On the Vibrations of a Spherical Shell. By Horace Lamb, M.A.

[*Read Dec. 14th*, 1882.]

1. The methods employed in a former communication, "On the Vibrations of an Elastic Sphere,"* can of course be readily adapted to the case of a shell bounded by two concentric spherical surfaces; the chief modification necessary being that we must now associate with every term of our formulæ involving a solid harmonic of degree n, a precisely similar term in which n is replaced by $-n-1$. The most interesting case is when the thickness of the shell is infinitely small compared with the radius. The results of the investigation then assume a very simple form, and may be stated as follows:—

The fundamental modes of vibration fall as before into two classes.

In the modes of the *First Class*, the motion at every point of the shell is wholly tangential. In the n^{th} species of this class, the lines of motion are the contour lines of a surface harmonic S_n, and the amplitude of the vibration at any point is proportional to the value of $dS_n/d\epsilon$, where $d\epsilon$ is the angle subtended at the centre by a linear element drawn on the surface of the shell at right angles to the contour line passing through that point. The frequency $(p/2\pi)$ is determined by the equation

$$k^2a^2 = (n-1)(n+2) \qquad\qquad (1),$$

where a is the radius of the shell, and $k^2 = p^2\rho/\mathfrak{n}$, if ρ denote the density, and $\mathfrak{n}$ the rigidity, of the substance.

In the vibrations of the *Second Class*, the motion is partly radial, and partly tangential. In the n^{th} species of this class the amplitude of the radial component is proportional to S_n, a surface harmonic of order n. The tangential component is everywhere at right angles to the contour lines of the harmonic S_n on the surface of the shell, and its amplitude is proportional to $\Lambda\, dS_n/d\epsilon$, where Λ is a certain constant, and $d\epsilon$ has the same meaning as before. It will appear further on (§ 3) that

$$\Lambda = -\frac{k^2a^2-4\gamma}{2n(n+1)\gamma} \qquad\qquad (2),$$

where k retains its former meaning, and $\gamma = (1+\sigma)/(1-\sigma)$, σ denoting, as before, Poisson's ratio. Corresponding to each value of n there are *two* values of k^2a^2, given by the equation

$$k^4a^4 - k^2a^2\{(n^2+n+4)\gamma+n^2+n-2\} + 4(n^2+n-2)\gamma = 0 \ldots (3).\dagger$$

* *Proceedings*, Vol. xiii., pp. 189—212.

† The case $n=0$ is exceptional. The vibrations are then purely radial, and the frequency is determined by $k^2a^2 = 4\gamma$.

Of the two roots of this equation, one is $<$ and the other $>4\gamma$. It appears, then, from (2) that the corresponding fundamental modes are of quite different characters. The mode corresponding to the lower root is always the more important.

When $n=1$, the values of k^2a^2 are 0 and 6γ. The zero root corresponds to a motion of translation of the shell as a whole parallel to the axis of the harmonic S_1. In the other mode the radial motion is proportional to $\cos\theta$, where θ is the colatitude measured from the pole of S_1; the tangential motion is along the meridian, and its amplitude (estimated in the direction of θ increasing) is proportional to $\frac{1}{2}\sin\theta$.

When $n=2$, the values of ka corresponding to various values of σ are given by the following table:—

$\sigma=0$	$\sigma=\frac{1}{4}$	$\sigma=\frac{3}{10}$	$\sigma=\frac{1}{3}$	$\sigma=\frac{1}{2}$
1·120	1·176	1·185	1·190	1·215
3·570	4·391	4·601	4·752	5·703

The most interesting variety is that of the zonal harmonic. Making $S_2=(3\cos^2\theta-1)/2$, we see that the polar diameter of the shell alternately elongates and contracts, whilst the equator simultaneously contracts and expands respectively. In the mode corresponding to the lower root, the tangential motion is towards the poles when the polar diameter is lengthening, and *vice versâ*. The reverse is the case in the other mode. We can hence understand the great difference in frequency.

2. To explain the manner in which the foregoing results were obtained, it may be sufficient to take the vibrations of the First Class. Omitting the time factor, the component displacements parallel to a system of rectangular axes having the centre of the shell as origin are given by the formulæ

$$\alpha=\psi_n(kr)\left(y\frac{d}{dz}-z\frac{d}{dy}\right)\chi_n+\psi_{-n-1}(kr)\left(y\frac{d}{dz}-z\frac{d}{dy}\right)\chi_{-n-1},\ \&c.,\ \&c. \qquad\text{.........(4)},$$

where χ_n, χ_{-n-1} are solid harmonics of the algebraical degrees indicated by the suffixes, and $\psi_n(kr)$ is defined by [3].* The surface conditions [18] become

$$\left(r\frac{d}{dr}-1\right)\alpha=0,\ \&c.,\ \&c.,$$

whence
$$P_n\chi_n+P_{-n-1}\chi_{-n-1}=0 \qquad\text{.......................(5)},$$

