1d_1e_1 \\ \frac{e_1}{e_2} & 1 & \frac{e_3}{e_2} & \frac{e_4}{e_2} & \frac{c_5}{c_2} & x_2 \cdot c_2d_1e_2 \\ \frac{e_1}{e_3} & \frac{e_2}{e_3} & 1 & \frac{e_4}{e_3} & \frac{d_5}{d_3} & x_3 \cdot c_1d_3e_3 \\ \frac{e_1}{e_4} & \frac{e_2}{e_4} & \frac{e_3}{e_4} & 1 & \frac{c_5}{c_4} & x_4 \cdot c_4d_1e_4 \\ \frac{d_1}{d_5} & \frac{c_2}{c_5} & \frac{d_3}{d_5} & \frac{c_4}{c_5} & 1 & x_5 \cdot c_5d_5e_1, \end{array} \quad b_5x_5 \cdot \begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline 1 & \frac{d_2}{d_1} & \frac{e_3}{e_1} & \frac{e_4}{e_1} & \frac{d_5}{d_1} & x_1 \cdot c_1d_1e_1 \\ \frac{d_1}{d_2} & 1 & \frac{d_3}{d_2} & \frac{c_4}{c_2} & \frac{d_5}{d_2} & x_2 \cdot c_2d_2e_1 \\ \frac{e_1}{e_3} & \frac{d_2}{d_3} & 1 & \frac{c_4}{e_3} & \frac{d_5}{d_3} & x_3 \cdot c_1d_3e_3 \\ \frac{e_1}{e_4} & \frac{c_2}{c_4} & \frac{e_3}{c_4} & 1 & \frac{c_5}{c_4} & x_4 \cdot c_4d_1e_4 \\ \frac{d_1}{d_5} & \frac{d_2}{d_5} & \frac{d_3}{d_5} & \frac{c_4}{c_5} & 1 & x_5 \cdot c_5d_5e_1, \end{array}$$

the square array of the fourth differing from E in the place (2, 5) and that of the fifth in the places (1, 2), (2, 3), (2, 4).

The cofactors of b_1x_1 and b_3x_3 , it will be seen, are identical,—a fact which would have appeared in the preceding paragraph if the expressions for them as determinants of the third order had been given in the case of both.

(22) The general character of the theorem which includes §§ 17, 20 as special cases is readily apparent, the number of terms in F_1 for the next case being 6^4 , and generally n^{n-2} ; and this is exactly the number which the supposed theorem of JACOBI required, as the following considerations will show:—The expansion of $u_1u_2 \dots u_n$ is an aggregate of n^n terms all positive: and as in $|a_1b_2c_3 \dots| \cdot x_1x_2 \dots x_n$ there are $\frac{1}{2}(n!)$ positive terms

and $\frac{1}{2}(n!)$ negative terms, all of which appear in the previous aggregate,—where in fact the cofactor of $x_1x_2 \dots x_n$ is $\begin{vmatrix} \alpha_1 & b_2 & c_3 & \dots \end{vmatrix}^+$,—it follows that the sum or difference of the two is, like the former, an aggregate of n^n positive terms, of which however $\frac{1}{2}(n!)$ are twins. If therefore this sum or difference is to be represented in the form $\sum u_1 \cdot D_1$, the number of terms in $u_1 \cdot D_1$ must be n^{n-1} and the number in D_1 must be n^{n-2} .

(23) The fact that the determinants of the type appearing in §§ 17, 20 have their terms all positive and n^{n-2} in number is a simple deduction from one of the following pair of theorems:—

(a) *In the final expansion of the determinant*

$$\begin{vmatrix} \alpha_1 + \dots + \alpha_n & -\alpha_1 & -\alpha_2 & \dots & \\ -\beta_1 & \beta_1 + \dots + \beta_n & -\beta_2 & \dots & \\ -\gamma_1 & & -\gamma_2 & \gamma_1 + \dots + \gamma_n & \dots & \\ \dots & \dots & \dots & \dots & \dots & \end{vmatrix}_n$$

all the terms are positive, and the number of them is

$$(z - n + 1)(z + 1)^{n-1}.$$

(b) *In the final expansion of the determinant*

$$\begin{vmatrix} \alpha_1 & -\alpha_1 & -\alpha_2 & \dots & \\ \beta_1 & \beta_1 + \dots + \beta_n & -\beta_2 & \dots & \\ \gamma_1 & & -\gamma_2 & \gamma_1 + \dots + \gamma_n & \dots & \\ \dots & \dots & \dots & \dots & \dots & \end{vmatrix}_n$$

all the terms are positive, and the number of them is

$$(z + 1)^{n-1}.$$

If for a particular value of n the two theorems be established, it is easy to show that the second will hold for the next higher value of n , and then from this to show that the first will also hold for this higher value. In the one case the determinant is expressed in terms of the elements of the first column and the complementary minors of these elements: in the other, each element of the first column is increased by all the elements in the same row with it, and the determinant then partitioned into determinants with monomial elements in the first column.