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VII. *Continuous Girders, Arched Ribs, and Tension Circles.*—By F. Guthrie.

[Read 31st July, 1878.]

In the following paper the terms *strain* and *stress* are used in the sense adopted by Rankine. A *strain* is a distortion of the form of an imperfectly rigid body; a *stress* is a force tending to produce a strain.

The subject of continuous girders has assumed some importance since the introduction of railways, for, though many engineers prefer, for certain reasons, to avoid their use when it can conveniently be done, still there are circumstances when the employment of such girders is desirable. There is some idea that the calculations of the strains and stresses of continuous girders presents peculiar difficulty. Such, however, is not the case. Supposing that the strains and stresses for each separate span can be calculated, the effect of continuity in altering these strains and stresses can be found with the utmost ease, and only involves elementary arithmetic. On a subject like this, which has already attracted the attention of able engineers and mathematicians, there is, of course, not much room for new and useful investigation, and the following observations are advanced with diffidence, as, with the multiplication of periodical and other works on engineering, it is almost impossible to say what is new and what is not.

A continuous girder is one which is supported on piers at intervals between the abutments, and is thus divided into two or more spans, and the principle of the method by which the strains and stresses of such a girder are calculated is as follows:—The girder is conceived as being rendered discontinuous at each pier. (Fig. 1.) The strains and stresses on each span being calculated; equal and opposite bending moments are then applied to

the conterminous ends until they are bent into continuity of position (Fig. 2.) Certain numerical equations are thus obtained for determining the required bending moments or stresses over the piers which are called into play owing to the continuity of the girder, and the effect of these moments is added to the strains and stresses of the various spans already calculated.

The general effect of these moments is easily seen. Thus the couple N_2 at A_2 (Fig. 2) acting on the first span diminishes the pressure on the abutment A_1 , and increases the pressure on the pier A_2 . The couple N_2 at A_2 acting on the second span decreases the pressure on A_2 and increases that on A_3 , while N_3 at A_3 on the same span diminishes the pressure on A_3 and increases that on A_4 , while N_3 at A_3 on the third span diminishes the pressure on the abutment A_4 , and increases the pressure on A_3 , so that the general effect of continuity is to diminish the pressure on the abutments and throw it on the intermediate piers. The effect of each couple in diminishing or increasing the pressures on the abutments and piers is obviously found by dividing the moment of the couple by the length of the span on which it is acting.

The case of a continuous uniform girder with equal spans seldom actually occurs in practice, but its consideration is useful as a preliminary to the investigation of more practical cases. The following example of a particular case will illustrate the extreme simplicity of the calculations involved.

Let n_1, n_2 , in (Fig. 3) stand for the moments at the point A_1, A_2 divided by half the product of the length and weight of each span. Let p stand for the pressure on each support due to continuity, divided by half the weight of a span, and let α stand for the angle of deflection of direction of the girder on the next span at each point of support, divided by $\frac{cwl^2}{12}$ when the number of spans is

even and by $\frac{cwl^2}{24}$ when odd; c being the modulus of curvature for the girder,—i.e., the curvature at any point produced by a unit moment on a unit of length.

The following table gives these several quantities at each point of support:—

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9
	$\frac{1}{2}$	$\frac{0}{1}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{3}{14}$	$\frac{4}{19}$	$\frac{11}{32}$	$\frac{15}{41}$	$\frac{41}{194}$
n	$\frac{0}{194}$	$\frac{41}{194}$	$\frac{30}{194}$	$\frac{33}{194}$	$\frac{32}{194}$	$\frac{33}{194}$	$\frac{30}{194}$	$\frac{41}{194}$	$\frac{0}{194}$
p	$-\frac{41}{194}$	$\frac{52}{194}$	$-\frac{14}{194}$	$\frac{4}{194}$	$\frac{-2}{194}$	$\frac{4}{194}$	$\frac{-14}{194}$	$\frac{52}{194}$	$\frac{-41}{194}$
a	$\frac{56}{194}$	$\frac{-15}{194}$	$\frac{4}{194}$	$\frac{-1}{194}$	$\frac{0}{194}$	$\frac{-1}{194}$	$\frac{4}{194}$	$\frac{-15}{194}$	$\frac{56}{194}$

The fractions in the first line are the convergents to the continued fraction

$$\frac{1}{4 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1} \&c.}}}$$

the law of their formation from each other is obvious.

