Periodic orbits.

(With a plate of figures.)

By

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§ 1.

Introduction.

This paper is an abridgement of one which appeared recently in vol. 21 of the *Acta Mathematica*, and the reader is referred to that Journal for details omitted here. The original numbering of the equations, has been retained, so that the numbers do not now run consecutively*).

Since the date of the original publication my attention has been drawn to a certain incompleteness in the series of curves traced. The nature of the difficulty is indicated below, and I hope that I shall soon be in a position to make good the deficiency, by the help of suggestions made to me by my friend Mr Hough.

The existing methods of treating the Problem of the three Bodies are only applicable to the determination, by approximation, of the path of the third body when the attraction of the first largely preponderates over that of the second. A general solution of the problem is accordingly not to be obtained by these methods.

In the Lunar and Planetary theories it has always been found necessary to specify the motion of the perturbed body by reference to a standard curve or intermediate orbit, of which the properties are fully known. The degree of success attained by any of these methods has always depended on the aptness of the chosen intermediate orbit for the object in view. It is probable that future efforts will resemble their precursors in the use of standard curves of reference.

*) Since the publication of my paper in the *Acta Mathematica*, I have seen two important papers by M. Burrau in the *Astronomischen Nachrichten*, nos 3230 and 3251. In these investigations 'Jove' is taken to have a mass equal to that of the 'Sun', and the orbits considered are those which belong to the family here denoted by the letter *b*. The orbits of 'ejection' from the Sun and Jove have doubtless a great importance in the complete classification.

Mr G. W. Hill's papers on the Lunar Theory*) mark an epoch in the history of the subject. His substitution of the Variational Curve for the ellipse as the intermediate orbit is not only of primary importance in the Lunar Theory itself, but has pointed the way towards new fields of research.

The variational curve may be described as the distortion of the moon's circular orbit by the solar attraction. It is one of that class of periodic solutions of the Problem of the three Bodies which forms the subject of the present paper.

Of these solutions M. Poincaré writes:

«Ce qui nous rend ces solutions périodiques si précieuses, c'est qu'elles sont, pour ainsi dire, la seule brèche par où nous puissions essayer de pénétrer dans une place jusqu'ici réputée inabordable»**).

The periodic orbits, considered in this paper, are those of the simplest character, for they arise when the perturbed body has infinitely small mass, when the two others revolve about one another in circles, and when the three bodies are coplanar.

M. Poincaré remarks that there is a quadruple infinity of periodic solutions, for there are four arbitrary constants viz. the period of the infinitesimal body, the constant of energy, the moment of conjunction, and the longitude of conjunction***). For the purpose of the present investigation this quadruple infinity may however be reduced to a single infinity, for the moment and longitude of conjunction need not be considered; and the scale on which we draw the circular orbit of the second body round the first is immaterial. Thus we are only left with the constant of relative energy of the motion of the infinitesimal body as a single arbitrary.

Notwithstanding the great interest attaching to periodic orbits, no suggestion has, up to the present time, been made by any writer for a general method of determining them. As far as I can see, the search resolves itself into the discussion of particular cases by numerical processes, and such a search necessarily involves a prodigious amount of work. It is not for me to say whether the enormous labour I have undertaken was justifiable in the first instance; but I may remark that I have been led on, by the interest of my results, step by step, to investigate more and again more cases. Now that so much has been attained I cannot but think that the conclusions will prove of interest both to astronomers and to mathematicians.

In my earlier work I received the greatest assistance from

*) *American Journal of Mathematics*, Vol. 1 pp. 5—29, 129—147, 245—260 and *Acta Mathematica*, T. 8 pp. 1—36.

**) *Mécanique Céleste*, T. I, p. 82.

***) *Mécanique Céleste*, T. I, p. 101.

Mr J. W. F. Allnutt; his early death has deprived me of a friend and of an assistant, whose zeal and care were not to be easily surpassed. Since his death Mr J. I. Craig (of Emmanuel College) and Mr M. J. Berry (of Trinity College) have rendered and are rendering valuable help. I have besides done a great deal of computing myself*).

The reader will see that the figures have been admirably rendered by Mr Edwin Wilson of Cambridge. The wood-cuts in the text are the same as those used in the *Acta Mathematica*, but the plates are reproductions of the original plates.

The first part of the paper is devoted to the mathematical methods employed, the second part contains the discussion of the results. Tables of numerical results are given in the original paper, but are omitted from this abridgement.

Part. I.

§ 2.

Equations of motion.

The particular case of the problem of the three bodies, considered in this paper, is where the mass of the third body is infinitesimal compared with that of either of the two others which revolve about one another in circles, and where the whole motion takes place in one plane.

For the sake of brevity the largest body will be called the Sun, the planet which moves round it will be called Jove, and the third body will be called the planet or the satellite, as the case may be.

Jove J , of unit mass, moves round the Sun S , of mass ν , in a circle of unit radius SJ , and the orbit to be considered is that of an infinitesimal body P moving in the plane of Jove's orbit.

Let S be the origin of rectangular axes; let SJ be the x axis, and let the y axis be such that a rotation from x to y is consentaneous with the orbital motion of J . Let x, y be the heliocentric coordinates of P , so that $x - 1, y$ are the jovicentric coordinates referred to the same x axis and a parallel y axis.

Let r denote SP , and θ the angle JSP ; let ρ denote JP , and let the angle SJP be $180^\circ - \psi$. Thus r, θ are the polar heliocentric coordinates, and ρ, ψ the polar jovicentric coordinates of P .

Let n denote Jove's orbital angular velocity, so that in accordance with Kepler's law

$$n^2 = \nu + 1.$$

*) About two thirds of the expense of these computations have been met by grants from the Government Grant and Donation Funds of the Royal Society.

The equations of motion of a particle referred to axes rotating with angular velocity ω , under the influence of forces whose potential is U , are

$$\begin{aligned}\frac{d}{dt}\left(\frac{dX}{dt} - \omega Y\right) - \omega\left(\frac{dY}{dt} + \omega X\right) &= \frac{\partial U}{\partial X}, \\ \frac{d}{dt}\left(\frac{dY}{dt} + \omega X\right) + \omega\left(\frac{dX}{dt} - \omega Y\right) &= \frac{\partial U}{\partial Y},\end{aligned}$$

where t is the time.

Now in the present problem, if the origin be taken at the centre of inertia of the Sun and Jove with SJ for the X axis, the coordinates of P are $X = x - \frac{1}{\nu+1}$, $Y = y$. Also the potential function is $\frac{\nu}{r} + \frac{1}{q}$. Hence the equations of motion are

$$\begin{aligned}\frac{d^2x}{dt^2} - 2n\frac{dy}{dt} - (\nu+1)\left(x - \frac{1}{\nu+1}\right) &= \frac{\partial}{\partial x}\left(\frac{\nu}{r} + \frac{1}{q}\right), \\ \frac{d^2y}{dt^2} + 2n\frac{dx}{dt} - (\nu+1)y &= \frac{\partial}{\partial y}\left(\frac{\nu}{r} + \frac{1}{q}\right).\end{aligned}$$

But $r^2 = x^2 + y^2$, $q^2 = (x-1)^2 + y^2$. Hence if we put

$$(1) \quad 2\Omega = \nu\left(r^2 + \frac{2}{r}\right) + \left(q^2 + \frac{2}{q}\right),^*)$$

the equations of motion may be written

$$(1) \quad \begin{cases} \frac{d^2x}{dt^2} - 2n\frac{dy}{dt} = \frac{\partial \Omega}{\partial x}, \\ \frac{d^2y}{dt^2} + 2n\frac{dx}{dt} = \frac{\partial \Omega}{\partial y}, \end{cases}$$

where $n^2 = \nu + 1$.

Let the second of (1) be multiplied by $2\frac{dx}{dt}$, and the third by $2\frac{dy}{dt}$, let the two be added together and integrated, and we have Jacobi's integral

$$(2) \quad V^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2\Omega - C,$$

where C is a constant, and V denotes the velocity of P relatively to the rotating axes.

Let s be the arc of the planet's relative orbit measured from any fixed point, and let φ be the inclination to the x axis of the outward normal of the orbit. Then

$$\frac{dx}{ds} = -\sin \varphi, \quad \frac{dy}{ds} = \cos \varphi.$$

*) It is perhaps worth noting that 2Ω may be written in the form

$$\nu(r-1)^2\left(1 + \frac{2}{r}\right) + (q-1)^2\left(1 + \frac{2}{q}\right) + 3(\nu+1).$$

Hence if P be the component of inward effective force,

$$(3) \quad P = -\frac{\partial \Omega}{\partial x} \cos \varphi - \frac{\partial \Omega}{\partial y} \sin \varphi.$$

Therefore

$$PV = -\frac{\partial \Omega}{\partial x} \frac{dx}{dt} + \frac{\partial \Omega}{\partial y} \frac{dy}{dt}.$$

Now if R denotes the radius of curvature at the point x, y , of the relative orbit of P ,

$$\frac{1}{R} = \frac{\frac{d^2 y}{dt^2} \frac{dx}{dt} - \frac{d^2 x}{dt^2} \frac{dy}{dt}}{\left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right]^{\frac{3}{2}}}.$$

On substituting for the second differentials from (1), we have

$$\frac{V^3}{R} = \frac{\partial \Omega}{\partial y} \frac{dx}{dt} - \frac{\partial \Omega}{\partial x} \frac{dy}{dt} - 2n \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right].$$

Hence by means of (2) and (3)

$$(4) \quad \frac{1}{R} = \frac{P}{V^2} - \frac{2n}{V}.$$

If the value of Ω in (1) be substituted in (3) we easily find

$$(4) \quad \left\{ \begin{array}{l} P = v \left(\frac{1}{r^2} - r \right) \cos (\varphi - \theta) + \left(\frac{1}{\varrho^2} - \varrho \right) \cos (\varphi - \psi), \\ \text{and} \\ V^2 = v \left(r^2 + \frac{2}{r} \right) + \left(\varrho^2 + \frac{2}{\varrho} \right) - C. \end{array} \right.$$

Thus the curvature at any point of the orbit is expressible in terms of the coordinates and of the direction of the normal. If $s_0, \varphi_0, x_0, y_0, t_0$ be the initial values of the same quantities, it is clear that

$$(5) \quad \left\{ \begin{array}{l} \varphi = \varphi_0 + \int_{s_0}^s \frac{ds}{R}, \\ x = x_0 - \int_{s_0}^s \sin \varphi \, ds, \\ y = y_0 + \int_{s_0}^s \cos \varphi \, ds, \\ n(t - t_0) = \int_{s_0}^s \frac{n}{V} \, ds. \end{array} \right.$$

Also the polar coordinates of P relatively to axes fixed in space with heliocentric origin are $r, \theta + n(t - t_0)$, and with jovian origin are $\varrho, \psi + n(t - t_0)$.

Hence the determination of x and y involves in each case two integrations, and another integration is necessary to find the time, and the orbit in space.

§ 3.

Partition of relative space according to the value of the relative energy*).

It may be easily shown that the function Ω arises from three sources, and that it is the sum of the rotation potential, the potential of the Sun and the disturbing function for motion relatively to the Sun. Hence Ω is the potential of the system, inclusive of the rotation potential. Thus the equation $V^2 = 2\Omega - C$ may be called the equation of relative energy.

For a real motion of the planet V^2 must be positive, and therefore 2Ω must be greater than C . Accordingly the planet can never cross the curve represented by $2\Omega = C$, and if this curve has a closed branch with P inside, it must always remain inside, or if P be outside, it must always remain so.

This is Mr Hill's result in his celebrated memoir** on the Lunar Theory, save that the value of Ω used here has not been reduced to an approximate form.

We shall now proceed to a consideration of the family of curves $2\Omega = C$. That is to say we shall find, for a given value of C , the locus of points for which the three bodies may move for an instant as parts of a single rigid body. We are clearly at the same time finding the curves of constant velocity relatively to the moving axes for other values of C .

Some idea of the nature of the family of curves may be derived from general considerations; for when r and ϱ are small the equation approximates to $\frac{2\nu}{r} + \frac{2}{\varrho} = C$, and the curves are like the equipotentials due to two attractive particles of masses 2ν and 2 .

Thus for large values of C they are closed ovals round S and J , the one round S being the larger. As C declines the ovals swell and coalesce into a figure-of-8, which then assumes the form of an hour-glass with a gradually thickening neck.

When on the other hand r and ϱ are large the equation approxi-

*) A somewhat similar investigation is contained in a paper by M. Bohlin, *Acta Math.* T. 10, p. 109 (1887). The author takes the Sun as a fixed centre, which is equivalent to taking the Sun's mass as very large compared with that of Jove; he thus fails to obtain the function Ω in the symmetrical form used above.

**) *Amer. Journ. of Math.* Vol. I, pp. 5—29.

mates to $\nu r^2 + \varrho^2 = C$, and this represents an oval enclosing both S and J , which decreases in size as C decreases.

It is thus clear by general reasoning that for large values of C the curve consists of two closed branches round S and J respectively, and of a third closed branch round both S and J . The spaces within which the velocity of the planet is real are inside of either of the smaller ovals, and outside of the larger one. Since the larger oval shrinks and the hour-glass swells, as C declines, a stage will be reached when the two curves meet and coalesce. This first occurs at the end of the small bulb of the hour-glass which encloses J . The curve is then shaped like a horse-shoe, but is narrow at the toe and broad at the two points.

For still smaller values of C , the horse-shoe narrows to nothing at the toe, and breaks into two elongated pieces. These elongated pieces, one on each side of SJ , then shrink quickly in length and slowly in breadth, until they contract to two points when C reaches its minimum.

This sketch of the sequence of changes shows that there are four critical stages in the history of the curves,

- (α) when the internal ovals coalesce to a figure-of-8;
- (β) when the small end of the hour-glass coalesces with the external oval;
- (γ) when the horse-shoe breaks;
- (δ) when the halves of the broken shoe shrink to points.

The points of coalescence and rupture in (α), (β), (γ) are obviously on the line SJ (produced either way), and the points in (δ) are symmetrically situated on each side of SJ .

We must now consider the physical meaning of the critical points, and show how to determine their positions.

In the first three cases the condition which enables us to find the critical point is that a certain equation derived from $2\Omega = C$ shall have equal roots.

