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XXVII. A new method of solving equations of partial differentials

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takes place in Germany more frequently in the order of S. through SW., W., NW., N., NE., E., and SE., than in the opposite order of S. through SE., E., NE., &c."

[To be continued.]

XXVII. *A new Method of solving Equations of partial Differentials.* By S. S. GREATHEED, Esq., B.A., of Trinity College, Cambridge.*

SEPARATION of the symbols of operation from those of quantity, has, as far as I know, been hitherto applied only to the calculus of finite differences, and to the differential calculus where both are involved. It appears to me that if any much greater eminence than that to which analysis has already been brought, remains to be attained by it, that process is the most obvious and likely path. The following pages will show how, by applying it, a large class of partial differential equations, including nearly all that occur in applied mathematics, may be reduced to differential equations of two variables.

The following known theorem is one which will be constantly made use of.

The expression $\varepsilon^h \frac{d}{dx} f(x)$ is equivalent to Taylor's series for $f(x+h)$.

I shall begin with the equation of the first order and degree with constant coefficients:

$$a \frac{dz}{dx} + b \frac{dz}{dy} = c.$$

If instead of $\frac{dz}{dy}$ we had nz , n being a constant, the equation would become

$$a \frac{dz}{dx} + nbz = c,$$

a linear equation between x and z , which may be solved by multiplying by the integrating factor $\varepsilon^{\frac{nbx}{a}}$, whence

$$\frac{d}{dx} \left(\varepsilon^{\frac{nbx}{a}} z \right) = \frac{c}{a} \varepsilon^{\frac{nbx}{a}},$$

therefore

$$\varepsilon^{\frac{nbx}{a}} z = \frac{c}{nb} \varepsilon^{\frac{nbx}{a}} + c,$$

$$z = \frac{c}{nb} + \varepsilon^{-\frac{nbx}{a}} c.$$

* Communicated by the Author.

My method, then, for solving the proposed partial differential equation is this: Substitute $\frac{d}{dy}$ for n , or let n represent $\frac{d}{dy}$, and proceed exactly as in the above solution of the equation between two variables, substituting for the arbitrary constant an arbitrary function of y . The result then will be

$$z = \frac{c}{b} \left(\frac{d}{dy} \right)^{-1} + \varepsilon^{-bx} \frac{d}{d \cdot ay} f(ay).$$

By the proposition which was premised, $\varepsilon^{-bx} \frac{d}{d \cdot ay} f(ay)$ is equivalent to $f(ay - bx)$. Also, since integration is the reverse operation to differentiation, $\left(\frac{d}{dy} \right)^{-1}$ is equivalent to $\int dy = y$. Hence, finally,

$$z = \frac{c y}{b} + f(ay - bx),$$

which is the correct solution of the proposed equation. I shall proceed to explain the meaning, and prove the legitimacy, of the several steps in the solution.

In the first place, multiplying by the factor $\varepsilon^{\frac{bx}{a} \frac{d}{dy}}$ is equivalent to changing all functions of y to which it is prefixed into the same functions of $y + \frac{bx}{a}$. Now z is an unknown function of x and y , suppose it equal to $\phi(x, y)$, then

$$\varepsilon^{\frac{bx}{a} \frac{d}{dy}} \frac{dz}{dx} + \frac{b}{a} \varepsilon^{\frac{bx}{a} \frac{d}{dy}} \frac{dz}{dy}$$

is equivalent to

$$\frac{d}{dx} \phi \left\{ x, y + \frac{bx}{a} \right\} + \frac{b}{a} \frac{d}{dy} \left\{ x, y + \frac{bx}{a} \right\},$$

which is the *total* differential coefficient of $\phi \left\{ x, y + \frac{bx}{a} \right\}$ with respect to x .

As for the other side of the equation, namely,

$$\varepsilon^{\frac{bx}{a} \frac{d}{dy}} \cdot \frac{c}{a},$$

*The more proper expression would be "prefixing the symbol $\varepsilon^{\frac{bx}{a} \frac{d}{dy}}$ ", but since it is convenient, and the operations are exactly analogous, I shall use the term "multiplying."

since $\frac{c}{a}$ does not contain y , the symbol prefixed to it has no effect, and

$$\varepsilon \frac{bx}{a} \frac{d}{dy} \frac{c}{a} = \frac{c}{a}.$$

But the integrals of these two expressions with respect to x are not the same. This may be explained by writing instead of $\frac{c}{a}$, $\frac{d}{dy} \cdot \frac{cy}{a}$; which, by prefixing $\varepsilon \frac{bx}{a} \frac{d}{dy}$, is changed

$$\text{to } \frac{d}{dy} \cdot \frac{c(y + \frac{bx}{a})}{a} = \frac{d}{dx} \frac{c(y + \frac{bx}{a})}{b}.$$