The numerators in line two are obtained by alternately adding and subtracting the alternate numerators in line 1, beginning from the end.

The numerators, except the first, in line three are the alternate denominators of the fractions in line 1 with alternate signs. The first is the last *numerator* in that line.

The numerators in line 4 are the sums of the consecutive pairs of numerators in line 1 with alternate signs.

As the strains and stresses of any detached span of such a girder are easily calculated, the whole condition of the girder is known.

The distances of the points of contrary flexure in each span from the end of that span, are the roots of the quadratic

$$a^2 - (1 + n_r - n_{r+1}) a + n_{r+1}$$

Another preliminary matter, the consideration of which essentially simplifies the calculations in practical cases, is the effect on a continuous girder of a bending stress at one end.

The effect of such a stress will be to produce strains which continually decrease in amount towards the other end (See Fig. 4.) The critical strains at each point of support are easily calculated from the fact that they must be such as to bend the end of each span into continuity of position with the end of the next span. This gives a series of equations connecting these moments with each other of the form

$$N_3 = 2 \frac{l_1 + l_2}{l_2} N_2$$

$$N_4 = 2 \frac{l_2 + l_3}{l_3} N_3 - \frac{l_2}{l_3} N_3$$

where l_1, l_2 are the breadths of the several spans. These equations enable us to express each moment in terms of the terminal moment N_6 , and the moments at the end of each span being known, the strains and stresses in that span can be set down at once.

When the spans are of equal length the calculations become expressly simple. Subjoined is an example for a five span girder:

	A_1	A_2	A_3	A_4	A_5	A_6
$\frac{a}{l}$	$\frac{6}{1}$	$\frac{1}{5}$	$\frac{4}{19}$	$\frac{15}{71}$	$\frac{56}{265}$	$\frac{200}{989}$
$\frac{N}{N_6}$	$\frac{20}{209}$	$\frac{19}{209}$	$\frac{4}{209}$	$\frac{15}{209}$	$\frac{56}{209}$	$\frac{200}{209}$
$\frac{6a}{clN_6}$	$\frac{21}{209}$	$\frac{2}{209}$	$\frac{7}{209}$	$\frac{26}{209}$	$\frac{67}{209}$	$\frac{362}{209}$
$\frac{Pl}{N_6}$	$\frac{1}{209}$	$-\frac{6}{209}$	$\frac{24}{209}$	$-\frac{90}{209}$	$\frac{336}{209}$	$\frac{265}{209}$

The first line gives the position of the point of contrary flexure relatively to the end of each span, its law of formation is obvious.

The second line gives the ratio of the bending moment at each point of support to the terminal moment

The fractions in the third line multiplied by $\frac{c l N_6}{6}$ give the deflections in directions at these points, and then if the fourth line multiplied by $\frac{N_6}{l}$ give the pressures at these points.

When equal and opposite moments act at the two ends of such a girder, the question is even more simplified, as will be seen from the following instance of a five span girder. (Fig. 5).

	A_1	A_2	A_3	A_4	A_5	A_6
	0	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{4}{7}$	$\frac{11}{19}$
$\frac{a}{l}$	$\frac{10}{24}$	$\frac{5}{6}$	..	$\frac{1}{6}$	$\frac{5}{24}$	..
$\frac{N}{N_6}$	$\frac{10}{19}$	$-\frac{5}{19}$	$+\frac{1}{19}$	$+\frac{1}{19}$	$-\frac{5}{19}$	$+\frac{10}{19}$
$\frac{2a}{c l N_6}$	$\frac{11}{19}$	$-\frac{3}{19}$	$+\frac{1}{19}$	$-\frac{1}{19}$	$+\frac{3}{19}$	$-\frac{11}{19}$
$\frac{P l}{6 N_6}$	$\frac{4}{19}$	$-\frac{5}{19}$	$+\frac{1}{19}$	$+\frac{1}{19}$	$-\frac{5}{19}$	$+\frac{4}{19}$

The first line contains the convergents to

$$1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1} \&c.}}$$

From which the pressures and other strains and stresses can be determined.

Where the spans are of irregular length, the calculation is still within the limits of the most elementary arithmetic. Subjoined is an example (Fig. 7).

