T H E

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[T H I R D S E R I E S .]

ART. XLIV.—*Elasticity of Ice*; by JOHN TROWBRIDGE and
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APPARENTLY the only previous experiments made to determine the modulus of elasticity of ice are those of Benjamin Bevan.* He went to a pond where the ice was about ten centimeters thick and sawed one end and two sides of a beam, leaving the other end attached to the main body of ice. He then loaded the free end and obtained the absolute modulus 54×10^9 . Nothing is said about the ice returning to its first position when the load was removed, nor of the deflection due to the weight of the beam itself. As the weight of the beam was four or five times the heaviest weight used, this deflection would be quite appreciable. The lower surface of the ice apparently rested on the water of the pond which prevented it from deflecting as far as it would have done. The upward pressure of the water would in part counterbalance the effect of the weight of the beam.

Methods of Observation.

We first endeavored to obtain the modulus by a direct pull on a bar. For this purpose water was put in brass tubes closed at one end and boiled so as to expel bubbles of air, then frozen, sometimes artificially, but generally by placing the tubes out of the window and allowing the water to freeze naturally. The

* Phil. Trans., 1826.

temperature of the air varying from -3° C. to -18° C. After it was thoroughly frozen, water from the water-pipes was allowed to flow over the tubes. This expanded the brass enough to allow the cylinder of ice to be slipped out without appreciable melting.

Some of the cylinders looked homogeneous throughout, but most of them appeared to be very porous along their axes.

The bar was hung by a clamp at its upper end. At the lower end there was another clamp to which a scale pan was attached. A small fibre was fastened to the bar at the upper mark and terminated in a pointed plumb-bob. The fiber hung clear of the ice so that its length remained invariable and the yielding of the upper support made no difference. The increased difference in the readings of the plumb-bob and of the lower mark gave the elongation for a given weight. The increased length being so small and the least wind causing the fiber and bar to swing, the errors of observation were so large that this method was abandoned.

It may be interesting to note that a bar left suspended an entire day (temperature -13° C.) changed from a cylinder to a frustrum of a cone.

The next experiments were made to get the transverse vibrations of the ice. The bars were obtained in the same manner as in the previous method. A sewing needle frozen transversely to one end of the bar served as a pointer. The other end was firmly clamped. The clamp was cooled to zero before the ice was put in. A tuning fork was placed by the side of the ice and so arranged that the two pointers were on a line. The tuning fork and bar were then set in vibration and a piece of smoked glass drawn under the two pointers received the two curves. Knowing the rate of the tuning fork, the number of transverse vibrations of the ice was obtained by comparing the number in equal distances along the two curves.

We experienced great difficulty from the ice breaking when it was clamped or when it was struck to set it in vibration, but by continued trials we succeeded in obtaining two well-defined curves. The height of the vibrations was about one-third of a millimeter. It is not certain that the ice always attained its proper pitch.

The results are expressed in the following formula :

$$E = \frac{n^2 l^4 s}{(.28)^2 r^2}$$

in which l = length of the bar, r = the radius, s = specific gravity and n = the number of transverse vibrations per second.

Experiment I.—Jan. 29, 1885.

$$l = 33 \text{ cm.} \quad r = .77 \text{ cm.} \quad s = .886 \quad n = 54.15 \\ \therefore E = 66 \times 10^9$$

Experiment II.—Jan. 31, 1885.

$$l = 35 \text{ cm} \quad r = .90 \text{ cm} \quad s = .886 \quad n = 51.4 \\ \therefore E = 55 \times 10^9.$$

Transverse Deflection.

We next went to a neighboring pond and sawed rectangular beams of ice and made measurements upon their elasticity at the pond. Two boards about 20 cm. wide and 60 cm. long were laid upon the ice. Two half cylinders of wood were placed upon these to form the supports for the beams. The cathetometer rested upon a board 30 cm. square placed upon the ice. It read to fiftieths of a millimeter. After the experiments the part of the scale used was compared with a standard centimeter, made by Prof. W. A. Rogers of the Harvard Observatory, and each millimeter was found to be not more than $.02^{\text{mm}}$ in error, so the readings were not corrected. A vertical rod with a needle at the top was inserted in a board 15 cm. square and served as an index. The index was placed on the middle of the beam and the weights were arranged symmetrically on each side.

The ice was 23 cm. thick, had no cracks or fissures and seemed to be free from air bubbles. There were about 3 cm. at the bottom which appeared to be of a later formation than the rest, it was therefore sawed off.

Care was taken to have all the apparatus below zero before the experiments began so as not to melt the ice. The wind was very light all day and did not shake the index. The results are given in tables I and II.