(α) The coalescence into a figure-of-8 must occur between S and J ; hence $r = 1 - \varrho$, and $2\Omega = C$ becomes

$$(6) \quad \nu \left[(1 - \varrho)^2 + \frac{2}{1 - \varrho} \right] + \varrho^2 + \frac{2}{\varrho} = C.$$

This equation must have equal roots, and by differentiation we find that ϱ must satisfy,

$$(\nu + 1)\varrho^5 - (3\nu + 2)\varrho^4 + (3\nu + 1)\varrho^3 - \varrho^2 + 2\varrho - 1 = 0,$$

An approximate solution for large values of ν is given by

$$(7) \quad \varrho = \frac{1}{(3\nu + 1)^{\frac{1}{3}} + \frac{1}{3}}.$$

If this value of ϱ be substituted in (6) we obtain the approximate result

$$(8) \quad C = 3\nu + \frac{2\nu}{3\nu+1} + 3(3\nu+1)^{\frac{1}{3}}.$$

In this paper the value adopted for ν is 10, and the solution of the quintic equation gives,

$$(9) \quad \varrho = .28249, \quad r = .71751, \quad C = 40.1821.$$

For even so small a value of ν as 10, the approximation is not far from the truth, and for such cases as actually occur in the solar system it would be accurate enough for every purpose.

The formula from which ϱ has been derived is equivalent to $\frac{\partial \Omega}{\partial x} = 0$, and since $y = 0$, we have also $\frac{\partial \Omega}{\partial y} = 0$. Hence the point is one of zero effective force at which the planet may revolve without motion relatively to the Sun and Jove.

This position of conjunction between the two larger bodies is obviously one of dynamical instability.

(β) The coalescence of the hour-glass with the external oval must occur at a point in SJ produced beyond J ; hence $r = 1 + \varrho$, and $2\Omega = C$ becomes

$$\nu \left[(1+\varrho)^2 + \frac{2}{1+\varrho} \right] + \varrho^2 + \frac{2}{\varrho} = C.$$

This equation must have equal roots, and by differentiation we find that ϱ must satisfy

$$(\nu+1)\varrho^5 + (3\nu+2)\varrho^4 + (3\nu+1)\varrho^3 - \varrho^2 - 2\varrho - 1 = 0.$$

An approximate solution, for large values of ν , is

$$(10) \quad \varrho = \frac{1}{(3\nu+1)^{\frac{1}{3}} - \frac{1}{3}},$$

$$(11) \quad C = 3\nu - \frac{2\nu}{3\nu+1} + 3(3\nu+1)^{\frac{1}{3}}.$$

When ν is 10,

$$(12) \quad \varrho = .34700, \quad r = 1.34700, \quad C = 38.8760.$$

The approximate solution would again be found to be near the truth, and in such cases as actually occur in the solar system the approximate formulae (10), (11) would lead to a high degree of accuracy.

This second critical point is another one at which the planet may revolve without motion relatively to the Sun and Jove, and such a motion is dynamically unstable.

(γ) The thinning of the toe of the horse-shoe to nothing must occur at a point in JS produced beyond S ; hence $\varrho = r + 1$, and $2\Omega = C$ becomes

$$\nu \left(r^2 + \frac{2}{r} \right) + (r+1)^2 + \frac{2}{r+1} = C.$$

This equation must have equal roots, and r must satisfy

$$(\nu+1)r^5 + (2\nu+3)r^4 + (\nu+3)r^3 - \nu(r^2+2r+1) = 0,$$

a quintic for finding r .

If we put $r = 1 - \xi$, the equation becomes

$$(\nu+1)\xi^5 - (7\nu+8)\xi^4 + (19\nu+25)\xi^3 - (24\nu+37)\xi^2 \\ + (12\nu+26)\xi - 7 = 0.$$

This equation may be solved by approximation, and the first approximation, which is all that I shall consider, gives

$$(13) \quad \xi = 1 - r = \frac{7}{12\nu + 26}.$$

Thus the approximate solution is $r = 1 - \frac{7}{12\nu + 26}$, and

$$(14) \quad C = 3\nu + 5 - \frac{7}{2}\xi + \left(3\nu + \frac{5}{4}\right)\xi^2.$$

The solution of the quintic equation gives

$$(15) \quad r = .94693, \quad \varrho = 1.94693, \quad C = 34.9054$$

This critical point is another one at which the three bodies may move round without relative motion, but as before the motion is dynamically unstable.

(δ) The fourth and last critical position occurs when C is a minimum so that $\frac{\partial C}{\partial r} = 0$, $\frac{\partial C}{\partial \varrho} = 0$, whence $r = 1$, $\varrho = 1$, and

$$C = 3\nu + 3.$$

If an equilateral triangle be drawn on SJ , its vertex is at this fourth critical point; and since this vertex may be on either the positive or negative side of SJ , there are two points of this kind.

It is well known that there is an exact solution of the problem of the three bodies in which they stand at the corners of an equilateral triangle, which revolves with a uniform angular velocity. The motion is stable provided that ν is sufficiently large. Mr Hough has however pointed out to me that if ν be less than 24.96, it is unstable.

Thus all the five critical points correspond with particular exact solutions of the problem, and of these solutions three are unstable and the symmetrical pair is generally stable.

Fig. 1 represents the critical curves of the family $2\Omega = C$, for the case $\nu = 10$. The points in the curves were determined by solutions of a cubic equation. I have only drawn the critical curves, because the addition of other members of the family would merely complicate the figure.

An important classification of orbits may be derived from this figure. When C is greater than 40.1821 the third body must be either a superior planet moving outside of the large oval, or an inferior planet moving inside of the larger internal oval, or a satellite moving inside of the smaller internal oval; and it can never exchange one of these parts for either of the other two. The limiting case $C = 40.1821$ gives superior limits to the radii vectores of inferior planets and of satellites, which cannot sever their connections with their primaries.

When C is less than 40.1821 but greater than 38.8760, the third body may be a superior planet, or an inferior planet or satellite, or a body which moves in an orbit which partakes of the two latter characteristics; but it can never pass from the first condition to any of the latter ones.

When C is less than 38.8760 and greater than 34.9054, the body may move anywhere save inside of a region shaped like a horse-shoe. The distinction between the two sorts of planetary motion and the motion as a satellite ceases to exist, and if the body is started in any one of these three ways it is possible for it to exchange the characteristics of its motion for either of the two other modes.

When C is less than 34.9054 and greater than 33, the forbidden region consists of two strangely shaped portions of space on each side of SJ .

Lastly when C is equal to 33, the forbidden regions have shrunk to a pair of infinitely small closed curves enclosing the third angles of a pair of equilateral triangles erected on SJ as a base. When C is less than 33 no portion of space is prohibited.

Fig.1

$$\text{Curves of zero velocity, } 10\left(r^2 + \frac{2}{r}\right) + \left(\rho^2 + \frac{2}{\rho}\right) = C.$$

§ 4.

A certain partition of space according to the nature of the curvature of the orbit.

It appears from (4) of § 2 that the curvature of an orbit is given by

$$\frac{V^2}{R} = P - 2nV,$$

where

$$P = -\frac{\partial \Omega}{\partial x} \cos \varphi - \frac{\partial \Omega}{\partial y} \sin \varphi.$$

Now if V_0 denotes any constant velocity, the equation $2\Omega = C + V_0^2$ defines a curve of constant velocity; it is one of the family of curves considered in § 3. We have seen that this family consists of a large oval enclosing two smaller ones, or of curves arising from the coalescence of ovals. In the mathematical sense of the term the 'interior' of the curve of constant velocity consists of the space inside of either of the smaller ovals or outside of the large one, or of the corresponding spaces when there is coalescence of ovals. It is a convenient and ordinary convention that when the circuit of a closed curve is described in a positive direction, the 'interior' of the curve is on the left-hand side. According to this convention the meaning of the 'inward' normal of one of these curves of constant velocity is clear, for it is directed towards the 'interior'. Similarly the inward normal of an orbit is towards the left-hand side, as the body moves along its path.

It is clear then that P is the component of effective force estimated along the inward normal of the orbit. Also if T be the resultant effective force $T^2 = \left(\frac{\partial \Omega}{\partial x}\right)^2 + \left(\frac{\partial \Omega}{\partial y}\right)^2$; and if χ be the angle between T and the inward normal to the orbit, $P = T \cos \chi$.

Hence

$$\frac{V^2}{R} = T \cos \chi - 2nV.$$

If we consider curvature as a quantity which may range from infinite positive to infinite negative, it may be stated that of all the orbits passing through a given point the curvature is greatest for that orbit which is tangential to the curve of constant velocity, when the motion takes place in a positive direction along that curve.

If χ lies between $\pm \chi_0$, where $\cos \chi_0 = \frac{2nV}{T}$, the orbit has positive curvature; if $\chi = \pm \chi_0$, there is a point of contrary flexure in the orbit; and if χ lies outside of the limits $\pm \chi_0$, the curvature is negative.

If however T be less than $2nV$, there are no orbits, passing through the point under consideration, which have positive curvature. Hence the equation $T = 2nV$ defines a family of curves which separate the regions in which the curvature of orbits is necessarily negative, from those in which it may be positive. These curves are very complicated, and as they have not been traced I omit the discussion of them from this abridgement.

§ 5.

Formulae of interpolation and quadrature.

The object of this paper is to search for periodic orbits, but no general method has been as yet discovered by which they may be traced. I have therefore been compelled to employ a laborious method of tracing orbits by quadratures, and of finding the periodic orbits by trial.

According to the usual notation of the calculus of finite differences, u_x is to denote a function of x , and the operator Δ is defined by

$$\Delta u_x = u_{x+1} - u_x.$$

Then the formula of integration, is

$$(24) \quad \int_0^n u_x dx = \Delta^{-1} u_{n+1} - \frac{1}{2} u_n - \frac{1}{12} \Delta u_{n-1} - \frac{1}{24} \Delta^2 u_{n-2} \\ - \frac{19}{720} \Delta^3 u_{n-3} - \frac{3}{160} \Delta^4 u_{n-4} - \frac{863}{60480} \Delta^5 u_{n-5}.$$

This is a convenient form when only the integral from n to 0 is wanted, and the integrals from $n-1$ to 0, $n-2$ to 0, etc. are not also wanted. But in the greater part of the work the intermediate integrals are also required. Now on applying the operator Δ to (24), we have

$$(25) \quad \int_n^{n+1} u_x dx = u_{n+1} - \frac{1}{2} \Delta u_n - \frac{1}{12} \Delta^2 u_{n-1} - \frac{1}{24} \Delta^3 u_{n-2} \\ - \frac{19}{720} \Delta^4 u_{n-3} \dots$$

If this be added to the integral from n to 0 we have the integral from $n+1$ to 0.

When the successive values of u depend on their precursors, it is necessary at the first stage of the integration to take Δx small, because it is only possible to take the first difference into account. At the second stage the second difference may be included, and at the third the third difference.

But in almost every case I begin integration with such a value of the independent variable (say $x = 0$), that we either have u_x an even function of x , or an odd function of x ; in the first case $u_x = u_{-x}$, in the second $u_x = -u_{-x}$. Both these cases present special advantages for the commencement of integration, for in the first integration we may take second differences into account. Thus when u_x is an even function, the second difference involved in the table of integration from 1 to 0 is $2\Delta u_0$; and when u_x is an odd function it is zero. In both cases third differences may be included in the second integration.

It is of course desirable to use the largest value of the increment of the independent variable consistent with adequate accuracy. If at any stage of the work it appears by the smallness of the second and third differences involved in the integrals, that longer steps may safely be employed, it is easy to double the value of Δx , by forming a new difference table with omission of alternate entries amongst the values already computed.

When on the other hand it appears by the growth of the second and third differences that Δx is becoming too large, Δx can be halved, and the new difference table must be formed by interpolation. The value of $u_{n-\frac{1}{2}}$ can be found from $u_n, u_{n-1}, u_{n-2}, \dots$ with sufficient accuracy for the purpose of obtaining the differences of $u_{n-\frac{3}{2}}, u_{n-1}, u_{n-\frac{1}{2}}, u_n$. The process of halving the value of Δx is therefore similar to that of doubling it.

In some of the curves which I have to trace there are sharp bends or quasi-cusps, and in these cases the process is very tedious, as it is sometimes necessary to repeatedly halve the increments of the independent variable, which is the arc s of the curve.

But the chief difficulty about these quasi-cusps arises when they are past, and when it is time to double the arc again. For the fact that the earlier values of the function to be used in the more open ranked difference tables are thrown back nearly to the cusp or even beyond it, makes the higher differences very large. Now the correctness of the formula of integration depends on the correctness of the hypothesis that an algebraic curve will give a good approximation to actuality. But in the neighbourhood of a quasi-cusp, and with increasing arcs this is far from correct. I have found then that in these cases of doubling the arc, a better result is obtained in the first and second integration by only including the second difference in the table of integration.

If we are tracing one member of a family of curves which are widely spaced throughout the greater part of their courses, but in one

region are closely crowded into quasi-cusps, it is difficult to follow one member of the family through the crowded region, and on emerging from the region we shall probably find ourselves tracing a closely neighbouring member, and not the original one. I have applied the method to trace the curve drawn by a point attached to a circle at nine tenths of its radius from the centre, as the circle rolls along a straight line. After the passage of the quasi-cusp I found that I was no longer exactly pursuing the correct line; nevertheless on a figure of the size of this page the difference between the two lines would be barely discernible. But the orbits which it is my object to trace do not quite resemble this case, since their cusps do not lie crowded together in one region of space. I believe therefore that these cases have been treated with substantial accuracy.

§ 6.

On the method of tracing a curve from its curvature.

It will be supposed that the curve to be traced is symmetrical with respect to the x axis, and starts at right angles to it so that $x = x_0$, $y = 0$, $\varphi = 0$, $s = 0$. This is not a necessary condition for the use of the method, but it appears from § 5 that the start is thus rendered somewhat easier than would be the case otherwise. The curvature at each point of the curve is supposed to be a known function of the coordinates x, y of the point, and of the direction of the normal defined by the angle φ .

The first step is to compute the initial curvature $\frac{1}{R_0}$; it is then necessary to choose such a value for the increment of arc δs as will give the requisite degree of accuracy.

I have found that it is well to take, as a rule, δs of such a size that $\frac{\delta s}{R_0}$ shall not be greater than about 8° ; but later, when all the differences in the tables of integration have come into use, I allow the increments of φ to increase to about 12° .

It is obvious that the curvature is even, when considered as a function of s . When nothing further is known of the nature of the curve, it is necessary to assume that the curvature is constant throughout the first arc δs , but it is often possible to make a conjecture that the curvature at the end of the arc δs will be say $\frac{1}{R_1}$. By the formula of integration with first and second differences we then compute $\varphi = \varphi_1$ at the end of the arc, by the first of equations (5) in § 2.