	A_2	A_3	A_4	A_5	A_6
$p_r = \frac{n_r l_r^2 + w_2 + 1}{4 l_{r+1}}$	9.45	14.2083	18.	17.05	11.8125
$a_r = 2 \frac{l_r + l_{r+1}}{2}$	3.6	3.6	4.	4.4	4.5
$\beta_r = \frac{l_r}{l_{r+1}}$	.81	.83	1.	1.2	1.25
$\gamma_r = a_r \gamma_{r-1} - \beta_r \gamma_{r-2}$	3.6	12.36	45.86	186.973	784.046
$\delta_r = p_r - a_r \delta_{r-1} - \beta_r \delta_{r-2}$	9.45	-20.4416	90.316	-355.813	1500.076
$N_r = \delta_{r-1} - \gamma_{r-1} N_{r-1}$	1.913	2.562	3.219	2.562	1.913

The various lines are calculated as indicated by the formulæ annexed.

Even the effect of a subsidence of one of the points of support can be determined with the greatest ease as follows :—

Calculate the strains and stresses on the girder, considering this support wholly removed, and find the changes of stress and strain throughout, and thence find the depression of the point of the girder above the support so removed, and then correct each strain and stress by adding or subtracting the same fraction of the changes of strain that the actual subsidence of the pier is of the subsidence due to the entire removal of that pier. Thus, in the above case, if the support A_4 were wholly removed (Fig. 8) the bending stresses would be as follows :—

A_2	A_3	A_5	A_6
$- \cdot 17$	10·09	10·09	$- \cdot 17$

whence the depression of the point A_4 is easily calculated to be

$$\frac{4 \cdot 91 cw \times 12^3}{96}$$

where c depends on the elasticity of the girder. If the actual subsidence of the pier A_4 is, say, one-third of this, then the actual correction for this subsidence will be one-third of the changes due to the total removal of A_4 .

Thus, for example, the pressure on the abutments in the above case would be affected as follows. The pressures in terms of w , the weight of a unit of length when A_4 is in position :—

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
Due to wt. ..	2	4.5	5.5	6.	5.5	4.5	2
Due to flexion ..	-.48	.35	.02	.22	.02	.35	-.48
Total ..	1.52	4.85	5.52	6.22	5.52	4.85	1.52

When A_4 is wholly removed:—

A_1	A_2	A_3	A_4	A_5	A_6	A_7
2.	4.5	8.5	..	8.5	4.5	2.
.04	-1.75	1.71	..	1.71	-1.75	.04
2.04	2.75	10.21	..	10.21	2.75	2.04
1.52	4.85	5.52	6.22	5.52	4.85	1.52
.52	-2.10	4.69	-6.22	4.69	-2.10	.52

If then the pier A_4 sank by half the amount required to take the entire weight off it, we should have (Fig. 9).

A_1	A_2	A_3	A_4	A_5	A_6	A_7
1.52	4.85	5.52	6.22	5.52	4.85	1.52
.26	-1.05	2.35	-3.11	2.35	-1.05	.26
1.78	3.80	7.87	3.11	7.87	3.80	1.78

all in terms of w , where w is the weight of a unit of length of the girder and is equal to $\frac{W}{30}$ where W is the weight of the entire girder.

The methods above indicated are applicable to determine the effect of a living load whether distributed or local.

Divide off the loaded span or spans, and calculate the several parts of the girder separately. Then apply to the continuous ends moments N_1 and N_2 , such as shall render the sections continuous in position and direction. Two simple arithmetical equations result for determining these moments, and the effect of these moments on each section of the girder are then calculated with the utmost ease by the method above indicated.

The subject of arched girders has perhaps hitherto been less satisfactorily investigated than that of continuous girders.

It may be approached as follows in any curve:—

$$\frac{1}{\rho} = \frac{1 - \frac{dr}{ds}^2 - r \frac{d^2r}{ds^2}}{\sqrt{1 - \frac{dr}{ds}^2}}$$

where ρ is the radius of curvature.