Hence the equation

$$\frac{d}{dx} \left(\varepsilon \frac{bx}{a} \frac{d}{dy} z \right) = \varepsilon \frac{bx}{a} \frac{d}{dy} \frac{c}{a}$$

is the same as

$$\frac{d}{dx} \phi \left\{ x, y + \frac{bx}{a} \right\} = \frac{d}{dx} c \left(\frac{y + \frac{bx}{a}}{b} \right)$$

(the first side representing the total differential coefficient with respect to x). By integration,

$$\phi \left\{ x, y + \frac{bx}{a} \right\} = \frac{c(y + \frac{bx}{a})}{b} + f(ay).$$

Finally, by multiplying both sides of the equation by $\frac{-bx}{a} \frac{d}{dy}$, that is, changing functions of y into functions of $y - \frac{bx}{a}$,

$$\phi(x, y), \text{ or } z = \frac{cy}{b} + f(ay - bx).$$

Hence it may be seen that the essential part of the method consists in making each member of a partial differential equation a total differential coefficient with respect to one of the variables.

The most general class of equations of the first order, which can (as far as I am at present aware) be solved by this method, are those which fall under the form

$$\frac{dz}{dx} + XY \frac{dz}{dy} = Pz + Q,$$

where X is a function of x only, Y of y only, and P and Q are functions of both x and y . This may easily be reduced to the form

$$\frac{dz}{dx} + X \frac{dz}{dy} = Pz + Q,$$

for if $y' = \int \frac{dy}{Y}$, $Y \frac{dz}{dy} = \frac{dz}{dy'}$. I shall then consider the latter equation.

Let it be put under the form

$$\frac{dz}{dx} + \left(X \frac{d}{dy} - P \right) z = Q,$$

and treated as a linear equation between two variables z and x .

The integrating factor is $\epsilon^{\int (X \frac{d}{dy} - P) dx} = \epsilon^{X' \frac{d}{dy}} \cdot \epsilon^{-\int P dx}$, supposing $\int X dx = X'$. The effect of the first part of the factor, namely, $\epsilon^{X' \frac{d}{dy}}$ is, as has before been stated, to change functions of y into functions of $y + X'$, and it affects, not only $\frac{dz}{dx}$ and z , but also the other part of the integrating factor, namely, $\epsilon^{-\int P dx}$, provided P contain y ; therefore y in P must be changed into $y + X'$ before the integration is performed, and afterwards, y is to be restored.

The equation will then stand :

$$\begin{aligned} \epsilon^{X' \frac{d}{dy}} \cdot \epsilon^{-\int P dx} \left\{ \frac{dz}{dx} + \left(X \frac{d}{dy} - P \right) z \right\} \\ = \epsilon^{X' \frac{d}{dy}} \epsilon^{-\int P dx} Q. \end{aligned}$$

The first side is the total differential coefficient of $\epsilon^{-\int P dx} z$ ($y + X'$ being substituted for y) with respect to x ; and the second side is the similar differential coefficient of some unknown function of x and y . It would require the solution of the original equation to exhibit this function, but this difficulty is avoided by integrating the expression on the second side by the common rules. It remains to integrate both sides with respect to x , add an arbitrary function of y , divide both sides by $\epsilon^{X' \frac{d}{dy}}$ (that is, change y into $y - X'$) and by $\epsilon^{-\int P dx}$, and then the value of z is obtained.

I shall adjoin a few examples, but previously I shall prove the following curious theorem, which is of use in several.

$$\frac{1}{a \pm \frac{d}{d \log y}} \cdot y^m = \frac{y^m}{a \pm m},$$

where a is any constant.

Let $y = \epsilon^t$, then $d \cdot \log y = d t$, and

$$\begin{aligned} \frac{1}{a \pm \frac{d}{d \log y}} &= \frac{1}{a \pm \frac{d}{d t}} \cdot \epsilon^{m t} \\ &= \frac{1}{a} \epsilon^{m t} \mp \frac{1}{a^2} \frac{d \epsilon^{m t}}{d t} + \frac{1}{a^3} \frac{d^2 \epsilon^{m t}}{d t^2} \mp \dots \\ &= \left(\frac{1}{a} \mp \frac{m}{a^2} + \frac{m^2}{a^3} \mp \dots \right) \epsilon^{m t} \\ &= \frac{\epsilon^{m t}}{a \pm m} = \frac{y^m}{a \pm m}. \end{aligned}$$

From this it may be shown that if $y, y', y'', \&c.$ be any number of independent variables, and $b, b', b'', \&c.$ constants,

$$\begin{aligned} \frac{1}{a + b \frac{d}{d \log y} + b' \frac{d}{d \log y'} + b'' \frac{d}{d \log y''} + \dots} y^m y'^{m'} y''^{m''} \dots \\ = \frac{y^m y'^{m'} y''^{m''}}{a + m b + m' b' + m'' b'' + \dots}. \end{aligned}$$