* The square brackets are used to distinguish references to the former paper.

where $$P_n = kr\psi_n'(kr) + (n-1)\psi_n(kr).$$

The equation (5) is to hold for $r=a$ and for $r=b$, where a and b respectively denote the inner and outer radii of the shell, which is not yet assumed to be infinitely thin. Eliminating the surface harmonics involved in the two equations of the form (5), we are led to the following equation to determine k:—

$$\begin{vmatrix} P_n a^n, & P_{-n-1}a^{-n-1} \\ \&c. & \&c. \end{vmatrix} = 0 \quad \text{.....................(6)}.$$

The second row is to be obtained from the first by writing b for a throughout. Passing now to the case of an infinitely thin shell, we write, in the second row, $b = a+da$, subtract the first row, and divide out by da. Now

$$a\frac{d}{da}(P_n a^n) = a^n\{k^2a^2\psi_n''(ka) + 2nka\psi_n'(ka) + n(n-1)\psi_n(ka)\}.$$

But, from [4] and [5], we have, identically,

$$k^2a^2\psi_n''(ka) + (2n+2)\,ka\psi_n'(ka) + k^2a^2\psi_n(ka) = 0.$$

Hence $$a\frac{d}{da}(P_n a^n) = a^n\{2ka\psi_n'(ka) + (k^2a^2 - n\,.\,n-1)\,\psi_n(ka),$$

and (6) becomes, after obvious reductions,

$$\begin{vmatrix} ka\psi_n'(ka) + (n-1)\,\psi_n(ka), & \&c. \\ (k^2a^2 - .\,n-1\,.\,n+2)\,\psi_n(ka), & \&c. \end{vmatrix} = 0 \quad \text{...............(7)},$$

where the second column is to be obtained from the first by changing n into $-n-1$. Since this change leaves the product $(n-1)(n+2)$ unaltered, the left-hand side of (7) contains $k^2a^2-(n-1)(n+2)$ as a factor. It will be shown immediately that

$$ka\psi_n'(ka)\,\psi_{-n-1}(ka) - ka\psi_{-n-1}'(ka)\,\psi_n(ka) \\ + (2n+1)\,\psi_n(ka)\,\psi_{-n-1}(ka) = 2n+1 \text{.........(8)};$$

so that the other factor is $2n+1$. Hence

$$k^2a^2 = (n-1)(n+2) \quad \text{..........................(1)}.$$

The relation (8) follows from Helmholtz's theorem,* that if u, v be any two functions satisfying the equation $(\nabla^2+k^2)\,\phi = 0$, where $\nabla^2 = d^2/dx^2 + d^2/dy^2 + d^2/dz^2$, then

$$\iint\left(u\frac{dv}{dn} - v\frac{du}{dn}\right)dS = 0,$$

where the integration extends over the boundary of any region not

* See Rayleigh's *Sound*, t. 2, §§ 294, 327.

containing singular points of u or v, and dn denotes an element of the inwardly directed normal to dS. Making

$$u = \psi_n(kr) \,.\, r^n T_n, \quad v = \psi_{-n-1}(kr) \,.\, r^{-n-1} T_n,$$

where T_n is a surface harmonic of order n, and taking as the region in question the space included between two spheres described about the origin as centre, we readily find that the value of the left-hand side of (8) is independent of a. Making a infinitesimal, we see that this value is $2n+1$.

The frequencies of the various vibrations of the Second Class, for a shell whose inner and outer radii are a and b, are determined by the equation

$$\begin{vmatrix} A_n, & C_n, & B_{-n-1}, & D_{-n-1} \\ \&c., & \&c., & \&c., & \&c. \\ B_n, & D_n, & A_{-n-1}, & C_{-n-1} \\ \&c., & \&c., & \&c., & \&c. \end{vmatrix} = 0 \quad \text{.....................(9),}$$

where A_n, B_n, C_n, D_n are defined by [52], [53], [55], [56]. The second and fourth rows of the determinant are to be obtained from the first and third respectively by the substitution of b for a. The reduction of the determinant when $b = a + da$ is much facilitated by the use of (8), and of the similar relation in which h* is written for k, but the details of the work would occupy too much space to be reproduced here.