Where the deviation from the circular form is very small $\frac{dr}{ds}^2$ may be omitted, and we get

$$\frac{1}{\rho} = \frac{1}{r} - \frac{d^2r}{ds^2}$$

If, now, a be the radius of a circle, and Δr its variation due to a slight deflection

$$\frac{1}{\rho} = \frac{1}{a} + \frac{\Delta r}{a^2} + \frac{d^2 \Delta r}{ds^2}$$

Where the *superadded* curvature due to a slight deflection is approximately

$$\frac{d^2 \Delta r}{ds^2} + \frac{\Delta r}{a^2}$$

and

$$\frac{d^2 \Delta r}{ds^2} + \frac{\Delta r}{a^2} = cm$$

where m is the moment of deflection, being generally a function of s , and c is a constant depending on the elasticity of the material of the girder and on its transverse section.

From this equation Δr can generally be easily determined in terms of s , and from Δr the overlap Δs is determined by the obvious relations.

$$\Delta s = \frac{1}{a} \int \Delta r ds. \quad (\text{Fig. 10.})$$

From Δr and Δs , the vertical and horizontal displacements of the end of the girder due to deflection are easily calculated.

Subjoined is a summary of some of the principal results for various forces:—

I.—Weight vertical (Fig. 11).

$$M = wa (\sin \phi - \sin \theta)$$

$$\Delta r = \frac{cwa^3}{2} (\sin 2\phi - \phi \cos \phi - \sin \phi)$$

$$\Delta s = \frac{cwa^3}{2} (3\phi \sin \phi + \cos \phi - \sin^2 \phi - 1)$$

$$a = \frac{cwa^2}{2} (\phi \sin \phi - 2 \sin^2 \phi)$$

when a is the deflection of the end of the girder in direction.

$$\Delta x = \frac{cwa^3}{2} \left(\phi (2 - \cos 2\phi) + \frac{3}{2} \sin 2\phi - 2 \sin \phi \right)$$

$$\Delta y = \frac{cwa^3}{2} \left(\phi \sin 2\phi - (1 + 3 \cos \phi) (1 - \cos \phi) \right)$$

$\frac{l}{a}$	$\frac{\Delta r}{cwa^3}$	$\frac{\Delta s}{cwa^3}$	$\frac{a}{cwa^2}$
$\frac{\pi}{2}$	$-\frac{1}{2}$	$\frac{3\pi}{4} - 2$	$\frac{\pi}{4} - 1$
π	$\frac{\pi}{2}$	-2	0
2π	$-\pi$	0	0
$n\pi$	$\pm \frac{n\pi}{2}$	$-\frac{2}{0}$	$0 \begin{matrix} n \text{ odd} \\ n \text{ even} \end{matrix}$

II.—Weight horizontal (Fig. 12):

$$M = \pi a (\cos \theta - \cos \phi)$$

$$\Delta r = \frac{c\pi a^3}{2} (\cos \phi - \phi \sin \phi - \cos^2 \phi)$$

$$\Delta s = \frac{c\pi a^3}{2} (\sin 2\phi + \sin \phi - 3\phi \cos \phi)$$

$$a = \frac{c\pi a^3}{2} (\sin 2\phi - \sin \phi - \phi \cos \phi)$$

$$\Delta x = \frac{c\pi a^3}{2} (1 - \cos \phi) (1 + 3 \cos \phi - \phi \sin 2\phi)$$

$$\Delta y = \frac{c\pi a^3}{2} \left(\left(\frac{3}{2} \sin 2\phi - \phi (2 + \cos 2\phi) \right) \right)$$

$\frac{l}{a}$	$\frac{\Delta r}{c\pi a^3}$	$\frac{\Delta s}{c\pi a^3}$	$\frac{a}{c\pi a^3}$
$\frac{\pi}{2}$	$-\frac{\pi}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$
π	-2	$\frac{3\pi}{2}$	$\frac{\pi}{2}$
2π	0	-3π	$-\pi$
$n\pi$	$-\frac{2}{0}$	$\pm \frac{3n\pi}{3}$	$\pm \frac{n\pi}{2}$ $\begin{matrix} n \text{ odd} \\ n \text{ even} \end{matrix}$