Example 1. $x \frac{dz}{dx} + y \frac{dz}{dy} = n z.$

$$\frac{dz}{dx} + \left(\frac{1}{x} \frac{d}{d \log y} - \frac{n}{x} \right) z = 0.$$

The integrating factor is

$$\epsilon^{\log x \frac{d}{d \log y} - n \log x} = x^{-n} \epsilon^{\log x \frac{d}{d \log y}}.$$

Multiply by this factor, and integrate, therefore,

$$x^{-n} \epsilon^{\log x \frac{d}{d \log y}} z = \phi(y) = \phi(\epsilon^{\log y})$$

and $z = x^n \epsilon^{-\log x \frac{d}{d \log y}} \phi(\epsilon^{\log y})$

$$= x^n \phi(\epsilon^{\log y - \log x})$$

$$= x^n \phi\left(\frac{y}{x}\right).$$

Example 2. $\frac{dz}{dx} = \frac{y}{x} \frac{dz}{dy} + \frac{x}{y},$

By transposition $\frac{dz}{dx} - \frac{y}{x} \frac{dz}{dy} = \frac{x}{y},$

or $\frac{dz}{dx} - \frac{1}{x} \frac{d}{d \log y} z = \frac{x}{y}.$

The integrating factor is $\epsilon^{-\log x \frac{d}{d \log y}}$. Multiplying by it, and integrating,

$$\begin{aligned} \epsilon^{-\log x \frac{d}{d \log y}} z &= \int \epsilon^{-\log x \frac{d}{d \log y}} \frac{x}{y} dx \\ &= \int \frac{x^{1 - \frac{d}{d \log y}}}{y} dx \\ &= \frac{x^{2 - \frac{d}{d \log y}}}{2 - \frac{d}{d \log y}} \cdot \frac{1}{y} + \phi(\epsilon^{\log y}). \end{aligned}$$

Divide by $\epsilon^{-\log x \frac{d}{d \log y}}$ or $x^{-\frac{d}{d \log y}},$

therefore $z = \frac{x^2}{2 - \frac{d}{d \log y}} \cdot y^{-1} + \epsilon^{\log x \frac{d}{d \log y}} \phi(\epsilon^{\log y}).$

By the theorem above,

$$\frac{1}{2 - \frac{d}{d \log y}} y^{-1} = \frac{y^{-1}}{3}.$$

Consequently $z = \frac{x^2}{3y} + \phi(xy).$

Example 3. $y \frac{dz}{dx} + x \frac{dz}{dy} = z,$

$$\frac{dz}{dx} + \left(\frac{x}{y} \frac{d}{dy} - \frac{1}{y} \right) z = 0,$$

or $\frac{dz}{dx} + \left(2x \frac{d}{d \cdot y^2} - \frac{1}{y} \right) z = 0.$

The integrating factor is

$$\epsilon \int \left(2x \frac{d}{d \cdot y^2} - \frac{1}{y} \right) dx = \epsilon^{x^2 \frac{d}{d \cdot y^2}} \cdot \epsilon^{-\int \frac{dx}{y}}.$$

The effect of the first part of the factor on the second is to change it into

$$\begin{aligned} \epsilon \int \frac{dx}{\sqrt{y^2+x^2}} &= \epsilon \log(x + \sqrt{x^2+y^2}) = \frac{1}{x + \sqrt{x^2+y^2}} \\ &= \epsilon x^2 \frac{d}{d \cdot y^2} \cdot \frac{1}{x+y}. \end{aligned}$$

Multiplying therefore by $\epsilon x^2 \frac{d}{d \cdot y^2} \frac{1}{x+y}$, and integrating,

$$\begin{aligned} \epsilon x^2 \frac{d}{d \cdot y^2} \frac{z}{x+y} &= \phi(y^2), \\ z &= (x+y) \epsilon^{-x^2 \frac{d}{d \cdot y^2}} \phi(y^2) \\ &= (x+y) \phi(y^2 - x^2). \end{aligned}$$

By the same principles the equation

$$\frac{dz}{dx} - \frac{dz}{dy} = \frac{z}{x+y}$$

may be integrated.

The method is equally applicable to equations of four or more variables, of which I shall take a single example.