3. By way of verification, I have worked out the results of § 1 by a different method, confining myself, however, for simplicity, to the cases where the vibration is symmetrical about a diameter. Taking this diameter as axis of reference, let the equilibrium position of a point of the shell be defined in the usual way by the polar coordinates θ, ϕ; and let us suppose that in consequence of the displacement these become $\theta+\Theta$, $\phi+\Phi$, whilst the radius vector becomes $a+R$. If ds, ds_1 respectively denote the lengths of a linear element lying in the surface of the shell before and after the displacement, we find, since Θ, Φ, R are by hypothesis independent of ϕ,

$$ds_1^2 = ds^2 + 2\left(\frac{R}{a} + \frac{d\Theta}{d\theta}\right) a^2 d\theta^2 + 2\sin\theta \frac{d\Phi}{d\theta} \,.\, a d\theta \,.\, a \sin\theta d\phi$$
$$+ 2\left(\frac{R}{a} + \Theta\cot\theta\right) a^2 \sin^2\theta \, d\phi^2.$$

Hence, if e, f denote the elongations along the meridian and the parallel of latitude, respectively, and c the shear parallel to the tan-

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* h is defined by [21].

gent plane, we have

$$\left.\begin{aligned} e &= \frac{R}{a} + \frac{d\Theta}{d\theta} \\ f &= \frac{R}{a} + \Theta\cot\theta \\ c &= \sin\theta\frac{d\Phi}{d\theta} \end{aligned}\right\} \quad \text{.......................(10).}$$

If g denote the elongation in the direction of the thickness, the condition that there is no stress perpendicular to the surface gives

$$(\mathfrak{m}+\mathfrak{n})\,g+(\mathfrak{m}-\mathfrak{n})\,(e+f)=0 \quad \text{..................(11).*}$$

Substituting the value of g, hence obtained, in the general expression† for the potential energy (w) per unit volume, we obtain

$$w = \frac{2\mathfrak{n}}{\mathfrak{m}+\mathfrak{n}}\{\mathfrak{m}e^2+\mathfrak{m}f^2+(\mathfrak{m}-\mathfrak{n})ef\}+\tfrac{1}{2}\mathfrak{n}c^2,$$

or, writing γ for $(3\mathfrak{m}-\mathfrak{n})/(\mathfrak{m}+\mathfrak{n})$,

$$w=\mathfrak{n}\left\{\frac{\gamma+1}{2}(e^2+f^2)+(\gamma-1)\,ef+\tfrac{1}{2}c^2\right\} \quad \text{...........(12).}$$

To obtain the vibrations of the First Class, we put $R=0$ and (in consequence of the restriction as to symmetry) $\Theta=0$. Hence $e=0$, $f=0$ and

$$w=\tfrac{1}{2}\mathfrak{n}\sin^2\theta\left(\frac{d\Phi}{d\theta}\right)^2.$$

The variational equation of motion, then, is

$$\rho\tau\int_0^\pi a\sin\theta\ddot{\Phi}\,.\,a\sin\theta\,\delta\Phi\,.\,2\pi a^2\sin\theta\,d\theta+\delta V=0,$$

where ρ is the density, τ the thickness of the shell, and V is the potential energy of the deformation, viz.,

$$V=2\pi a^2\tau\int_0^\pi w\sin\theta\,d\theta \quad \text{.......................(13).}$$

The usual method then leads to the equation

$$\rho a^2\sin^3\theta\ddot{\Phi}=\mathfrak{n}\frac{d}{d\theta}\left(\sin^3\theta\frac{d\Phi}{d\theta}\right).$$

Hence, if $\Phi \propto \sin pt$, we have, writing $p^2\rho/\mathfrak{n}=k^2$, $\cos\theta=\mu$,

$$\frac{d}{d\mu}\left\{(1-\mu^2)^2\frac{d\Phi}{d\mu}\right\}+k^2a^2(1-\mu^2)\,\Phi=0,$$

* $\mathfrak{m}$ is a constant such that $\mathfrak{m}-\frac{1}{3}\mathfrak{n}$ is the resilience of volume.
† Thomson and Tait, *Natural Philosophy*, § 695.

or $$(1-\mu^2)\frac{d^2\Phi}{d\mu^2}-4\mu\frac{d\Phi}{d\mu}+k^2a^2\Phi = 0 \qquad\text{(14)}.$$

But the well-known equation satisfied by the zonal harmonic Q_n yields on differentiation

$$(1-\mu^2)\frac{d^3Q_n}{d\mu^3}-4\mu\frac{d^2Q_n}{d\mu^2}+(n-1)(n+2)\frac{dQ_n}{d\mu} = 0 \qquad\text{(15)}.$$

Hence (14) is satisfied by

$$\Phi = \frac{dQ_n}{d\mu} \qquad\text{(16)}.$$

provided $$k^2a^2 = (n-1)(n+2) \qquad\text{(1)}.$$

It may be shown that (16) and (1) together constitute the only solution of (14) which is finite all over the shell.