With this value of φ we find $\sin \varphi_1$, $\cos \varphi_1$, and observing that $\sin \varphi_0 = 0$, $\cos \varphi_0 = 1$, we compute x_1, y_1 by means of the second and third of (5), using first and second differences.

We next compute $\frac{1}{R_1}$ with these values of x, y , and if it agrees with the conjecture the work is done; and if not so, the work is repeated until there is agreement between the initial and final values of the curvature.

After the first arc, a second is computed, and higher differences are introduced into the tables of integration. We thus proceed by steps along the curve.

The approximation to the final result is usually so rapid, that in the recalculation it commonly suffices to note the changes in the last significant figure of the numbers involved in the original computation, without rewriting the whole.

The correction of the tables of integration is also very simple; for suppose that the first assumed value of the function to be integrated is u , and that the second approximation shows that it should have been $u + \delta u$; then all the differences in the column of the table have to be augmented by δu , and therefore the integral has to be augmented by

$$\left(1 - \frac{1}{2} - \frac{1}{12} - \frac{1}{24} - \dots\right) \delta u \delta s.$$

If we stop with third differences, this gives the simple rule that the integral is to be augmented by $\frac{3}{8} \delta u \delta s$.

It has been shown in § 5 how the chosen arc δs is to be increased or diminished according to the requirements of the case.

This method is the numerical counterpart of the graphical process described by Lord Kelvin in his Popular Lectures*), but it is very much more accurate, and when the formula for the curvature is complex it is hardly if at all more laborious. In the present investigation it would have been far more troublesome to use the graphical method, with such care as to attain the requisite accuracy, than to follow the numerical method.

In order to trace orbits I first computed auxiliary tables of $r^2 + \frac{2}{r}$, and of $\log\left(\frac{1}{r^2} - r\right)$ for $r < 1$, and of $\log\left(r - \frac{1}{r^2}\right)$ for $r > 1$; the tables extend from $r = 0$ to 1.5 at intervals of .001, but they will ultimately require further extension.

The auxiliary tables of logarithms are computed to 5 figures, but the last figure is not always correct to unity, and the fifth figure is principally of use in order to make correct interpolation possible.

*) Popular Lectures, vol. 1, 2nd ed. pp. 31—42; Phil. Mag. vol. 34, 1892, pp. 443—448.

The conversion of φ from circular measure to degrees and the values of $\sin \varphi$ and $\cos \varphi$ are obtained from Bottomley's four-figured table.

Most of the work has been done with these tables, but as it appears that the principal source of error lies in the determination of r and ρ , five figured logarithms have generally been used in this part of the work, and the values of θ and ψ are written down to 0.1.

In order to test the method, I computed an unperturbed elliptic orbit by means of the curvature, and found that the results were far more accurate than I should have expected.

In some cases, where the orbits have very sharp curvature or cusps, it was found necessary to calculate the coordinates by means of series proceeding by powers of the time. The series show that where an orbit has a cusp, its form is that of a semi-cubical parabola.

§ 7.

Variation of orbit.

Our object is not only to discover periodic orbits but also to consider their stability.

Now the stability of a periodic orbit is determinable by discovering whether the motion is oscillatory or not, when the path varies by infinitely little from that of the periodic orbit. The variation of an orbit may be of two kinds, for the constant of relative energy may be varied, or the planet may be displaced from the periodic orbit.

Suppose that the constant C undergoes a small variation and becomes $C + \delta C$; then there must be a periodic orbit, corresponding to $C + \delta C$, which differs by very little from that corresponding to C .

Now if a planet be moving in a periodic orbit, and if C suddenly becomes $C + \delta C$, we may henceforth refer the motion to the varied periodic orbit, and may consider the constant of relative energy as $C + \delta C$ and invariable. The periodic orbit of reference then varies *per saltum*, but the instantaneous position of the planet is unvaried, and therefore the planet is now displaced from its orbit of reference. Hence the result of a variation of C will virtually be determined by regarding C as constant, and by supposing the planet to be displaced from the periodic orbit. This subject is considered in the present section.

The whole of the following investigation is founded on the work of Mr Hill*), but it is presented in a different form.

*) On the part of the motion of the moon's perigee etc. *Acta Mathem.* Vol. 8, pp. 1-36.

If the Jacobian integral (2) be differentiated with respect to the time, and if the equations $\frac{dx}{dt} = -V \sin \varphi$, $\frac{dy}{dt} = V \cos \varphi$ be used in the result, we obtain

$$(28) \quad \frac{dV}{dt} = -\sin \varphi \frac{\partial \Omega}{\partial x} + \cos \varphi \frac{\partial \Omega}{\partial y}.$$

Again if the first of the equations of motion (1) be multiplied by $-\cos \varphi$, and the second by $-\sin \varphi$, and if the two be added together, the result may be written

$$(29) \quad V \left(\frac{d\varphi}{dt} + 2n \right) = -\cos \varphi \frac{\partial \Omega}{\partial x} - \sin \varphi \frac{\partial \Omega}{\partial y}.$$

Let s be the arc of the orbit, and p the arc of an orthogonal trajectory of the orbit, estimated in the direction of the outward normal of the orbit; then

$$(30) \quad \begin{cases} \frac{\partial}{\partial s} = -\sin \varphi \frac{\partial}{\partial x} + \cos \varphi \frac{\partial}{\partial y}, \\ \frac{\partial}{\partial p} = \cos \varphi \frac{\partial}{\partial x} + \sin \varphi \frac{\partial}{\partial y}. \end{cases}$$

Accordingly (28), (29) and the Jacobian integral become

$$(30) \quad \begin{cases} \frac{dV}{dt} = \frac{\partial \Omega}{\partial s}, \\ V \left(\frac{d\varphi}{dt} + 2n \right) = -\frac{\partial \Omega}{\partial p}, \\ V^2 = 2\Omega - C. \end{cases}$$

The equations (30) are equivalent to (1) and (2).

Now suppose that x, y are the coordinates of a point on an orbit, and that $x + \delta x, y + \delta y$ are the coordinates of a point on an adjacent orbit. Then if we put

$$\begin{aligned} \delta p &= \delta x \cos \varphi + \delta y \sin \varphi, \\ \delta s &= -\delta x \sin \varphi + \delta y \cos \varphi, \end{aligned}$$

$\delta p, \delta s$ are the distances, measured along the outward normal and along the arc of the unvaried orbit, from the original point x, y to the adjacent point $x + \delta x, y + \delta y$.

If, with x, y as origin, rectangular axes be drawn along the outward normal and along the arc of the unvaried orbit, we may regard $\delta p, \delta s$ as the coordinates of the new point relatively to the old one. The new axes rotate with angular velocity $\frac{d\varphi}{dt} + n$, the first term representing the angular velocity of the normal and the second that of our original axes of x and y .

The well-known formulae for the component accelerations of a point along two directions, which instantaneously coincide with a pair

of rotating rectangular axes by reference to which the position of the point is determined, give the accelerations

$$(31) \quad \begin{cases} \frac{d^2}{dt^2} \delta p - \delta p \left(\frac{d\varphi}{dt} + n \right)^2 - 2 \frac{d\delta s}{dt} \left(\frac{d\varphi}{dt} + n \right) - \delta s \frac{d^2\varphi}{dt^2}, \\ \text{along the normal} \\ \frac{d^2}{dt^2} \delta s - \delta s \left(\frac{d\varphi}{dt} + n \right)^2 + 2 \frac{d\delta p}{dt} \left(\frac{d\varphi}{dt} + n \right) + \delta p \frac{d^2\varphi}{dt^2}, \\ \text{along the tangent.} \end{cases}$$

These are the accelerations of the new point relatively to the old, estimated along lines fixed in space which coincide instantaneously with the normal and tangent of the unvaried orbit.

The function Ω includes the potential of the rotation n of the original axes of x and y . Hence $\Omega - \frac{1}{2} n^2 r^2$ is the true potential of the forces under which the body moves in the unvaried orbit, and

$$\frac{\partial}{\partial p} \left(\Omega - \frac{1}{2} n^2 r^2 \right), \quad \frac{\partial}{\partial s} \left(\Omega - \frac{1}{2} n^2 r^2 \right)$$

are the components of force in the unvaried orbit along the normal and along the arc.

Therefore the excess of the forces in the varied orbit above those in the unvaried orbit are

$$\left(\delta p \frac{\partial^2}{\partial p^2} + \delta s \frac{\partial^2}{\partial p \partial s} \right) \left(\Omega - \frac{1}{2} n^2 r^2 \right)$$

and

$$\left(\delta p \frac{\partial^2}{\partial p \partial s} + \delta s \frac{\partial^2}{\partial s^2} \right) \left(\Omega - \frac{1}{2} n^2 r^2 \right).$$

Now by considering the meaning (30) of the operations $\frac{\partial}{\partial p}$, $\frac{\partial}{\partial s}$, it is easy to prove that

$$\frac{1}{2} \frac{\partial^2 r^2}{\partial p^2} = \frac{1}{2} \frac{\partial^2 r^2}{\partial s^2} = 1, \quad \frac{1}{2} \frac{\partial^2 r^2}{\partial p \partial s} = 0.$$

Hence the excess of the forces in the varied orbit above those in the unvaried orbit are

$$\delta p \frac{\partial^2 \Omega}{\partial p^2} + \delta s \frac{\partial^2 \Omega}{\partial p \partial s} - n^2 \delta p,$$

and

$$\delta p \frac{\partial^2 \Omega}{\partial p \partial s} + \delta s \frac{\partial^2 \Omega}{\partial s^2} - n^2 \delta s,$$

along the normal and along the arc of the unvaried orbit.

But these are necessarily equal to the accelerations (31) of which they are the cause. Then transferring $-n^2 \delta p$, $-n^2 \delta s$ to the left hand sides of the equations, we have

$$(32) \quad \begin{cases} \frac{d^2}{dt^2} \delta p + \delta p \left[n^2 - \left(\frac{d\varphi}{dt} + n \right)^2 \right] - 2 \frac{d\delta s}{dt} \left(\frac{d\varphi}{dt} + n \right) - \delta s \frac{d^2\varphi}{dt^2} \\ \qquad \qquad \qquad = \delta p \frac{\partial^2 \Omega}{\partial p^2} + \delta s \frac{\partial^2 \Omega}{\partial p \partial s}, \\ \frac{d^2}{dt^2} \delta s + \delta s \left[n^2 - \left(\frac{d\varphi}{dt} + n \right)^2 \right] + 2 \frac{d\delta p}{dt} \left(\frac{d\varphi}{dt} + n \right) + \delta p \frac{d^2\varphi}{dt^2} \\ \qquad \qquad \qquad = \delta p \frac{\partial^2 \Omega}{\partial p \partial s} + \delta s \frac{\partial^2 \Omega}{\partial s^2}. \end{cases}$$

These are the equations of motion in the varied orbit.

The variation of the last of (30), the Jacobian integral, gives

$$(33) \quad V \delta V = \delta p \frac{\partial \Omega}{\partial p} + \delta s \frac{\partial \Omega}{\partial s}.$$

Now δV is the tangential velocity of the point $x + \delta x, y + \delta y$ in the varied orbit, relatively to the original point x, y . But as we only want to consider a velocity relatively to the axes of x and y , which themselves rotate with angular velocity n , our p, s axes must be regarded as rotating with angular velocity $\frac{d\varphi}{dt}$, instead of $\frac{d\varphi}{dt} + n$.

Accordingly

$$(34) \quad \delta V = \frac{d}{dt} \delta s + \delta p \frac{d\varphi}{dt}.$$

The formula (34) enables us to get rid of δV in (33), but we may also get rid of $\frac{\partial \Omega}{\partial p}$ and $\frac{\partial \Omega}{\partial s}$ by means of the equations of motion (30). Thus the variation of the Jacobian integral leads to

$$V \left(\frac{d}{dt} \delta s + \delta p \frac{d\varphi}{dt} \right) = -V \left(\frac{d\varphi}{dt} + 2n \right) \delta p + \frac{dV}{dt} \delta s.$$

Therefore

$$(35) \quad \begin{cases} \frac{d}{dt} \delta s + 2\delta p \left(\frac{d\varphi}{dt} + n \right) - \frac{1}{V} \frac{dV}{dt} \delta s = 0, \\ \text{or} \\ V \frac{d}{dt} \left(\frac{\delta s}{V} \right) + 2\delta p \left(\frac{d\varphi}{dt} + n \right) = 0, \end{cases}$$

The equations (35) are two forms of the varied Jacobian integral.

A great simplification of the equations of motion (32) is possible by reference to the unvaried motion.

Let us suppose then that $\delta p, \delta s$ are no longer displacements to a varied orbit, but are the actual displacements occurring in time δt in the unvaried orbit. Thus $\delta p = 0, \delta s = V \delta t$.

The equations (32) then give

$$(36) \quad \begin{cases} -2 \frac{dV}{dt} \left(\frac{d\varphi}{dt} + n \right) - V \frac{d^2\varphi}{dt^2} = V \frac{\partial^2 \Omega}{\partial p \partial s}, \\ \frac{d^2 V}{dt^2} + V \left[n^2 - \left(\frac{d\varphi}{dt} + n \right)^2 \right] = V \frac{\partial^2 \Omega}{\partial s^2}. \end{cases}$$

The first of (36) may be written

$$\frac{d^2 \varphi}{dt^2} + \frac{\partial^2 \Omega}{\partial p \partial s} = -\frac{2}{V} \frac{dV}{dt} \left(\frac{d\varphi}{dt} + n \right).$$

These two terms, multiplied by δs , occur in the first of (32), which may therefore be written

$$\begin{aligned} \frac{d^2 \delta p}{dt^2} + \delta p \left[n^2 - \left(\frac{d\varphi}{dt} + n \right)^2 \right] - 2 \frac{d\delta s}{dt} \left(\frac{d\varphi}{dt} + n \right) \\ + \frac{2\delta s}{V} \frac{dV}{dt} \left(\frac{d\varphi}{dt} + n \right) - \delta p \frac{\partial^2 \Omega}{\partial p^2} = 0. \end{aligned}$$

The terms in this which involve δs may now be eliminated by the first of (35) and we have

$$\frac{d^2 \delta p}{dt^2} + \delta p \left[n^2 - \left(\frac{d\varphi}{dt} + n \right)^2 + 4 \left(\frac{d\varphi}{dt} + n \right)^2 - \frac{\partial^2 \Omega}{\partial p^2} \right] = 0.$$

If then we put

$$(37) \quad \Theta = n^2 + 3 \left(\frac{d\varphi}{dt} + n \right)^2 - \frac{\partial^2 \Omega}{\partial p^2},$$

we have

$$(37) \quad \begin{cases} \frac{d^2 \delta p}{dt^2} + \Theta \delta p = 0, \\ \frac{d}{dt} \left(\frac{\delta s}{V} \right) + 2 \frac{\delta p}{V} \left(\frac{d\varphi}{dt} + n \right) = 0. \end{cases}$$

The differential equation for δp is Mr Hill's well-known result

We have now to consider the form of the function Θ .