Example 4.

$$u \frac{dz}{du} + x \frac{dz}{dx} + y \frac{dz}{dy} = az + \frac{xy}{u}.$$

$$\frac{dz}{du} + \frac{1}{u} \left(\frac{d}{d \log x} + \frac{d}{d \log y} - a \right) z = \frac{xy}{u^2}.$$

Multiply by $\epsilon^{\log u} \left(\frac{d}{d \log x} + \frac{d}{d \log y} - a \right) = u^{\frac{d}{d \log x} + \frac{d}{d \log y} - a}$, and integrate, therefore

$$\epsilon^{\log u} \left(\frac{d}{d \log x} + \frac{d}{d \log y} - a \right) z = \int u^{\frac{d}{d \log x} + \frac{d}{d \log y} - a - 2} xy du$$

$$= u^{\frac{\frac{d}{d \log x} + \frac{d}{d \log y} - a - 1}{\frac{d}{d \log x} + \frac{d}{d \log y} - a - 1}} xy + \phi \left(\epsilon^{\log x}, \epsilon^{\log y} \right)$$

$$z = \frac{u^{-1}}{\frac{d}{d \log x} + \frac{d}{d \log y} - a - 1} xy$$

$$+ u^a \phi (\epsilon^{\log x - \log u}, \epsilon^{\log y - \log u})$$

$$= - \frac{xy}{(a-1)u} + u^a \phi \left(\frac{x}{u}, \frac{y}{u} \right).$$

There are several equations which are not of a form to which this method is immediately applicable, but which become so by a slight transformation. For instance, in the equation

$$X \frac{dz}{dx} + Y \frac{dz}{dy} = Z,$$

where X, Y, Z are functions of x, y , and z , respectively, assume

$$\int \frac{dz}{Z} = z'$$

and it becomes

$$X \frac{dz'}{dx} + Y \frac{dz'}{dy} = 1.$$

In the equation

$$x \frac{dz}{dx} + z \frac{dz}{dy} + y = 0$$

make x the dependent variable, then

$$\frac{dz}{dx} = \frac{1}{\frac{dx}{dz}}, \quad \frac{dz}{dy} = - \frac{\frac{dx}{dy}}{\frac{dx}{dz}}$$

and the equation becomes

$$y \frac{dx}{dz} - z \frac{dx}{dy} + x = 0$$

which may be solved in the same manner as Example 3.

The equation

$$(x+y) \frac{dz}{dx} + (y-x) \frac{dz}{dy} = z,$$

may be transformed so that this method shall be applicable, by taking $x+y = u$, and x and u for the independent variables. It then becomes

$$u \frac{dz}{dx} - 2x \frac{dz}{du} = z.$$

As this article has already been extended to some length, I shall defer to a succeeding Number the application of my method to equations of the second and higher orders. It is in those chiefly that it claims a superiority to the method of Lagrange; and in cases to which his does not apply it gives,

very readily, an abbreviated expression for the series, in which the solutions may be exhibited, but which would often be hard to determine by indirect methods.

Trin. Coll., Camb., July 20, 1837.

S. S. GREATHEED.

XXVIII. *Memoranda on the Origin of the Botanical Alliances.*

By Sir EDW. FF. BROMHEAD, Bart., M.A., F.R.S. L. & E.*

AGARDH mentions Batsch as the first writer who attempted to form alliances on a sound principle, though unsuccessfully; and Dr. Lindley justly notices "*Agardhii opera æstumatissima Bartlingiique, qui viam ad meliores res aperuere.*" Agardh's work and a selection of others, which I in vain endeavoured to procure from the Continent, Dr. Lindley most kindly forwarded to me from his own library. By this I have been enabled to institute a comparison with the alliances proposed by botanists of every school, a test very useful in confirming sound assemblages, ascertaining their true limits, and developing the ground of their formation. Such a reference is moreover a necessary act of justice to the founders.

USNEALES.—These are the *Rhizophyta* or *Sporidiaceæ* of Rudolphi: "*Sporangia spuria vel nulla, sporidia substantiæ vel superficiæ immersa, cotyledonidia nulla.*" Linnæus doubted whether the Fungi should form part of the series in an arrangement of plants. They have been made sufficiently numerous to form one or more distinct alliances; some have represented them as passing into Lichens, others as passing into Algæ.

USNEOSÆ are nearly the *Sporidifera* of Agardh, and the *Fungales* of Dr. Lindley.

JUNGERMANNIALES.—These are the *Musci* (*hepatici* and *frondosi*) of Hedwig, who omits Riccia. They are the *Foliaceæ* of De Candolle, the *Musci* of Bartling, and the *Muscales* of Dr. Lindley.

JUNGERMANNIOSÆ are nearly Dr. Lindley's second section of Acrogens.

LYCOPODIALES.—The *Tetradidymæ* of Wahlenbergh, founded "*quaternatâ dispositione sporarum,*" add to this alliance Ophioglossaceæ, and doubtfully Equisetaceæ. Agardh excluded the latter; Dr. Lindley excluded also the former from his *Lycopodales*. *Lepidodendrum* is now added: there cannot be any reason why extinct genera should be overlooked in botany more than in other branches of natural history. All

* Communicated by the Author.