III.—A simple bending stress at the end (Fig. 13):

$$M = cN$$

$$\Delta r = ca^2 N (\cos \phi - 1)$$

$$\Delta s = ca^2 N (\phi - \sin \phi)$$

$$a = -caN (\sin \phi)$$

$$\Delta x = ca^2 N (\phi \sin \phi + \cos \phi - 1)$$

$$\Delta y = ca^2 N (\phi \cos \phi - \sin \phi)$$

l	$\frac{\Delta r}{cNu^2}$	$\frac{\Delta s}{cNa^2}$	$\frac{a}{cna}$
$\frac{\pi}{2}$	- 1	π	- 1
π	- 2	$\frac{\pi}{2} - 1$	0
2π	0	π	0
$n\pi$	$-\frac{2}{0}$	$n\pi$	0 $\begin{matrix} n \text{ odd} \\ n \text{ even} \end{matrix}$

IV.—Circular arc deflected by its own weight (Fig. 14.)

$$M = wa \frac{a}{l} (\theta \sin \theta + \cos \theta - \phi \sin \theta - \cos \phi)$$

$$\Delta r = cwa^3 \left\{ \left(\frac{1}{\phi} - \frac{\phi}{4} - \frac{\cos \phi}{\phi} \right) \cos \phi - \frac{\sin \phi}{4} \right\}$$

$$\Delta s = cwa^3 \left\{ \left(\frac{\phi}{4} + \frac{5}{4\phi} + \frac{\cos \phi}{\phi} \right) \sin \phi - \frac{5}{4} \cos \phi - 1 \right\}$$

$$a = cwa^2 \left\{ \left(\frac{\phi}{4} + \frac{3}{4\phi} + \frac{\cos \phi}{\phi} \right) \sin \phi - \frac{\cos \phi}{4} \right\}$$

$$\Delta x = cwa^3 \left\{ \frac{\phi}{4} + \frac{1}{4\phi} (1 - \cos \phi) (5 + 9 \cos \phi) - \frac{1}{2} \sin 2\phi - \sin \phi \right\}$$

$$\Delta y = cwa^3 \left\{ \frac{9}{8\phi} \sin 2\phi - (\cos \phi + \frac{1}{2})^2 \right\}$$

$\frac{l}{a}$	$\frac{\Delta r}{cwa^3}$	$\frac{\Delta s}{cwa^3}$	$\frac{\alpha}{cwa^3}$
$\frac{\pi}{2}$	$-\frac{1}{4}$	$\frac{\pi}{8} + \frac{5}{2\pi} - 1$	$\frac{\pi}{8} - \frac{3}{2\pi}$
π	$\frac{\pi}{4} - \frac{2}{\pi}$	$\frac{1}{4}$	$-\frac{1}{4}$
2π	$-\frac{\pi}{2}$	$-\frac{9}{4}$	$\frac{1}{4}$
$n\pi$	$-\left(\frac{1}{n\pi} \pm \left(\frac{n\pi}{4} - \frac{1}{n\pi}\right)\right)$	$-1 \mp \frac{5}{4}$	$\pm \frac{1}{4}$ $n \text{ odd}$ $n \text{ even}$

From these data the thrust of a circular arc which is not allowed to expand, caused by a weight $2w$ at the crown, is

$$w_1 = w \frac{2(1 - \cos \phi)(1 + 3 \cos \phi) - 2\phi \sin 2\phi}{2\phi(2 + \cos 2\phi) - 3 \sin 2\phi}$$

where ϕ is half the angular expanse of the arc (Fig. 15) and the thrust due to evenly distributed weight is

$$\begin{aligned} w_1 &= w \frac{\sin 2\phi \left(\frac{9}{29} - 2\phi \right) - 4 - 5 \cos 2\phi}{2\phi(2 + \cos 2\phi) - 3 \sin 2\phi} \\ &= w \frac{\sin \psi \left(\frac{9}{\psi} - \psi \right) - 5 \cos \psi - 4}{\psi(2 + \cos \psi) - 3 \sin \psi} \end{aligned}$$

when w is half the entire weight, and $\psi = 2\phi$.