Let us write $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial p^2} + \frac{\partial^2}{\partial s^2}$; then adding $V \frac{\partial^2 \Omega}{\partial p^2}$ to each side of the second of (36), we have

$$\frac{1}{V} \frac{d^2 V}{dt^2} + n^2 - \left(\frac{d\varphi}{dt} + n \right)^2 + \frac{\partial^2 \Omega}{\partial p^2} = \nabla^2 \Omega,$$

so that

$$n^2 - \frac{\partial^2 \Omega}{\partial p^2} = \frac{d}{dt} \left(\frac{dV}{V dt} \right) + \left(\frac{dV}{V dt} \right)^2 - \left(\frac{d\varphi}{dt} + n \right)^2 + 2n^2 - \nabla^2 \Omega.$$

Substituting in (37),

$$\Theta = 2n^2 - \nabla^2 \Omega + 2 \left(\frac{d\varphi}{dt} + n \right)^2 + \frac{d}{dt} \left(\frac{dV}{V dt} \right) + \left(\frac{dV}{V dt} \right)^2.$$

If we put $u = x + yi$, $s = x - yi$, $\frac{d}{dt} = \iota D$, where $\iota = \sqrt{-1}$, it is easy to show that $Du = Ve^{\varphi \iota}$, $Ds = -Ve^{-\varphi \iota}$, and

$$2 \frac{d\varphi}{dt} = \frac{D^2 u}{Du} - \frac{D^2 s}{Ds}, \quad 2 \frac{dV}{V dt} = \iota \left(\frac{D^2 u}{Du} + \frac{D^2 s}{Ds} \right).$$

Mr Hill's form for the function Θ follows as once from these transformations.

Another form for Θ , deducible directly from (37), is

$$\Theta = n^2 - \frac{1}{2} \nabla^2 \Omega - \frac{1}{2} \left(\frac{\partial^2 \Omega}{\partial x^2} - \frac{\partial^2 \Omega}{\partial y^2} \right) \cos 2\varphi - \frac{1}{2} \frac{\partial^2 \Omega}{\partial x \partial y} \sin 2\varphi + 3 \left(\frac{d\varphi}{dt} + n \right)^2,$$

whence

$$\Theta = \frac{v}{r^3} + \frac{1}{q^3} - \frac{3v}{r^3} \cos^2(\varphi - \theta) - \frac{3}{q^3} \cos^2(\varphi - \psi) + 3V^2 \left(\frac{1}{R} + \frac{n}{V} \right)^2.$$

§ 8.

Change of independent variable from time to arc of orbit.

For the purpose of future developments it is now necessary to change the independent variable from the time t to the arc s .

Let

$$(38) \quad \delta q = \delta p V^{\frac{1}{2}}.$$

Then

$$\begin{aligned} \frac{d^2 \delta p}{dt^2} &= V \frac{d}{ds} \left(V \frac{d}{ds} \left(\frac{\delta q}{V^{\frac{1}{2}}} \right) \right) = V \frac{d}{ds} \left(V^{\frac{1}{2}} \frac{d \delta q}{ds} - \frac{1}{2V^{\frac{1}{2}}} \delta q \frac{dV}{ds} \right) \\ &= V^{\frac{3}{2}} \frac{d^2 \delta q}{ds^2} - \frac{1}{2} \delta q V \frac{d}{ds} \left(\frac{1}{V^{\frac{1}{2}}} \frac{dV}{dt} \right). \end{aligned}$$

But

$$\begin{aligned} V \frac{d}{ds} \left(\frac{1}{V^{\frac{1}{2}}} \frac{dV}{ds} \right) &= \frac{d}{dt} \left(\frac{1}{V^{\frac{3}{2}}} \frac{dV}{dt} \right) = -\frac{3}{2V^{\frac{5}{2}}} \left(\frac{dV}{dt} \right)^2 + \frac{1}{V^{\frac{3}{2}}} \frac{d^2 V}{dt^2} \\ &= -\frac{3}{2V^{\frac{1}{2}}} \left(\frac{dV}{ds} \right)^2 + \frac{1}{V^{\frac{3}{2}}} \frac{d^2 V}{dt^2}. \end{aligned}$$

Hence

$$\frac{d^2 \delta p}{dt^2} = V^{\frac{3}{2}} \frac{d^2 \delta q}{ds^2} + \frac{3}{4V^{\frac{1}{2}}} \left(\frac{dV}{ds} \right)^2 \delta q - \frac{\delta q}{2V^{\frac{3}{2}}} \frac{d^2 V}{dt^2}.$$

Also

$$\Theta \delta p = \frac{\Theta \delta q}{V^{\frac{1}{2}}}.$$

If these two be added together, and divided by $V^{\frac{3}{2}}$, we obtain

$$(39) \quad \begin{cases} \frac{d^2 \delta q}{ds^2} + \Psi \delta q = 0, \\ \text{where} \\ \Psi = \frac{\Theta}{V^{\frac{3}{2}}} + \frac{3}{4} \left(\frac{dV}{V ds} \right)^2 - \frac{1}{2V^{\frac{3}{2}}} \frac{d^2 V}{dt^2}. \end{cases}$$

It remains to obtain the expression for the function Ψ .

Since

$$\begin{aligned} \frac{d\varphi}{ds} &= \frac{1}{R}, \quad \text{and} \quad n^2 = v + 1, \\ \Theta &= v + 1 + 3 \left(\frac{V}{R} + n \right)^2 - \frac{\partial^2 \Omega}{\partial p^2}. \end{aligned}$$

Now from the first of (30) and the second of (36),

$$V \frac{dV}{ds} = \frac{\partial \Omega}{\partial s},$$

$$\frac{1}{V} \frac{d^2 V}{dt^2} = \left(\frac{V}{R} + n \right)^2 + \frac{\partial^2 \Omega}{\partial s^2} - \nu - 1.$$

Then by substitution in the second of (39);

$$\Psi V^2 = \frac{3}{2}(\nu + 1) + \frac{5}{2} \left(\frac{V}{R} + n \right)^2 + \frac{3}{4} \left(\frac{dV}{ds} \right)^2 - \frac{\partial^2 \Omega}{\partial p^2} - \frac{1}{2} \frac{\partial^2 \Omega}{\partial s^2}.$$

Also

$$\frac{\partial^2 \Omega}{\partial p^2} + \frac{1}{2} \frac{\partial^2 \Omega}{\partial s^2} = \frac{1}{2} \nabla^2 \Omega + \frac{1}{2} \frac{\partial^2 \Omega}{\partial p^2}.$$

Now

$$2\Omega = \nu \left(r^2 + \frac{2}{r} \right) + \left(\rho^2 + \frac{2}{\rho} \right),$$

and

$$\frac{\partial^2 \Omega}{\partial x^2} = \nu + 1 - \frac{\nu}{r^3} - \frac{1}{\rho^3} + \frac{3\nu}{r^3} \cos^2 \theta + \frac{3}{\rho^3} \cos^2 \psi,$$

$$\frac{\partial^2 \Omega}{\partial x \partial y} = \frac{3\nu}{r^3} \sin \theta \cos \theta + \frac{3}{\rho^3} \sin \psi \cos \psi,$$

$$\frac{\partial^2 \Omega}{\partial y^2} = \nu + 1 - \frac{\nu}{r^3} - \frac{1}{\rho^3} + \frac{3\nu}{r^3} \sin^2 \theta + \frac{3}{\rho^3} \sin^2 \psi.$$

Hence

$$\nabla^2 \Omega = 2(\nu + 1) + \frac{\nu}{r^3} + \frac{1}{\rho^3},$$

and

$$\frac{\partial^2 \Omega}{\partial p^2} = \cos^2 \varphi \frac{\partial^2 \Omega}{\partial x^2} + 2 \sin \varphi \cos \varphi \frac{\partial^2 \Omega}{\partial x \partial y} + \sin^2 \varphi \frac{\partial^2 \Omega}{\partial y^2},$$

$$= \nu + 1 - \frac{\nu}{r^3} - \frac{1}{\rho^3} + \frac{3\nu}{r^3} \cos^2 (\varphi - \theta) + \frac{3}{\rho^3} \cos^2 (\varphi - \psi).$$

Therefore

$$(40) \quad \Psi = \frac{5}{2} \left(\frac{1}{R} + \frac{n}{V} \right)^2 - \frac{3}{2V^2} \left[\frac{\nu}{r^3} \cos^2 (\varphi - \theta) + \frac{1}{\rho^3} \cos^2 (\varphi - \psi) \right] + \frac{3}{4} \left(\frac{dV}{ds} \right)^2.$$

Also since

$$V \frac{dV}{ds} = \frac{\partial \Omega}{\partial s} = - \sin \varphi \frac{\partial \Omega}{\partial x} + \cos \varphi \frac{\partial \Omega}{\partial y},$$

$$(40) \quad \frac{dV}{ds} = \frac{\nu}{V^2} \left(\frac{1}{r^2} - r \right) \sin (\varphi - \theta) + \frac{1}{V^2} \left(\frac{1}{\rho^2} - \rho \right) \sin (\varphi - \psi).$$

This completes the formula for Ψ in terms of the coordinates, the velocity, the curvature and of φ .

It may be useful to obtain the expressions for δs and $\delta \varphi$ in terms of the new independent variable s .

The second of (37) may be written down at once, namely

$$(41) \quad \frac{d}{ds} \left(\frac{\delta s}{V} \right) = - \frac{2\delta q}{V^{\frac{3}{2}}} \left(\frac{1}{R} + \frac{n}{V} \right).$$

Also it is clear from geometrical considerations that

$$\delta \varphi = -\frac{d}{ds} \delta p + \frac{\delta s}{R},$$

whence

$$(42) \quad \delta \varphi = -\frac{1}{V^{\frac{1}{2}}} \left[\frac{d \delta q}{ds} - \frac{1}{2} \delta q \left(\frac{ds}{V ds} \right) \right] + \frac{\delta s}{R}.$$

§ 9.

The solution of the differential equation for δq .

The function Ψ has a definite value at each point of a periodic orbit whose complete arc is S . Therefore Ψ is a function of the arc s of the orbit, measured from any point therein, and when s has increased from zero to S , Ψ has returned to its initial value. Also since a periodic orbit is symmetrical with respect to the x -axis, Ψ is an even function of the arc s , when s is measured from an orthogonal intersection of the orbit with the x -axis. If the periodic orbit only goes once round S or J , or round both, all the intersections with the x -axis are necessarily orthogonal. I call such an orbit simply periodic, but the term must have its meaning extended so as to embrace the possibility of loops. But when there are loops the intersections with the x -axis are not necessarily orthogonal, and if the orbit is only periodic after several revolutions some of the intersections cannot be orthogonal.

With the understanding that s is measured from an orthogonal intersection with the x -axis, Ψ is an even function of s and is expressible by the Fourier series

$$\Psi = \Psi_0 + 2\Psi_1 \cos \frac{2\pi s}{S} + 2\Psi_2 \cos \frac{4\pi s}{S} + \dots$$

Now multiply the differential equation (39) for δq by $\frac{S^2}{\pi^2}$, write σ for $\frac{\pi s}{S}$, and put $\Phi = \frac{S^2}{\pi^2} \Psi$, and we have

$$(43) \quad \frac{d^2}{d\sigma^2} \delta q + \Phi \delta q = 0.$$

Also if $\Phi_j = \frac{S^2}{\pi^2} \Psi_j$,

$$\Phi = \Phi_0 + 2\Phi_1 \cos 2\sigma + 2\Phi_2 \cos 4\sigma + \dots$$

If then we write $\xi = e^{\sigma\sqrt{-1}}$,

$$\xi \frac{d}{d\xi} = \frac{1}{\sqrt{-1}} \frac{d}{d\sigma},$$

and the equation (43) becomes

$$(44) \quad \left(\xi \frac{d}{d\xi} \right)^2 \delta q = \Phi \delta q,$$

where $\Phi = \sum_j \Phi_j \xi^{2j}$, the summation being taken from $j = +\infty$ to $j = -\infty$, and Φ_{-j} being equal to Φ_j .

Let us assume as the solution of (44)

$$\begin{aligned} \delta q &= \sum_j [(b_j + e_{-j}) \cos (c+2j)\sigma + (b_j - e_{-j}) \sqrt{-1} \sin (c+2j)\sigma] \\ &= \sum_j [b_j \xi^{c+2j} + e_j \xi^{-c+2j}]. \end{aligned}$$

The equation (44) must be separately satisfied for the terms involving b and for those involving e ; hence we need only regard one series of terms.

On substituting in (44) the assumed expression for δq , and equating to zero the coefficients of the several powers of ξ , we have

$$b_j (c+2j)^2 = \sum_i b_{j-i} \Phi_i, *$$

written *in extenso* this is

$$\dots - b_{j-2} \Phi_2 - b_{j-1} \Phi_1 + b_j [(c+2j)^2 - \Phi_0] - b_{j+1} \Phi_1 - b_{j+2} \Phi_2 - \dots = 0.$$

There are an infinite number of equations like the above, but the infinity must be regarded as an odd number.

If from these equations the b 's be eliminated, we have an infinite determinantal equation for determining c . If we write

$$(c+2j)^2 - \Phi_0 = \{j\},$$

the equation is

$$\begin{vmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \{ -1 \} & , & -\Phi_1 & , & -\Phi_2 & \dots \\ \dots & -\Phi_1 & , & \{ 0 \} & , & -\Phi_1 & \dots \\ \dots & -\Phi_2 & , & -\Phi_1 & , & \{ 1 \} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} = 0.$$

This is the same in form as Mr Hill's determinantal equation.