The difference in the modulus of the same bar with the same weight at different times may in part be due to slight variations in the positions of the weights. The index was simply placed on the beam and may have been jarred by putting on or taking off the weights, although as much care as possible was taken to avoid these errors. There did not seem to be a limit of perfect elasticity to any of the beams we used, but all gradually bent under their own weight. The readings were taken before, during and after the weights were on. The downward deflection is used in calculating E' and the return deflection E .

The formula used was obtained in the following manner:

Let P = weight divided in halves placed equidistant from the center of the beam, a = distance between the weights, l = length of span, wl = weight of the beam. Take the origin at

the centre of the curve. Then taking the moments about any section between one of the weights and the nearest support the elastic curve $E \frac{d^2y}{dx^2} = f(x)$ becomes

$$\begin{aligned}
 E \frac{d^2y}{dx^2} &= -\left(\frac{P+wl}{2}\right) \left(\frac{l}{2} - x\right) + w \left(\frac{l}{2} - x\right) \left(\frac{l}{2} - x\right) \\
 &= -\frac{P}{2} \left(\frac{l}{2} - x\right) - \frac{w}{2} \left(\frac{l^2}{4} - x^2\right) \\
 E \frac{dy}{dx} &= -\frac{P}{2} \left(\frac{lx}{2} - \frac{x^2}{2}\right) - \frac{w}{2} \left(\frac{l^2x}{4} - \frac{x^3}{3}\right) + C \\
 &\text{but } C=0 \\
 Ey &= -\frac{P}{2} \left(\frac{lx^2}{4} - \frac{x^3}{6}\right) - \frac{w}{2} \left(\frac{l^2x^2}{8} - \frac{x^4}{12}\right) + B \\
 &\text{but } B=0 \\
 \text{limits } x=0, x=\frac{l}{2} \\
 Es &= \frac{l^3}{4bd^3} (P + \frac{5}{8}W) \quad \text{since } I = \frac{1}{12}bd^3 \\
 &\text{and } W=wl.
 \end{aligned}$$

s = the deflection at the centre if the weight was there.

limits $x=\frac{\alpha}{2}$ $x=\frac{l}{2}$

$$Es' = \frac{l^3}{4bd^3} (P + \frac{5}{8}W) - \frac{P}{4bd^3} \left(\frac{\alpha^2}{2} (3l - \alpha)\right) - \frac{w}{4bd^3} \left(\frac{\alpha^2}{8} (6l^2 - \alpha^2)\right)$$

s' = the deflection at $x=\frac{\alpha}{2}$ when the weight is at the center.

The weight of the beam was acting all the time so that we only measured the deflection due to the weight P . Therefore the part containing W in the preceding formula will disappear. By the principle of reciprocal relations a weight at a distance from the centre $x=\frac{\alpha}{2}$ will produce the same deflection at the centre as the same weight placed at the center would produce at $x=\frac{\alpha}{2}$. If the weight had been at the centre we should have

$$E = \frac{Pl^3}{4bd^3s}$$

but we have seen that

$$s = \frac{s'l^3}{l^3 - \frac{\alpha^2}{2}(3l - \alpha)}$$

$$\therefore E = \frac{P}{4bd^3s'} \left(l^3 - \frac{\alpha^2}{2}(3l - \alpha) \right)$$

At another time we went where the old ice had been cut and a new crop about 5 cm. thick had formed. It was perfectly limpid and free from air bubbles. We sawed out half a dozen bars and conveyed them to the laboratory to make more careful measurements. The cathetometer rested on the stone steps of the laboratory. The same kind of supports was used as before. The index consisted of a needle run through a piece of cork which was pasted on the ice and could not be jarred. A similar index was placed upon the supports to see if they gave way under the load. It was found that they were firm and did not vary. All the apparatus was placed out in the air long enough to acquire its temperature before the experiments began.

The bars were smaller than the former ones and had a limit of perfect elasticity. In two cases the limit was exceeded, then smaller weights were put on and the bar was again perfectly elastic. The density of this ice was greater than that of the ice used in former experiments. It therefore appears that ice has a limit of perfect elasticity if small bars are used, but that large beams, although elastic to a certain extent, gradually bend under their own weight. The results are given in tables III and IV.

The symbols in the tables are:

L = length of the span,

T = temperature of the ice,

Δ = density of the ice,

b = breadth of the beam,

d = depth of the beam,

P = weight applied,

S' = deflection of the beam from zero under a given load,

S = the rise of the beam when the load is removed,

ϵ' = modulus of elasticity using S',

ϵ = modulus of elasticity using S,

α = distance between the weights.

The measurements are made in the C. G. S. system of units. The temperature is given in Centigrade degrees.

The values of ϵ and ϵ' are given in absolute measure.