In the symmetrical vibrations of the Second Class, we have $\Phi = 0$, $v = 0$. Hence the variational equation of motion is

$$\rho\tau\int_0^\pi (\ddot{R}\delta R+a^2\ddot{\Theta}\delta\Theta)\,2\pi a^2\sin\theta d\theta+2\pi a^2\tau\int_0^\pi \delta w\sin\theta\,d\theta = 0 \quad\text{...(17)},$$

where $\delta w = \mathfrak{n}\,[\{(\gamma+1)\,e+(\gamma-1)f\}\,\delta e+\{(\gamma-1)\,e+(\gamma+1)f\}\,\delta f]$.

Assuming that R and Θ both $\propto \sin pt$, introducing the values of e and f from (10), and writing as before $p^2\rho/\mathfrak{n} = k^2$, we find, by equating to zero the coefficient of δR in (17),

$$k^2a^2\frac{R}{a}\sin\theta = 2\gamma\left(2\frac{R}{a}\sin\theta+\Theta\cos\theta+\sin\theta\frac{d\Theta}{d\theta}\right) \quad\text{......(18)}.$$

Again, equating (after a partial integration) the coefficient of $\delta\Theta$ to zero,

$$k^2a^2\Theta\sin\theta = -\frac{d}{d\theta}\left[\left\{2\gamma\frac{R}{a}+(\gamma+1)\frac{d\Theta}{d\theta}+(\gamma-1)\,\Theta\cot\theta\right\}\sin\theta\right]$$
$$+2\gamma\frac{R}{a}\cos\theta+(\gamma-1)\frac{d\Theta}{d\theta}\cos\theta+(\gamma+1)\,\Theta\frac{\cos^2\theta}{\sin\theta}.$$

This reduces to

$$k^2a^2\Theta\sin\theta = -\frac{2\gamma}{a}\frac{dR}{d\theta}\sin\theta-(\gamma+1)\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)$$
$$-2\Theta\sin\theta+(\gamma+1)\frac{\Theta}{\sin\theta} \qquad\text{(19)}.$$

Equations (18) and (19) may be written

$$(k^2a^2-4\gamma)\frac{R}{a}\sin\theta = 2\gamma\frac{d}{d\theta}(\sin\theta\,.\,\Theta) \qquad\text{(20)},$$

$$2\frac{\gamma}{a}\frac{dR}{d\theta}\sin\theta = -(k^2a^2+2)\sin\theta\,.\,\Theta$$
$$-(\gamma+1)\sin\theta\frac{d}{d\theta}\left\{\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\,.\,\Theta)\right\} \quad\text{......(21)}.$$

Now the zonal harmonic Q_n satisfies the equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dQ_n}{d\theta}\right)+n(n+1)\,Q_n=0.$$

Hence (20) and (21) are satisfied by

$$\frac{R}{a}=Q_n,\quad \Theta=\Lambda\frac{dQ_n}{d\theta},$$

provided $$k^2a^2-4\gamma=-2\gamma\,.\,n(n+1)\,\Lambda \quad\text{.................(22)},$$

and $$2\gamma=\{(\gamma+1)\,n(n+1)-(k^2a^2+2)\}\,\Lambda \quad\text{...........(23)}.$$

Eliminating Λ, we find

$$(k^2a^2-4\gamma)\,\{k^2a^2-n(n+1)\,\gamma-(n^2+n-2)\}=4n(n+1)\,\gamma^2,$$

which agrees with (3). When the values of ka have been found, the corresponding values of Λ are given by (22).

4. It is hardly necessary to point out that the vibrations of a *complete* spherical shell are in no way analogous to the flexural vibrations of an open shell or bowl, which have been discussed by Lord Rayleigh (*Proceedings*, Vol. xiii., p. 4). By a theorem due to Jellett, any deformation of a closed convex surface involves extension or contraction in some part of it. Hence, in our problem, that part of the potential energy which is due to the flexure may, if the shell be sufficiently thin, be neglected in comparison with that due to the extensions and contractions.

I find that a thin glass globe 20 centimètres in diameter should, in its gravest mode, make about 5350 vibrations per second.*

On Polygons circumscribed about a Tricuspidal Quartic.

By R. A. Roberts, M.A.

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1. In a recent number of the *Proceedings*, I arrived at some results concerning polygons circumscribed about a cuspidal cubic and inscribed in another curve. I propose, in this paper, to treat the tricuspidal quartic in a similar manner. As a tricuspidal quartic is the reciprocal of a nodal cubic, the problem is the same as to inscribe polygons in a nodal cubic; but I find it more convenient to consider the tricuspidal quartic.

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* The data employed in the calculation are taken from Everett's *Units and Physical Constants*, § 61; viz., $n = 2.40\times10^{11}$, $\rho = 2.942$, $\sigma = \cdot258$.