As much has been written on the subject, it is unnecessary to reproduce the arguments by which it may be shown that if

$$[j] = \Phi_0 - 4j^2,$$

and

*) The equation of condition for the e 's is easily shown to be

$$e_{-j} (c+2j)^2 = \sum_i e_{i-j} \Phi_{-i};$$

and since $\Phi_i = \Phi_{-i}$, this is exactly the same as that for the b 's save that e_{-j} corresponds with b_j .

$$(45) \quad \Delta = \begin{vmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & 1 & , & \frac{\Phi_1}{[1]} & , & \frac{\Phi_2}{[1]} & \dots \\ \dots & \frac{\Phi_1}{[0]} & , & 1 & & \frac{\Phi_1}{[0]} & \dots \\ \dots & \frac{\Phi_2}{[1]} & , & \frac{\Phi_1}{[1]} & , & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix},$$

the solution of the determinantal equation is given by

$$(45) \quad \sin^2 \frac{1}{2} \pi c = \Delta \sin^2 \frac{1}{2} \pi \sqrt{\Phi_0}.$$

§ 10.

On the stability or instability of an orbit.

When c is real, δq is expressible by a series of sines and cosines of multiples of the arc. Since V is an even function of the arc, it is expressible by a series of cosines of the same form as that for Φ : hence δp , which is equal to $V^{\frac{1}{2}} \delta q$, is expressible in a series, similar in form to that for δq .

But δp denotes normal displacement from the periodic orbit, and therefore the motion in the varied orbit is oscillatory with reference to the periodic orbit. In other words the periodic orbit is stable.

If c_0 be any one value of c , all its infinite values are comprised in the formula $\pm c_0 \pm 2i$, where i is an integer. It is however convenient to choose one value of c as fundamental. When the choice has been made we may refer to the terms in the series for δq of which the argument is c_0 as the principal terms, although it does not appear to be necessary that these terms should have the largest coefficients. In fact since two arbitrary constants are involved in the specification of a definite variation of orbit, it is probable that the terms which are numerically the most important in one variation, will not be so in another.

If the body be considered as moving in an elliptic orbit, it will be at its pericentre or apocentre, when δp is a negative or positive maximum, respectively. The principal terms of δq , and therefore also of δp , have the argument $c\sigma$ or $\frac{c\pi s}{S}$; hence if we may assume that the principal term is also the most important, the body has passed through a complete anomalistic circuit when s has increased from zero to $2 \frac{S}{c}$. Since S is the synodic arc in the relative orbit, $\frac{1}{2} c$ is the ratio of the synodic to the anomalistic arc, both arcs being measured on the orbit as drawn with reference to the moving axes.

Now I propose to adopt as a convention that the fundamental value of c shall be that value which lies nearest to $\sqrt{\Phi_0}$, where Φ_0 denotes the mean value of Φ . This convention certainly attributes to $\frac{1}{2}c$ a physical meaning, which is correct in all those cases which have any resemblance to the motion of an actual satellite in the solar system. I shall accordingly use the value of c which lies nearest to $\sqrt{\Phi_0}$ as fundamental.

We have just arrived at a physical meaning for c by considering the principal term in the series; now in so doing we were in effect considering only the mean motion of the body with reference to the moving axes; therefore $\frac{1}{2}c$ is also the ratio of the synodic to the anomalistic period*).

If T denotes the synodic period, the mean motion of the body referred to axes fixed in space is $\frac{2\pi}{T} + n$; and if $\frac{d\omega}{dt}$ denotes the mean angular velocity of the pericentre with reference to axes fixed in space, the mean motion of the body with reference to the pericentre is $\frac{2\pi}{T} + n - \frac{d\omega}{dt}$. Then, since angular velocities vary inversely as periods,

$$\frac{1}{2}c = \frac{\frac{2\pi}{T} + n - \frac{d\omega}{dt}}{\frac{2\pi}{T}},$$

where

$$n^2 = \nu + 1.$$

Therefore

$$(46) \quad \begin{cases} \frac{d\omega}{dt} = n - \frac{2\pi}{T} \left(\frac{1}{2}c - 1 \right), \\ \text{or} \\ T \left(n - \frac{d\omega}{dt} \right) = 2\pi \left(\frac{1}{2}c - 1 \right). \end{cases}$$

Mr Hill's c is equal to one half of my c , and accordingly the first of (46) is identical with the formula from which Mr Hill derives 'a part of the motion of the lunar perigee'**).

The angular velocity of regression of the pericentre being $n - \frac{d\omega}{dt}$, it follows from (46) that $2\pi \left(\frac{1}{2}c - 1 \right)$ is the amount of that regression with respect to the moving axes in the synodic period.

*) It may be observed that when V is constant (as is the case when we only consider mean motion) $V^2\Psi = \Theta$, and Mr Hill's equation for δp becomes identical with the present one for δq . It is well to remark that what I denote by c is $2c$ of Mr Hill's notation.

**) *Acta Mathem.* vol. 8.

Whilst the pericentre regresses with reference to the moving axes, it advances with reference to fixed axes; the advance in the synodic period is $nT - 2\pi\left(\frac{1}{2}c - 1\right)$, and in the sidereal period the advance is

$$2\pi \left[1 - \frac{\frac{1}{2}c}{1 + \frac{nT}{2\pi}} \right].$$

In the numerical treatment of stable periodic orbits I tabulate the apparent regression $2\pi\left(\frac{1}{2}c - 1\right)$, and the actual advance $nT - 2\pi\left(\frac{1}{2}c - 1\right)$ in the synodic period; also $2\pi \left[1 - \frac{\frac{1}{2}c}{1 + \frac{nT}{2\pi}} \right]$ the advance in the sidereal period.

Let us now consider the case where c is imaginary, so that the motion is no longer oscillatory with respect to the periodic orbit, and the periodic orbit is unstable.

The form of (45) shows that c becomes imaginary either when $\Delta \sin^2 \frac{1}{2} \pi \sqrt{\Phi_0}$ is negative, or when it is greater than unity; this function will therefore be described below as the criterion of stability.

If Φ_0 were negative it would indicate that the mean force of restitution towards the periodic orbit was negative. Hence it seems obvious that the body would then depart from the periodic orbit, which would therefore be unstable. If however Δ were negative as well as Φ_0 , it would seem as if it were possible to have a real value for c ; but it is not easy to see how this condition could lead to a stable orbit.

I have not yet come on any case where Φ_0 is negative and accordingly that condition is left out of consideration for the present. We are left then with the two conditions, Δ negative or $\Delta \sin^2 \frac{1}{2} \pi \sqrt{\Phi_0}$ greater than unity; these lead to two kinds of instability.

In instability of the first kind Δ is negative; for reasons which will appear below, I shall call this 'even instability'.

In this case let us put

$$\Delta \sin^2 \frac{1}{2} \pi \sqrt{\Phi_0} = -D^2,$$

so that (45) becomes $\sin \frac{1}{2} \pi c = \pm D \sqrt{-1}$.

The sine in this case is hyperbolic, and if we write

$$c = 2i + k\sqrt{-1},$$

where i is an integer, the equation for k becomes $\sinh \frac{1}{2} \pi k = \pm D$.

Since the values of c occur in pairs, equal in magnitude and opposite in sign, it is only necessary to consider the upper sign and the result may be written

$$(47) \quad e^{\frac{1}{2}\pi k} = \sqrt{(D^2 + 1)} + D.$$

I shall return in § 11 to the form of solution adapted to the case of 'even instability'.

Turning to the instability of the second kind, which I shall call 'uneven instability', we have

$$\sin^2 \frac{1}{2} \pi c = \Delta \sin^2 \frac{1}{2} \pi \sqrt{\Phi_0} = D^2,$$

where D^2 is greater than unity, so that c is imaginary.

The sine in this case also becomes a hyperbolic function, and if we write $c = 2i + 1 + k\sqrt{-1}$, where i is an integer: we have

$$\sin \frac{1}{2} \pi c = (-)^i \cosh \frac{1}{2} \pi k,$$

a hyperbolic cosine.

Hence

$$\cosh \frac{1}{2} \pi k = \pm D.$$

Taking only the upper sign as before, this may be written

$$(18) \quad e^{\frac{1}{2}\pi k} = \sqrt{(D^2 - 1)} + D.$$

I shall return in § 11 to the form of solution adapted to the case of 'uneven instability', but I wish now to consider the nature of the transitions from instability to stability.

Suppose that we are considering a family of periodic orbits, the members of which are determined by the continuous increase or decrease of the constant C of relative energy; and let us suppose that $\Delta \sin^2 \frac{1}{2} \pi \sqrt{\Phi_0}$, being at first negative, increases and reaches the value zero. At the moment of the transition of this function from negative to positive, there is transition from even instability to stability. If on the other hand this function were positive and less than unity, and were to increase up to and beyond unity there would be a transition from stability to uneven instability.

In all the cases of stability which I have investigated, except one*), the fundamental value of c lies between 2 and 3, and the apparent regression of pericentre in the synodic period, namely $2\pi(\frac{1}{2}c - 1)$, lies between 0 and 180° , these extreme values corresponding with transitional stages.

*) The orbit in question is $C = 40.0$, $z_0 = 1.0334$.

It will now conduce to brevity to regard c as lying between 2 and 3, instead of regarding it as a multiple-valued quantity.

If we refer back to the form of solution assumed for the equation (41), we see that when $c = 2$, the solution is

$$\delta q = (b_{-1} + e_1) + (b_0 + e_0 + b_{-2} + e_2) \cos \frac{2\pi s}{S} \dots \\ + (b_0 - e_0 - b_{-2} + e_2) \sqrt{-1} \sin \frac{2\pi s}{S} \dots,$$

and that when $c = 3$, it is

$$\delta q = (b_1 + e_{-1} + b_{-2} + e_2) \cos \frac{\pi s}{S} + (b_0 + e_0 + b_{-3} + e_3) \cos \frac{3\pi s}{S} \dots \\ + (b_1 - e_{-1} - b_{-2} + e_2) \sqrt{-1} \sin \frac{\pi s}{S} \\ + (b_0 - e_0 - b_{-3} + e_3) \sqrt{-1} \sin \frac{3\pi s}{S} \dots$$

In the first case it is clear that when $s = S$, δq has gone through a complete period and has returned to its initial value; but in the second case whilst δq is equal in value, it is opposite in sign to what it was at first.

Consider then the first case where $c = 2$, and suppose that the body is displaced from the periodic orbit along the normal, at a conjunction. Then the body starts moving at right angles to the line of syzygies, and when $s = S$ it has again returned to the same point, and is again moving at right angles to the line of syzygies.

Hence it follows that we have found a new periodic orbit differing by infinitely little from the original one. Thus the original orbit is a double solution of the problem, and the interpretation to be put on the result $c = 2$ is, that we have found a periodic orbit which is a member of two distinct families.

The $\Delta \sin^2 \frac{1}{2} \pi \sqrt{\Phi_0}$ corresponding to our family of orbits has been supposed to be increasing from a negative to a positive value; at the instant of transition the same function for the other family must also be passing through the value zero.

If C be the value of the constant of relative energy for the critical orbit which gives $c = 2$, there must be *two* orbits, infinitely near to one another, for which the constant is $C - \delta C$.

If the orbits were classified according to values of the parameter $\Delta \sin^2 \frac{1}{2} \pi \sqrt{\Phi_0}$, instead of according to values of C , these two families would have to be regarded as a single family, and the critical stage would be that in which C reached a maximum or minimum value.

But when the classification is according to values of C , we say that there are two families which coalesce at the critical value of C ;

it is also clear that, as the orbit we were following was unstable up to this critical value, the other must have been stable.

An interesting example of this will be found below, where the families of orbits *B* and *C* spring from a single orbit.

Now reverting again to the question of the transition from instability to stability, let us suppose that as the constant *C* varies, $\Delta \sin^2 \frac{1}{2} \pi \sqrt{\Phi_0}$, being at first greater than unity, diminishes, passes through the value unity and continues diminishing. Then the orbit was at first unstable with uneven instability and *c* of the form $3 + k\sqrt{-1}$; it becomes stable at the critical stage with *c* less than 3. But there is now no real double solution at the moment of transition and no coalescence of families*). It is probable that there is coalescence with another family of imaginary orbits at this crisis, but I do not discuss this, since I am not looking at the subject from the point of view of the theory of differential equations. Accordingly in our figures of orbits there will be nothing to mark the transition from uneven instability to stability, and it will only be by the consideration of the function $\Delta \sin^2 \frac{1}{2} \pi \sqrt{\Phi_0}$ that we shall be aware of the change.

The conclusions arrived at in this section seem to accord with those of M. Poincaré in his *Mécanique Céleste*, who remarks that periodic orbits will disappear in pairs.

It is clear from this discussion that uneven instability can never graduate directly into even instability, but the transition must take place through a range of stability.

§ 11.

Modulus of instability, and form of solution.

The cases of instability will now be considered.

When the instability is of the first or even kind, we have

$$c = 2i + k\sqrt{-1},$$

and

$$(49) \quad \begin{cases} e^{\frac{1}{2} \pi k} = \sqrt{(D^2 + 1)} + D, \\ e^{-\frac{1}{2} \pi k} = \sqrt{(D^2 + 1)} - D, \end{cases}$$

where $D^2 = -\Delta \sin^2 \frac{1}{2} \pi \sqrt{\Phi_0}$.

*) When I explained the results at which I have arrived to M. Poincaré, he suggested that there may be coalescence between a doubly periodic orbit and a singly periodic one, when the two circuits of the former become identical with one another and with the latter.