I. FEBRUARY 12, 1885.

Bar.	L	α	T	Δ	b	d	P	S'	S	E	E'
A	118	30	-10.0	.886	22.6	6.2	10,000	.016	.012	57×10^9	41×10^9

II. FEBRUARY 13, 1885.

Bar.	L	α	T	Δ	b	d	P	ϑ'	ϑ	E	E'
B	148	30	-3.0	.886	10.0	11.0	14,000	.0285	.014	56×10^9	26×10^9
C	205	"	"	"	10.0	8.7	6,000	.046	.035	53×10^9	41×10^9
"	"	"	"	"	"	"	"	.0404	.0302	62	46
"	"	"	"	"	"	"	8,000	.0495	.0423	59	50
"	"	"	"	"	"	"	10,000	.0704	.0494	63	44
"	"	"	"	"	"	"	12,000	.0705	.0535	70	53
"	"	"	"	"	"	"	10,000	.0708	.049	63	44
"	"	"	"	"	"	"	"	.055	.046	68	56
"	"	"	"	"	"	"	14,000	.085	.076	57	51
"	"	"	"	"	"	"	"	.100	.075	58	44
"	"	"	"	"	"	"	10,000	.054	.040	78	58
"	"	"	"	"	8.7	10.0	10,000	.049	.0344	68	48
"	"	"	"	"	"	"	14,000	.0568	.049	67	58
"	"	"	"	"	"	"	16,000	.0599	.0519	73	63
D	212	"	"	"	17.5	6.7	10,000	.155	.054	80	28
"	"	"	"	"	"	"	"	.0998	.0578	75	43
E	213	"	"	"	17.5	7.8	10,000	.100	.036	77	28
"	"	"	"	"	"	"	14,000	.121	.065	60	32
"	"	"	"	"	"	"	18,000	.155	.092	54	32
Average -----										65×10^9	

III. FEBRUARY 19, 1885.

Bar.	L	α	T	Δ	b	d	P	ϑ'	ϑ	E	E'
H	107	20	-7.0	.920	4.5	6.6	4,000	.010	.010	88×10^9	88×10^9
"	"	"	"	"	"	"	6,000	.022	.022	60	60
"	"	"	"	"	"	"	10,000	.039	.039	57	57
"	"	"	"	"	"	"	16,000	.059	.059	60	60
"	"	"	"	"	"	"	25,459	.129	.069	81	44
"	"	"	"	"	6.6	4.5	10,000	.060	.060	79	79
"	"	"	"	"	"	"	25,459*	broke			"
I	94	"	"	"	6.6	4.5	22,730*	"			"
K	54	"	"	"	4.5	6.6	48,000*	"			"
Average -----										71×10^9	

* Weight required to break the bar.

IV. FEBRUARY 20, 1885.

Bar.	L	α	T	Δ	b	d	P	ϑ'	ϑ	E	E'
M	131	20	-5.0	.920	5.0	6.8	10,000	.035	.035	97×10^9	97×10^9
"	"	"	"	"	"	"	"	.031	.031	109	109
"	"	"	"	"	"	"	14,000	.059	.059	80	80
"	"	"	"	"	"	"	24,543	.135	.088	95	62
"	"	"	"	"	"	"	25,000*	broke			
N	"	"	"	"	6.8	5.0	10,000	.090	.058	108×10^9	70×10^9
"	"	"	"	"	"	"	4,000	.028	.028	90	90
Average -----										96	

* Weight required to break the bar.

Longitudinal Vibrations.

Before using bar M table IV we compared its note with that of a pitch pipe which had been graduated by a siren.

The ice gave three-tenths of a semitone of C sharp.

American Pitch.

Pitch Pipe C,	67.93	135.9	271.7	543.5	1087
C#,	71.97	143.9	287.9	575.8	1152

ϵ = modulus of elasticity,

n = number of longitudinal vibrations per second,

l = length of bar,

Δ = density of bar,

$n = 1107$, $l = 138$, $\Delta = .920$.

$\therefore \epsilon = 4l^2 n^2 \Delta = 86 \times 10^9$.

Summation.

Average of table I	-	-	-	-	-	57×10^9
" " " II	-	-	-	-	-	65 "
" " " III	-	-	-	-	-	71 "
" " " IV	-	-	-	-	-	96 "
" " all the observations	-	-	-	-	-	72 "

We assign greater value to tables III and IV than to I and II.

Average of tables III and IV	-	-	-	84×10^9
" " transverse vibrations	-	-	-	61 "
" " longitudinal	"	-	-	86 "

Velocity of Sound in Ice.

$$v = \sqrt{\frac{\epsilon}{\Delta}} = \sqrt{\frac{84 \times 10^9}{.920}} = 290,000 \text{ cm. per sec.} = 2900 \text{ m. per sec.}$$

or about nine times the velocity of sound in air.

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