When ϕ is a small angle this gives

$$w_1 = \frac{25w}{16\phi} \text{ for load.}$$

$$w_1 = \frac{w}{\phi} \text{ for distributed weight}$$

When $\phi = \frac{\pi}{2}$

$$w_1 = \frac{2w}{\pi} \text{ for load at crown}$$

$$w_1 = \frac{w}{\pi} \text{ for distributed weight.}$$

By combining together the deflection caused by weight, that caused by the vertical pressures from the abutments and that caused by horizontal thrust, when the arc is not allowed to spread, we get for the *depression* of the crown of a semicircular arc

$$cma^3 \left(\frac{5\pi}{8} - \frac{3}{\pi} - 1 \right) = \cdot 0085657 cma^3$$

showing that the crown of a semicircular arc of uniform weight is depressed, not elevated, as is sometimes stated.

The deflection of the crown of a circular arc of small angular magnitude will be :

$$(1) \text{ an elevation due to weight of } \frac{cwl^3}{8}$$

$$\text{a depression due to abutments of } \frac{cwl^3}{3}$$

$$\text{an elevation due to thrust of } \frac{5cwl^3}{24}$$

and since the sum of these deflections is zero, it follows that the crown of a circular arc of small rise due to bending stresses is nothing.

This suggests that the depression due to compression should be calculated, and the result is as follows :

$$\Delta = \frac{a}{an} \left(w_1 + \frac{w}{\phi} \right) \sin \phi - w \cos \phi$$

$$\cot \frac{\phi}{2} + \frac{\phi}{2} \left(1 - \cot \frac{2\phi}{2} \right)$$

where ϕ is small this comes to

$$\Delta = \frac{cwl\phi}{a\phi^2}$$

where l is the length of the arc, a the area of a section, ϕ the moment of inertia of a section about a neutral axis and w the entire weight.

In the case of a semicircular arc

$$\Delta = \frac{3can\phi}{a\pi}$$

from which it can be calculated that in this case the depression of the crown due to bending stresses is to its depression due to compression as a^2 to $(5.2\bar{b})^2$ nearly. When b is the depth of the girder, supposed to consist of two equal flanges with a tension web.

An interesting development of the circular arc is the elastic ring.

An elastic ring, acted on by two equal and opposite radial forces, will obviously be compressed in the direction of the diameter along which those forces act, and expanded in the opposite direction. (Fig. 16.)

Calling the force $2w$, and the bending stress at A and A_1 due to compression N ,

$$w \text{ at } A \text{ gives } \Delta r = -\frac{\pi}{2} cwa^3, \Delta s = 2cwa^3$$

$$N \text{ „ } A \text{ „ } \Delta r = 2ca^2N, \Delta s = -\pi ca^2M$$

And since A does not move laterally,

$$\pi ca^2N = 2cwa^3$$

$$N = \frac{2wa}{\pi}.$$

Whence A is elevated by

$$\frac{\pi}{2} cwa^3 - 2ca^2N = 2cwa^3\left(\frac{\pi}{4} - \frac{2}{\pi}\right)$$

Whence each of the points A and A_1 are moved towards the centre by

$$cwa^3\left(\frac{\pi}{4} - \frac{2}{\pi}\right).$$

The deflection at any point will be

$$\begin{aligned} \Delta r &= cwa^3\left(\frac{\phi}{2}\cos\phi - \frac{\sin\phi}{2}\right) - ca^2N(\cos\phi - 1), \\ &= cwa^3\left(\frac{\phi}{2}\cos\phi - \frac{\sin\phi}{2} + \frac{2}{\pi}(1 - \cos\phi)\right), \end{aligned}$$

Or, removing the origin to the centre of symmetry,

$$\begin{aligned} \Delta r &= cwa^3\left(\frac{\phi}{2}\sin\frac{\phi}{2} - \frac{\sin\phi}{2} - \frac{\pi}{4}\cos\phi + \frac{2}{\pi}\right), \\ &= cwa^3\left(\left(\frac{\phi}{2} - \frac{\pi}{4}\right)\cos\phi - \frac{\sin\phi}{2} + \frac{2}{\pi}\right). \end{aligned}$$

This value of Δr is nothing when $\phi = 43^\circ, 53', 29'$ very nearly, which gives the point of the circumference whose position remains undisturbed.

The points of maximum and minimum curvature are found from $\sin \phi \left(\phi - \frac{\pi}{2} \right) = 0$, and the neutral points from

$$\begin{aligned} na \sin \phi &= N \\ \sin \phi &= \frac{2}{\pi} \\ \phi &= 39^\circ 32' 24''.9 \end{aligned}$$

When $\phi = \frac{\pi}{2}$

$$\Delta r = \left(\frac{2}{\pi} - \frac{1}{2} \right) cna^3 = cna^3 \times .1365198.$$

These results enable us to treat generally the deflections of a tension wheel, that is, of a wheel the rigidity of whose tire is secured by radial ties instead of by rigid spokes or struts.