The solution of (44) was

$$\delta q = \sum_j [(b_j + e_{-j}) \cos (c + 2j)\sigma + (b_j - e_{-j})\sqrt{-1} \sin (c + 2j)\sigma].$$

Now if we take the integer i involved in the expression for c as zero,

$$\cos (c + 2j)\sigma = \cosh k\sigma \cos 2j\sigma - \sqrt{-1} \sinh k\sigma \sin 2j\sigma,$$

$$\sqrt{-1} \sin (c + 2j)\sigma = -\sinh k\sigma \cos 2j\sigma + \sqrt{-1} \cosh k\sigma \sin 2j\sigma.$$

Therefore when the sign of summation only runs from ∞ to 0, instead of to $-\infty$, and when b_0 and e_0 are supposed to be the halves of their values when the summation ran from $+\infty$ to $-\infty$, the solution may be written

$$\begin{aligned} \delta q = \sum_0^\infty \{ & \cosh k\sigma [(b_j + e_{-j} + b_{-j} + e_j) \cos 2j\sigma \\ & + (b_j - e_{-j} - b_{-j} + e_j)\sqrt{-1} \sin 2j\sigma] \\ & + \sinh k\sigma [-\sqrt{-1} (b_j + e_{-j} - b_{-j} - e_j) \sin 2j\sigma \\ & - (b_j - e_{-j} + b_{-j} - e_j) \cos 2j\sigma] \}. \end{aligned}$$

Putting

$$\begin{aligned} b_j + b_{-j} &= B_j, & e_{-j} + e_j &= E_j, \\ b_j - b_{-j} &= \beta_j \sqrt{-1}, & e_{-j} - e_j &= \varepsilon_j \sqrt{-1}, \end{aligned}$$

and writing the hyperbolic functions as exponentials, we have

$$(50) \quad \delta q = \sum_0^\infty \{ e^{k\sigma} (E_j \cos 2j\sigma + \varepsilon_j \sin 2j\sigma) + e^{-k\sigma} (B_j \cos 2j\sigma - \beta_j \sin 2j\sigma) \}.$$

By means of (49) this may be written

$$(50) \quad \delta q = \sum_0^\infty \left\{ \left(\sqrt{(D^2 + 1)} + D \right)^{\frac{2\sigma}{\pi}} (E_j \cos 2j\sigma + \varepsilon_j \sin 2j\sigma) + \left(\sqrt{(D^2 + 1)} - D \right)^{\frac{2\sigma}{\pi}} (B_j \cos 2j\sigma - \beta_j \sin 2j\sigma) \right\}.$$

In (50) it is not safe to assume that the most important term is that for which $j = 0$; indeed this will usually not be the case. All that we know is that the series contains sines and cosines of even multiples of σ , that one set of terms increases without limit and that the other set diminishes.

In the numerical treatment of unstable periodic orbits it will be well to have a modulus of the degree of instability; and these considerations afford a convenient means of obtaining such a modulus.

This modulus may be taken to be the number of synodic revolutions in which the augmenting factor doubles its initial value; that is to say we are to put

$$e^{k\sigma} = [\sqrt{(D^2+1)} + D]^{\frac{2\sigma}{\pi}} = 2.$$

Therefore

$$(51) \quad \frac{s}{S} = \frac{\sigma}{\pi} = \frac{\log \sqrt{2}}{\log [\sqrt{(D^2+1)} + D]}.$$

This is the modulus of instability, when it is of the even kind.

A consideration of the form of the series for δq shows that it increases without limit, and that the planet or satellite crosses and recrosses the periodic orbit an even number of times in a single circuit; it is on this account that I have called this 'even instability'.

When the instability is of the second or uneven kind, we have $c = 2i + 1 + k\sqrt{-1}$, or if we take i as zero, $c = 1 + \sqrt{-1}$; also

$$(52) \quad \begin{cases} e^{\frac{1}{2}\pi k} = D + \sqrt{(D^2-1)}, \\ e^{-\frac{1}{2}\pi k} = D - \sqrt{(D^2-1)}, \end{cases}$$

where $D^2 = \Delta \sin^2 \frac{1}{2} \pi \sqrt{\Phi_0}$.

Then by a series of transformations, similar to those followed before, we find

$$(53) \quad \sum_0^\infty \left\{ (D + \sqrt{(D^2-1)})^{\frac{2\sigma}{\pi}} (E_j \cos (2j+1)\sigma + \epsilon_j \sin (2j+1)\sigma) \right. \\ \left. + (D - \sqrt{(D^2-1)})^{\frac{2\sigma}{\pi}} (B_j \cos (2j+1)\sigma - \beta_j \sin (2j+1)\sigma) \right\}.$$

In this case again the terms for which $j = 0$ are not usually the most important ones, but we see that the series contains sines and cosines of odd multiples of σ ; and that one set of terms increases without limit and that the other diminishes. As in the first sort of instability, a convenient modulus is the number of synodic revolutions in which the amplitude of the increasing oscillation doubles its initial value; that is to say we put

$$e^{k\sigma} = (D + \sqrt{(D^2-1)})^{\frac{2\sigma}{\pi}} = 2.$$

Therefore

$$(54) \quad \frac{s}{S} = \frac{\sigma}{\pi} = \frac{\log \sqrt{2}}{\log [D + \sqrt{(D^2-1)}]},$$

where

$$D^2 = \Delta \sin^2 \frac{1}{2} \pi \sqrt{\Phi_0}.$$

This is the modulus of instability, when it is of the uneven kind. A consideration of the principal term has shown us that there is an oscillation, whose amplitude increases without limit. The planet or satellite crosses and recrosses the periodic orbit an odd number of

times in a single circuit, making ever increasing excursions on each side; it is on this account that I have called this 'uneven instability'.

It is easy to write down the forms which the equations of condition assume in the two sorts of instability, but I do not reproduce them here.

§ 12.

Numerical determination of stability.

When a periodic orbit has been found by quadratures, it is not obvious by mere inspection whether it is stable or not, and we must consider the numerical processes requisite to obtain an answer to the question.

The points which are determined by quadratures in a periodic orbit do not divide the arc S into a number of equal parts. The distance along the arc from the first orthogonal crossing of the x axis to the second orthogonal crossing is $\frac{1}{2} S$; this may be determined by interpolation, for we may find what value of s makes y vanish.

In general there are two orbits computed, which differ from exact periodicity in opposite directions by small amounts. The arc $\frac{1}{2} S$, measured from the first orthogonal crossing to the second, which is not exactly orthogonal, is determined in each of these cases. The subsequent proceedings are then carried out in duplicate, and the final step is an interpolation between the two results to obtain the result for the exactly periodic orbit. In many cases however the computed orbit differs from a truly periodic one by an amount which is so small, that it may be attributed to the errors inherent to the method of calculation. In such cases the duplicate computation is unnecessary, and since the operations on the approximately periodic orbits are exactly like those on the truly periodic ones, we may henceforth speak as if the true orbit had been found.

The work already carried out in the quadratures affords many of the data requisite for the computation of Φ , and the next step is the computation of Φ corresponding to each computed point of the orbit.

It would be tedious to find the Fourier's series for Φ from its computed values, and it is best to find interpolated values of Φ at exact submultiples of the arc S . I therefore interpolate Φ at the points for which the arc is $\frac{1}{24} S, \frac{2}{24} S \dots \frac{12}{24} S$, 13 values in all.

The next step is the harmonic analysis of these 13 values of Φ , which is an even function of the arc, and the work may be conveniently arranged in the form of a schedule. I have usually proceeded as far as the term of the eighth order.

It may be remarked that if the harmonic expansion of Φ is convergent, the determinant from which the stability is determinable is also convergent.

But if the representation of Φ by the harmonic expansion up to the 8th harmonic is very imperfect, it is necessary to give up the attempt to determine the stability numerically. In such cases however it is nearly always possible to see that the orbit is unstable, although it may not sometimes be so easy to perceive whether the instability is even or uneven.

We next have to calculate the several members of the determinant Δ by the formula

$$\frac{\Phi_i}{\Phi_0 - 4j^2}.$$

This is the entry for the j^{th} row above or below the centre of the determinant, and it is the i^{th} member to the right and to the left of the leading diagonal, all the members on the diagonal being unity. The values of Φ_i computed to the eighth order suffice to enable us to write down 17 columns and rows of Δ . The method of computing Δ will be considered in the next section.

§ 13.

The calculation of a determinant of many columns and rows.

The following transformation contains the principle by which the number of columns and rows of a determinant may be diminished by unity;

$$\begin{aligned} \Delta = \begin{vmatrix} a_1, a_2, a_3, \dots \\ b_1, b_2, b_3, \dots \\ c_1, c_2, c_3, \dots \\ \dots \end{vmatrix} &= a_1 \begin{vmatrix} 1, & \frac{a_2}{a_1}, & \frac{a_3}{a_1}, & \dots \\ 0, & b_2 - b_1 \frac{a_2}{a_1}, & b_3 - b_1 \frac{a_3}{a_1}, & \dots \\ 0, & c_2 - c_1 \frac{a_2}{a_1}, & c_3 - c_1 \frac{a_3}{a_1}, & \dots \\ \dots \end{vmatrix} \\ &= a_1 \begin{vmatrix} b_2 - b_1 \frac{a_2}{a_1}, & b_3 - b_1 \frac{a_3}{a_1}, & \dots \\ c_2 - c_1 \frac{a_2}{a_1}, & c_3 - c_1 \frac{a_3}{a_1}, & \dots \\ \dots \end{vmatrix}. \end{aligned}$$

Now if we write $b_2' = b_2 - b_1 \frac{a_2}{a_1}$, and so on, and then extract the factor b_2' , another column and row may be removed, and the process may be repeated until the determinant is reduced to a single member, say z_n ; then

$$\Delta = a_1 b_2' c_3'' \dots z_n.$$

If the determinant is convergent and if the rows and columns be removed in proper succession, the factors tend to unity.

By interchanges of columns and rows any member of a determinant may be brought to stand at a corner, but if the number of interchanges is odd the sign of the determinant is changed.

It is not therefore necessary to work from a corner, as in the above example, but any column and any row may be chosen for elimination. The member which stands at the intersection of the chosen column and row may be called the centre of elimination. Then if the centre of elimination be at an odd or even number of moves from a corner, the sign of the whole is or is not changed.

In the determinants which arise in this investigation the centre of elimination is always taken on the diagonal, and thus no change of sign is introduced.

Let us suppose that the determinant to be evaluated is a symmetrical one, and that the columns and rows are numbered as in the following example:

$$\begin{array}{rccccc} & -2 & -1 & 0 & 1 & 2 \\ \begin{array}{l} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{array} & \left| \begin{array}{ccccc} C, & c_1, & c_2, & c_3, & c_4 \\ b_1, & B, & b_1, & b_2, & b_3 \\ a_2, & a_1, & A, & a_1, & a_2 \\ b_3, & b_2, & b_1, & B, & b_1 \\ c_4, & c_3, & c_2, & c_1, & C \end{array} \right| \end{array}$$

Let $(-1, -1)$ be the first centre of elimination, and $(1, 1)$ the second; then if the double elimination be carried out and algebraic reductions effected, it will be found that the result is

$$\begin{array}{rcc} & -2 & 0 & 2 \\ B^2 \left(1 - \frac{b_2^2}{B^2}\right) & \left| \begin{array}{ccc} B', & b_1', & b_2' \\ a_1', & A', & a_1' \\ b_2', & b_1', & B' \end{array} \right| & \begin{array}{l} -2 \\ 0 \\ 2 \end{array} \end{array}$$

Where

$$B' = C - \frac{b_3 c_1 + b_1 c_3}{B + b_2} - \frac{(b_1 - b_3)(c_1 - c_3)}{B - \frac{b_2^2}{B}}, \quad b_1' = c_2 - \frac{b_1(c_1 + c_3)}{B + b_2},$$

$$b_2' = c_4 - \frac{b_1 c_1 + b_3 c_3}{B + b_2} + \frac{(b_1 - b_3)(c_1 - c_3)}{B - \frac{b_2^2}{B}}, \quad a_1' = a_2 - \frac{a_1(b_1 + b_3)}{B + b_2},$$

$$A' = A - \frac{2a_1 b_1}{B + b_2}.$$

If the determinant is convergent, with an odd number of columns and rows, $(0, 0)$ is the heart of the determinant; if the elimination pro-

ceeds away from the heart, at any stage of the process the approximation consists of the product of all the factors extracted, multiplied by $(0, 0)$, the heart of the remaining determinant.

Thus in the above example after one double elimination the approximation is

$$B^2 \left(1 - \frac{b_2^2}{B^2}\right) \left(A - \frac{2a_1b_1}{B + b_2}\right).$$

This is in fact the full expression for the determinant

$$\begin{vmatrix} B, & b_1, & b_2 \\ a_1, & A, & a_1 \\ b_2, & b_1, & B \end{vmatrix}.$$

I have found it most convenient in practice first to extract a squared factor, such as B^2 (thus reducing $(-1, -1)$ and $(1, 1)$ to unity), and afterwards to extract a single factor, such as $1 - \frac{b_2^2}{B^2}$.

This process cannot of course be applied with advantage, when the work is algebraical, but some process of the kind seems to be practically necessary, when the approximate numerical value is to be found of a determinant of a large number of columns and rows.

It will be noticed that after each pair of eliminations the primitive symmetry is restored; but the work might equally well be arranged otherwise. For we might first eliminate from the centre $(0, 0)$, which would not affect the symmetry, and we might then take the pair $(-1, -1)$ and $(1, 1)$. This variation of procedure would afford a valuable check on the arithmetic.

Where the outer fringe of the determinant obviously has but little influence on the final result, and where we are in any case going to use all the members in the original determinant, I have found it best to begin from the outside. In such a case four or five columns and rows may, as it were, be shelled off the outside, with scarcely any alteration of the central entries.

The actual numerical work of evaluating a determinant may be arranged as follows:

The number of decimal places to be retained is first fixed on. A paper is then marked with a gridiron of columns and rows, numbered from zero at the centre upwards and downwards. Each square should be large enough to contain four or five rows of figures. The original determinant is then written in the squares, the numbers being put as near the top of each square as possible. I have found it convenient to omit decimal points, and to express the numbers in units of the last decimal place retained. In most of my work, where only a rough

result was required, I have adopted three places of decimals; thus the unit in which the entries are expressed is '001, and the diagonal members are all written as 1000.

The pair of symmetrical diagonal members, which is to form the first pair of centres, is then chosen. As stated above, I have in my later work usually worked from the outside. In the first pair of eliminations these diagonals are already unity, but this is not so subsequently, and we first reduce them to unity by dividing the rows on which they stand by their values, and by extracting a squared factor.

It will be found convenient to run a red line through the column and row to be removed. If the red lines be regarded as coordinate axes, the row being x and the column y , any member of the determinant may be specified by its x and y . If the member of the determinant whose coordinates are x, y be a ; and if the member whose coordinates are $x, 0$ be b ; and if the member whose coordinates are $0, y$ be c ; then the number which has to be substituted for a is $a - bc$.

In other words each number on the horizontal red line has to be multiplied by each number on the vertical red line, and the products have to be subtracted from the numbers which stand at the remote corners of the rectangles.

In effecting this process I form a separate table of the subtrahends, and write down the differences immediately under the numbers which they displace.

After the first elimination, which has rendered the determinant unsymmetrical, a single factor corresponding to the other chosen diagonal member is extracted, its row is correspondingly altered, red lines are drawn to mark the column and row to be removed, and the similar process is repeated. The symmetry of the determinant should now be restored, but any pair of numbers which should agree are arrived at by different numerical processes.

The restoration of symmetry affords a very valuable check on arithmetical processes which I have found singularly difficult to work correctly.

As only a limited number of decimal places are employed there is often a discrepancy of unity in the last significant figure between two numbers which ought to agree. It is sometimes possible to determine by inspection which of the two numbers is arrived at by the less risky series of operations, and I then adopt that number to represent both entries. But where there is no obvious reason for choosing one result more than the other, I choose one or other at hasard, and restore the perfect symmetry.

The process of elimination is continued until the determinant is

reduced to $(0, 0)$, but in the last two or three stages it is well to increase the number of decimals retained.

If at any stage the factor to be extracted becomes small, the whole row to which it belongs becomes large, and the symmetry may perhaps be seriously affected. In this case it is well not to choose this pair of centres of elimination, but to take another pair, leaving this pair to a later stage in the calculation.

If the determinant is negative, a negative factor will be extracted at some stage. In all the cases which have been worked out it is easy to see that no other negative factor will ever arise, and thus the determinant will remain clearly negative. Most of the determinants have been written with 17 columns and rows; then beginning with $(-8, -8)$ and $(8, 8)$ I find that it is often possible to erase 8 columns and 8 rows on a single sheet of paper, with scarcely any modification of the central part of the determinant. Thus the determinant which at first had 289 spaces (although many only contain zeros) is reduced to 81 spaces, with but little labour.

The multiplications have been done with Crelle's table, but a specially computed auxiliary table of products, from $\cdot 000 \times \cdot 000$ up to $\cdot 040 \times \cdot 040$ to three places of decimals, has rendered the work much more rapid.

I believe that the values obtained by this process are correct to within about one per cent. For the same determinant when reduced with different order of elimination agrees with its previous determination within less than that amount of discrepancy.

Part. II.

§ 14.

Periodic Orbits.

An orbit in which the third body can continually revolve, so as always to present the same character relatively to the two other bodies, is said to be periodic. If the motion is referred to a plane which is carried round with Jove and revolves about the Sun as a centre, any re-entrant orbit of the third body is periodic. Periodic orbits may consist of any number of revolutions round either of the primaries, or round other points in space. Periodic orbits, which are only re-entrant after several circuits, are much more difficult to discover than those which only make a single one; as hardly anything is known up to

the present time about this subject, I determined to confine my attention to 'simple periodic orbits', which are re-entrant after a single circuit. This definition of a simply periodic orbit must not preclude the consideration of orbits with loops, for the inclusion of such loops is necessary to the comprehension of the subject.

It appears from the differential equation of motion that periodic orbits must in general be symmetrical with respect to the line of syzygy; or if any periodic orbit consists of a closed circuit round a point which does not lie on this line, there must be a similar closed circuit round a symmetrical point on the other side of it.

Periodic orbits are critical cases which separate the orbits of one class from those of another, and the chief difficulty in tracing them consists in the fact that it is necessary to trace the gradual change of an orbit, as its parameters change, and to discover its form at the instant of its transformation into an orbit of a different character.

The partition of space derived from the Jacobian integral (§ 3) shows that the constant of relative energy C is of primary importance in the classification of orbits. The work of this investigation being numerical, I was compelled to assume a definite ratio for the mass of the Sun in terms of that of Jove; this ratio is taken as 10. The mass of the actual Sun in terms of that of the actual planet Jupiter is about 1000, and accordingly all the phenomena of perturbation are greatly exaggerated in our figures as compared with the real solar system. This exaggeration appeared to me advantageous for the purpose of giving a clear view of the phenomena.

The mass of the Sun being 10, that of Jove being unity and the distance between them being unity, we found in (9) that when C is greater than 40.1821 the third body must be either a superior planet, or an inferior planet, or a satellite, but cannot change from one of these conditions to another.

These larger values of C then bring us to those cases which are treated in the Planetary and Lunar Theories; I therefore cease my consideration of the problem for all values of C which are greater than 40.5. Hence the field to be treated is covered by values of C less than 40.5, and the problem is to obtain a complete synopsis of simply periodic orbits and of their stabilities*).

The field of investigation is however so large that in the present paper I am compelled to make further restrictions. In the first place, the case of superior planets has not been touched at all; although, at

*) In the original paper I inadvertently stated that C could not be less than 33. I should of course have said that $C = 33$ is the limiting value of C for which any portion of space is prohibited. This inadvertence has caused no error, since the smaller values of C have not yet been considered.

the point at which I have now arrived, they must soon be taken into account.

Secondly all the orbits considered are direct; the retrograde orbits would afford an interesting field of research.

Lastly the present paper only covers the field from C equal to 38 to 40.5; and even this has occupied me for three years.

The slowness with which results are attained by arithmetical processes has been very tantalising, but the interest of the work has been sustained by the fact that the results have presented a succession of surprises. I have, over and over again, been deceived when I imagined I could foresee the shape which would be assumed by the next orbit to be treated, and thus the subject was continually presenting itself under a new light. Nevertheless a point has, I think, been now reached at which some forecasts are possible, and I shall venture to say something hereafter in § 18 on this head, with the full knowledge however that the conjectures may prove erroneous.

Being ignorant of the nature of the orbits of which I was in search, I determined to begin by a thorough examination of one case. It seemed likely that the most instructive results would be obtained from cases in which it should be possible for an inferior planet and satellite to interchange their parts. Now when C is greater than 38.8760 but less than 40.1821, the two interior ovals of the curve of zero velocity coalesce into the shape of an hour-glass, and thus interchange of parts is possible. I therefore began by the consideration of the case where C is 39, and traced a large number of orbits which start at right angles to SJ , and in some cases I also traced the orbit with reference to axes fixed in space.

The two curves, which represent the orbit in space and with reference to the moving plane, contain a complete solution of the problem. For if the curve on the moving plane be drawn as a transparency, and if the Sun in the two figures be made to coincide, and if the transparent figure be made to revolve uniformly about the sun, the intersection of the two curves will give the position of the body both in time and place.

In order to exhibit this I show in fig. 2 a certain orbit with reference to axes fixed in space and also the same orbit referred to rotating axes. In the former figure the simultaneous positions of the planet and of Jove are joined by dotted lines. It is interesting to observe how the body hangs in the balance between the two centres, before the elliptic form of the orbit asserts itself, as the body approaches the Sun.

This figure, and others of the same sort, are instructive as illustrating the usual sequence of events in orbits of this class.

If a planet be started to move about the Sun in an orbit of a certain degree of eccentricity, it will at first move with more or less exactness in an ellipse with advancing perihelion. But as the aphelion approaches conjunction with Jove the perturbations will augment at each passage of the aphelion. At length the perturbation becomes so extreme that the elliptic form of the orbit is entirely lost for a time, and the body will either revert to the Sun, or it will be drawn off

and begin a circuit round Jove. In either case after the approximate concurrence of aphelion with conjunction, the orbit will have lost all resemblance to its previous form.

The figure 2 exhibits the special case in which the body only makes a single circuit round Jove, and where the heliocentric elliptic orbit before and after the crisis has the same form; the perihelion has however advanced through twice the angle marked ω on the figure. In general the body would, after parting from the Sun, move several times round Jove until a concurrence of apojove with conjunction produced a severance of the connection, but in the figure this concurrence happens after the first circuit. If the neck of the hour-glass defining the curve of zero velocity be narrow, the body may move hundreds of times round one of the centres before its removal to the other.

It seems likely that a body of this kind would in course of time find itself in every part of the space within which its motion is confined. Sooner or later it must pass indefinitely near either to the Sun or to Jove, and as in an actual planetary system those bodies must have finite dimensions, the wanderer would at last collide with one of them and be absorbed. We thus gain some idea of the process by which stray bodies are gradually swept up by the Sun and planets.

It might be supposed that all possible orbits for any value of C will pass through a similar series of changes and that the bodies which move in them will be thus finally absorbed. Lord Kelvin is of opinion that this must be the case, and that all orbits are essentially unstable*). This may be so when sufficient time is allowed to elapse, but we shall see later that, even when the hour-glass has an open neck, there are still stable orbits, as far as our approximation goes. The only approximation permitted in this investigation is the neglect of the perturbation of Jove by the planet. For a very small planet the instability must accordingly be a very slow process, and I cannot but believe that the whole history of a planetary system may be comprised in the interval required for the instability to render itself manifest. Henceforward then I shall speak as though the stability of stable orbits were absolute, instead of being, as it probably is, only approximate.

§ 15.

Non-periodic orbits; $C = 39.0$.

(a) *Orbits round Jove.* Fig. 3.

The Sun S is outside of the figure towards the left. A small portion of the curve $2\Omega = 39$ is shown to the right of J , and another

*) Sir William Thomson, *On the Instability of Periodic Motion*, *Philosophical Magazine*, vol. 32, 1891, p. 555. M. Poincaré also considers that orbits may have a temporary, but not a secular stability. *Acta Mathem.* T. 13, 1890, p. 101.

Fig. 3

Non-periodic orbits round force, $C=2890$

portion at the narrowing of the neck of the hour-glass. The two points of zero force given by $\frac{\partial \Omega}{\partial x} = 0$, $\frac{\partial \Omega}{\partial y} = 0$ (see § 3) are also marked.

The complete circuits are shown in order to obtain a better idea of the nature of the orbits, although this is unnecessary for the search for periodic orbits.

The satellite is supposed to be started at right angles to SJ at the conjunction-remote from the sun, and enough of the orbits are shown to obtain a synopsis of the class. Here and elsewhere I define the orbits by the initial value of x , which is denoted by x_0 ; in this case the final value of x after the complete circuit may be called x_2 .

The first on the right (dotted-line) starts with $x_0 = 1.3$, and x_2 is much less than x_0 . The second (chain-dotted) has $x_0 = 1.26$, and x_2 has considerably increased so as to approach x_0 . The third (broken-line) has $x_0 = 1.22$, and x_2 has now become greater than x_0 ; therefore we have passed an orbit for which x_2 was equal to x_0 , and such an orbit is periodic.

In the fourth (full-line) with $x_0 = 1.18$, x_2 exceeds x_0 by more than it did in the third orbit. But in the fifth (dotted) with $x_0 = 1.14$, x_2 has again become less than x_0 ; therefore we have passed another periodic orbit.

In the sixth orbit, (broken-line) $x_0 = 1.12$, x_2 has decreased very much, and in the seventh (full-line) $x_0 = 1.10$, x_2 has become quite small. This last has very nearly a cusp. It is not so accurately computed as the preceding ones, having been the first difficult orbit undertaken, and my methods at that time were not quite so satisfactory as they became subsequently. In this seventh orbit at the final intersection φ has just passed through the value zero, and I think it is probable that there is an orbit of very nearly this form, with the final φ exactly zero. Such an orbit would be periodic, but as it would not be simply periodic, it falls outside the scope of this paper.

The first part of the eighth orbit (chain-dot) was derived by interpolation between $x_0 = 1.1$ and $x_0 = 1.09$ (shown in a future figure); the beginning of this orbit, which I call $x_0 = 1.095$, is not shown. It is a very remarkable curve, for after the loop, the body recrosses SJ , and going directly towards J , passes so close to it that it is impossible without more accurate computation to say what would happen subsequently. This orbit was so unexpected that I have thought it well to show in Fig. 4 its form with respect to axes fixed in space; in this figure (which does not claim close accuracy) the interpolated portion has been inserted. I do not think that any one could

have conjectured how the body should have been projected so as to fall into Jove.

For smaller values of x_0 the bodies are no longer simple satellites as they part company with Jove and pass away towards the Sun.

Fig.4

Orbit round Jove referred to axes fixed in space ($x_0 = 1.095$, $C = 39.6$)

(β) *Orbits passing from Jove to Sun.* Fig. 5.

The curve of zero velocity for $C = 39$ having been computed, it is shown in this figure, although it is not necessary.

The starting points are again from conjunction remote from the Sun. The first orbit (broken-line) is the one with which we ended in Figure 3, viz. $x_0 = 1.095$; the interpolated portion is however now drawn, as well as the computed portion. The body in this case does not pass away to the Sun.

We next come to an orbit (dotted) of which the first part was found by interpolation and which I call $x_0 = 1.09375$; the earlier portion of the curve is not drawn.

Where the orbit $x_0 = 1.095$ crosses SJ for the third time, φ is clearly negative, but where the orbit $x_0 = 1.09375$ crosses for the third time φ is positive. There must therefore be an intermediate case for which φ vanishes, and this will give us a third periodic orbit round J . The orbit $x_0 = 1.09375$ passes away to the sun; and we then come to four more orbits $x_0 = 1.09, 1.08, 1.06, 1.04$ which follow a similar course, but with diminishing depression towards the negative side of SJ . The next orbit is $x_0 = 1.02$, in which the depression has disappeared. This curve has a slight hump in the place of the depression; it is the sort of feature which would present itself in a computed curve, when there has been an arithmetical error in the calculation, but we shall soon see that this hump is not explicable in this way.

The next curve which is traced (although others have been computed) starts with $x_0 = 1.001$ (chain-dot); in a figure of this scale, it apparently starts actually from J . It will be observed that we now have a remarkable cusp, and it becomes obvious that the hump referred to above was an incipient elevation towards the cusp.

Passing now to the other end of the figure where the body passes round the Sun, we see from the incidence of the perihelia (which are indicated by radii from the Sun) that there can be no periodic orbit which is partly the path of a satellite and partly that of a planet; for such an orbit must have the longitude of the perihelion 180° .

The positions of the perihelia and the perihelion-distances seem to be almost chaotic in the figure, but I believe that the calculations are substantially correct. But as each one of these curves represents three or four weeks hard work, I have not thought it good economy of labour to pursue the inquiry further in this respect.

(γ) orbits round the Sun; $C = 39.0$. Fig. 6.

These curves are drawn with less accuracy than the others, being computed with three-figured logarithms. I thought that sufficient accuracy would be attainable with this degree of approximation, but when I found that the saving of labour was not considerable, whilst the loss of accuracy was very great, I returned to the use of four-figured tables. It did not however seem necessary to recompute these curves.

The complete circuit is drawn for four of the curves, but the rest are only carried half way round.

The orbits start to the left of the Sun at the conjunction remote from Jove. The first orbit is $x_0 = -.6$ (full-line), and at the second

All the succeeding orbits viz. $x_0 = -.337, -.3, -.2, -.1, -.04, -.001$ have φ positive and successively increasing at the second crossing; and thus there is no other periodic orbit. The last two of these orbits have loops.

The orbit $x_0 = -.337$ was found in part by interpolation. It has been inserted because the third crossing of the line SJ appears to be orthogonal, and therefore the orbit is periodic, but not *simply* periodic. No search was being made for this sort of orbit, and the discovery was accidental.