It is obvious that all that is required is to calculate the deflection from the centre of each critical point due to the different pairs of forces, and then by equating to nothing the deflections of those points which are rigidly connected, equations are found for determining the strains on the various ties.

Subjoined are the results of various particular cases.

In figure 17 the ring is compressed by equal and opposite forces w at A and A_1 , and prevented from expanding by the tie $B B_1$. The tension in the tie is $.918 w$. Figures 18, 19, 20, and 21 are cases where the number of ties is greater.

The following points call for special attention. 1. That *all* the ties are in tension even those nearest to the points of application of the compressing forces. 2. That the tensions are alternately greater and less with a constant approach to equality; that is to say, the least tension is found in the tie nearest to the points of application of the compressing forces, the greatest in the next tie, the next least in the next tie, and so on.

If the applied force be at the nave of the wheel instead of at the circumference the calculation is somewhat more complicated. The following plan, however, may then be adopted.

Imagine a complete ring acted on by various forces towards the centre.

Where each force acts the circumference will be flattened. Now let the ring be cut close by one force, and one end be held in position, it is then obvious that the overlap will be as in Fig. 22.

Now apply to the extremity three forces, one tangential, the other radial, and the third a bending stress, so that the two ends may be brought into continuity of position. Three equations result for determining these forces. Then calculate the deflections caused by each circumferential force, and by the terminal forces representing the thrust, shear and bending stress at the point of imaginary section. Equate to nothing the sum of the deflections of each point whose distance from the centre remains constant, and we get a number of equations by which the different forces can be determined.

The equation for determining the deflection of any point A_2 , is as follows :

$$\Delta r = \frac{cwa^3}{\pi} \left(\pi \Sigma_{o,r} w \sin(\phi_r - \phi) - \Sigma_{o,r} w (\phi_r - \phi) \cos(\phi_r - \phi) \right. \\ \left. - (\Sigma_{o,n} w \phi \cos \phi) \left(\frac{\phi_r}{2} \cos \phi_r - \frac{\sin \phi_r}{2} \right) - (\Sigma_{o,n} w \phi \sin \phi) \right. \\ \left. \left(\frac{\phi_r}{2} \sin \phi_r \right) - (\Sigma w) (1 - \cos \phi) \right)$$

The stresses in the example shewn in Fig. 23 are calculated from this formula.

The weight w is applied to the nave of a tension wheel with 8 radial tires and the strains are then

$$\begin{aligned} w_0 &= \cdot 89 \, w \\ w_1 &= \cdot 86 \, w \\ w_2 &= \cdot 92 \, w \\ w_3 &= \cdot 78 \, w \end{aligned}$$

It will be observed that, as before, the stresses are alternately greater and less, but tend to equality, the least is that next to w , the greatest next to that, the next to the least comes next, and the next to the greatest follows, and so on, thus reproducing the law of stresses in the continuous girder.

The case of an elastic ring deflected by its own weight is perhaps more interesting theoretically than practically useful. Two cases may be considered, one in which the ring is suspended from its circumference, the other in which it rests on a horizontal support. Neither case presents any practical difficulty.

In the first case (Fig. 24) consider the ring as discontinuous at the base, and calculate the strains at the extremity by the foregoing formula. Then apply horizontal forces and moments so as to bring the extremities into continuity of position. The horizontal force so required is $\frac{w}{4\pi}$ when w is the entire weight of the ring,

and $N = \frac{w}{4\pi}$ and the consequent radial deflection at any point P is :

$$\Delta r = \frac{cw a^3}{\pi} \left(\left(\frac{\phi^2}{4} - \frac{\phi \pi}{2} + 1 \right) \cos \phi - \left(\frac{\phi}{2} - \frac{2}{\pi} \right) \sin \phi - 1 \right)$$

The neutral point is given by the equation $\tan \phi = 2(\pi - \phi)$.

The case of an elastic ring supported on a horizontal plane is treated in a similar manner, and gives parallel results.