§ 16.

Periodic Orbits classified according to values of C .

Plate.

Plate, fig. 1. $C = 40.0$.

When C is greater than 40.18, the inner branch of the curve of zero velocity, $2\Omega = C$, consist of two ovals, as seen in fig. 1; the periodic orbits then consist of two approximately circular orbits round S and J respectively. These cases may be treated by the methods of the Planetary and Lunar Theories, and fall outside the scope of this paper.

When $C = 40.18$ there is a third periodic orbit consisting of the point $x = .7175, y = 0$. At this point a body is in unstable equilibrium, and this point is the beginning of a family of orbits; for, whilst in general periodic orbits begin in pairs, a single orbit may begin at a point.

In discussing these figures I shall denote the initial value of any function by the suffix 0; the suffix 1 will denote the value after the completion of a half circuit, and the suffix 2 the value on the completion of the whole circuit.

The planet A starts from $x_0 = -.414, \varphi_0 = \pi$, and φ increases.

When $x \leq -.414, \varphi_1 \leq 0, x_2 \geq x_0$ ($x_0 < -.414$ of course denotes a starting point more remote from S , with x_0 numerically greater than .414).

This orbit is stable with $c = 2.81$.

The satellite A starts from $x_0 = 1.03341, \varphi_0 = 0$, and φ increases.

This orbit changes its shape rapidly with changes of C , as will appear below in the classification by families. Great care was bestowed on this case, and it was very troublesome to compute, since a considerable variation of x_0 corresponded with a small variation of φ_1 .

When $x_0 \geq 1.03341, \varphi_1 - \pi \leq 0, x_2 \leq x_0$.

The orbit is stable, but borders closely on instability, with $c = 3.7$.

The third orbit is the oscillating satellite a , moving slowly with a retrograde revolution round the point of zero force $x = .7175$, $y = 0$, which was described above as the commencement of a series of orbits.

The orbit a starts from $x_0 = .705$, $\varphi_0 = 0$, and φ diminishes.

When $x_0 \geq .705$, $\varphi_1 - \pi \geq 0$. That is to say if the body starts too near to Jove the change of direction at the sharp turn is not quite sufficient for periodicity; and if it starts too near the Sun the converse is true. In the first case after one or more circuits the body passes away towards J , and in the second case towards S .

This orbit is very unstable, and the instability is almost certainly of the even type.

Plate, fig. 2. $C = 39.5$ and 39.3 .

The planetary orbit A ($C = 39.5$) differs little from the proceeding case.

It starts from $x_0 = -.424$, $\varphi_0 = \pi$, and φ increases.

When $x_0 \leq -.424$, $\varphi_1 \leq 0$, $x_2 \geq x_0$.

The orbit is stable with $c = 2.90$; but it is less stable than when $C = 40$.

The classification by families below shows that as C falls below 40.0 , the orbit of the satellite A stretches out rapidly towards S , and at the same time the oval a expands. The manner in which the figure-of-8 orbit takes its origin from the orbits A and a is still obscure, as I shall point out below.

When C has diminished to 39.5 the orbit A is a figure-of-8, whilst the orbit a remains a closed oval.

The satellite A starts from $x_0 = 1.0650$, $\varphi_0 = 0$, and φ begins increasing. When the body has passed half round J so that y vanishes, φ is equal to $\pi - 15^\circ 37'$; shortly after this φ diminishes and continues doing so until when y again vanishes $\varphi_1 = 0$.

We have $x_0 \geq 1.0650$, $\varphi_1 \leq 0$. When the body starts too far from J , it will move in some orbit round J , and when it starts too near J it will pass away to S .

This orbit is very unstable with even instability.

The oscillating orbit a was not computed for $C = 39.5^*)$; during one part of its course it would be indistinguishable from part of A , and the rest is shown conjecturally by a dotted line.

This orbit is very unstable, with even instability.

It has already been remarked that after the first half circuit of satellite A φ was $\pi - 15^\circ 37'$, or as we may now write it $\pi - \varphi_1 = 15^\circ 37'$.

*) At least the computation was not completed, for it was found to be so troublesome, that it appeared that the work could be better bestowed elsewhere.

Now when x_0 is made to increase from 1.0650 until it reaches the curve $2\Omega = C$, $\varphi_1 - \pi$ will always be negative, or $\pi - \varphi_1$ positive. It appears however that $\pi - \varphi_1$ has a minimum value, which very nearly reaches zero. In fact when $x_0 = 1.140$, $\varphi_1 = \pi - 0^{\circ}20'$.

Since $\pi - \varphi_1$ is large when x_0 approaches $2\Omega = C$, and is $15^{\circ}37'$ when $x_0 = 1.0650$, it follows that if it vanishes at all, it must vanish twice. That is to say if there is another periodic orbit, there must be two.

As C diminishes the minimum value of $\pi - \varphi_1$ falls, and I found that when $C = 39.4$ the minimum is reached when x_0 is about 1.15; for this value of x_0 , $\pi - \varphi_1$ is $0^{\circ}9'$, and there is still no value of x_0 for which $\pi - \varphi_1$ vanishes.

But when $C = 39.3$ I computed the four orbits $x_0 = 1.18, 1.17, 1.16, 1.15$ and found that for the two middle ones $\pi - \varphi_1$ was negative. By interpolation the pair of periodic orbits B and C were found.

The orbit B is given by

$$x_0 = 1.1575, \quad \varphi_0 = 0;$$

and the orbit C by

$$x_0 = 1.1751, \quad \varphi_0 = 0.$$

In both cases φ increases.

The relationship to the neighbouring orbits is given by the inequalities

$$\begin{aligned} x_0 &> 1.1751, & \varphi_1 - \pi &< 0, & x_2 &< x_0, \\ &< 1.1751 \\ x_0 &> 1.1575, & \varphi_1 - \pi &> 0, & x_2 &> x_0, \\ &< 1.1575 \\ x_0 &< 1.1575, & \varphi_1 - \pi &< 0, & x_2 &< x_0. \end{aligned}$$

The orbit B is slightly unstable, with even instability, and

$$c = .156\sqrt{-1};$$

the orbit C is stable, but approaches instability, and $c = 2.163$.

Plate, fig. 3. $C = 39.0$

These are the periodic orbits which belong to the families of non-periodic orbits shown in figs. 3, 4, 6 above.

The planetary orbit A starts from $x_0 = -.434$, $\varphi_0 = \pi$, and φ increases. The incidence amongst the neighbouring orbits is shown by the inequalities

$$x_0 \leq -.434, \quad \varphi_1 \leq 0, \quad x_2 \geq x_0.$$

This orbit is unstable with slight uneven instability and

$$c = 1 + .10\sqrt{-1}.$$

It thus appears that for some value of C between 39.5 and 39.0 we

should find the passage of the planetary orbit A from stability to instability. It is certainly surprising to find that the instability of the planet sets in when the planet is a little less than half way to Jove at conjunction.

The satellite A starts from $x_0 = 1.0941$, $\varphi_0 = 0$, and φ increases until when y vanishes it is equal to about $\pi - 13^\circ 30'$; it then diminishes to zero,

Its incidence among neighbouring orbits (figs. 3 and 4) is given by the inequalities

$$x_0 \geq 1.0941, \quad \varphi_1 \leq 0.$$

When it starts too far from Jove it will move in some orbit round J , and when it starts too near Jove it will pass away towards S .

This orbit is very unstable, with even instability and $c = .46\sqrt{-1}$. The orbit of the oscillating satellite a is indistinguishable from A throughout part of its course, but falls more remote from J on the side towards S . It starts from $x_0 = .687$, $\varphi_0 = 0$, and φ diminishes.

When $x_0 \geq .687$, $\varphi_1 = \pi \geq 0$; thus if the body starts too near Jove the total change of direction is insufficient for periodicity; and if it starts too near the Sun the converse is true. In the first case it passes away towards Jove, and in the second towards the Sun.

This orbit is very unstable with even instability, and c is about $2\sqrt{-1}$.

The satellite B starts with $x_0 = 1.1500$, $\varphi_0 = 0$, and φ increases. When $x_0 \geq 1.1500$, $\varphi_1 - \pi \geq 0$, $x_2 \geq x_0$.

This orbit is unstable, with even instability, and $c = .38\sqrt{-1}$.

The satellite C starts with $x_0 = 1.2338$, $\varphi_0 = 0$, and φ increases. When $x_0 \geq 1.2338$, $\varphi_1 - \pi \leq 0$, $x_2 \leq x_0$.

This orbit is stable, with $c = 2.46$.

Plate, fig. 4. $C = 38.5$.

The planet A starts from $x_0 = -.444$, $\varphi_0 = \pi$, and φ increases. When $x_0 \leq -.444$, $\varphi_1 \leq 0$, $x_2 \geq x_0$.

The orbit is unstable, with uneven instability and $c = 1 + .18\sqrt{-1}$.

The satellite A starts from $x_0 = 1.1164$, $\varphi_0 = 0$, and φ increases until when y vanishes it is equal to about $\pi - 12^\circ$; it then diminishes to zero. It will be observed that at the first vanishing of y , the curve cuts the axis more nearly at right angles than was the case when $C = 39.0$ and 39.5 . When $x_0 \geq 1.1164$, $\varphi_1 \leq 0$. When it starts too far from Jove it will move in some orbit round J , and when it starts too near Jove it will pass away to the Sun. The orbit is very unstable, with even instability.

The oscillating satellite *a* starts with $x_0 = .6814$, $\varphi_0 = 0$, and φ diminishes. When $x_0 \geq .6814$, $\varphi_1 - \pi \geq 0$. In the first case it passes away towards Jove, in the second towards the Sun. The orbit is very unstable with even instability.

The satellite *B* starts with $x_0 = 1.1497$, $\varphi_0 = 0$ and φ increases. When $x_0 \geq 1.1497$, $\varphi_1 - \pi \geq 0$, $x_2 \geq x_0$. The orbit is unstable with even instability, and $c = .70 \sqrt{-1}$.

The satellite *C* starts with $x_0 = 1.2760$, $\varphi_0 = 0$, and φ increases. When $x_0 \geq 1.2760$, $\varphi_1 - \pi \leq 0$, $x_2 \leq x_0$.

This orbit is very unstable, and as will appear below the instability is uneven. There has in fact been a passage from stability to uneven instability for some value of *C* between 39.0 and 38.75.

This orbit is interesting because it corresponds almost exactly to the cusped orbit described by Mr Hill as the moon of greatest lunation. It would seem however that this description is incorrect, for the satellite *C* moves with a still longer period when the cusp is replaced by a loop. Mr Hill's orbit was, on the account of his approximation, necessarily a symmetrical one with reference to the line of quadratures, but it will be observed that when the solar parallax is taken into account the orbit is very unsymmetrical.

When $C = 38.88$ a new periodic orbit arises in the point $x_0 = 1.3470$, $y = 0$ marked in the figure. This is the beginning of a second family of oscillating satellites, referred to here as *b*.

When $C = 38.5$ this orbit begins with $x_0 = 1.2919$, $\varphi_0 = 0$, and φ diminishes. When $x_0 \geq 1.2919$, $\varphi_1 - \pi \geq 0$. That is to say if the body starts too far from Jove for periodicity, it will pass away in an orbit as a superior planet; if on the other hand it starts too near Jove for periodicity, it will pass to some orbit about Jove. This orbit is very unstable.

Plate, fig. 5. $C = 38.0$.

The planet *A* starts from $x_0 = -.455$, $\varphi_0 = \pi$, and φ increases. When $x_0 \leq -.455$, $\varphi_1 \leq 0$, $x_2 \geq x_0$. The orbit is unstable, with uneven instability, and

$$c = 1 + .193 \sqrt{-1}.$$

The satellite *A* starts from $x_0 = 1.1305$, $\varphi_0 = 0$, and φ increases. When $x_0 \geq 1.1305$, $\varphi_1 \leq 0$. The remarks concerning this orbit in previous cases apply again here.

At the point where the orbit crosses the axis of *x* for the second time $\pi - \varphi$ is less than it was in the preceding case.

The oscillating satellite *a* starts from $x_0 = .6760$, $\varphi_0 = 0$ and φ decreases. When $x_0 \geq .6760$, $\varphi_1 - \pi \geq 0$. It is very unstable, with even instability.

The satellite *B* starts from $x_0 = 1.1470$, $\varphi_0 = 0$, and φ increases. When $x_0 \geq 1.1470$, $\varphi_1 - \pi \geq 0$, $x_2 \geq x_0$. The orbit is very unstable with even instability, and $c = .96 \sqrt{-1}$.

The orbit *B* is rapidly approaching to a close resemblance to part of the orbit *A*, for the crossing point of the figure-of-8 in *A* is tending to become nearly perpendicular to *SJ*, and the two curves nearly coincide.

The satellite *C* starts from $x_0 = 1.2480$, $\varphi_0 = 0$, and φ increases. When $x_0 \geq 1.2480$, $\varphi_1 - \pi \leq 0$, $x_2 \leq x_0$.

This orbit was very troublesome, and is not computed with a high degree of accuracy. A very small variation of *C* would make a large change in the size of the loops in the curve.

The orbit is very unstable with uneven instability.

The oscillating satellite *b* starts with $x_0 = 1.2595$, $\varphi_0 = 0$, and φ decreases. When $x_0 \geq 1.2595$, $\varphi_1 - \pi \geq 0$. The remarks made concerning this curve for *C* = 38.5 apply again here.

This orbit is very unstable.

The orbit *C* seems to be gaining a close resemblance, in part of its course, with the loop *b*.

§ 17.

Classification of orbits by families.

Several orbits are given in this classification which were not included in § 16.

Table of results.

Constant of Energy	Coord. of starting point	Synodic Period	Criterion of Stability	Apparent advance of pericentre in synodic period	Regression of pericentre in sid. period	Description of instability	Modulus of instability	Remarks
C	x_0	nT	$d \sin^2 \frac{1}{2} \pi \sqrt{E_0}$	$2\pi \left(\frac{1}{2} e - 1 \right)$	$2\pi \left(1 - \frac{\frac{1}{2} e}{1 + \frac{nT}{2\pi}} \right)$		$\frac{\log \sqrt{E}}{\log [D + \sqrt{D^2 + 1}]}$	
Satellite A, Plate, fig. 8.								
40.5	1.1185	61°20'	+ .112	39° 0'	22° 20'			{ minimum of criterion maximum of x_0
40.25	1.1150	65°40'	+ .063	29° 0'	31° 0'			
40.2	1.1090	66°50'	+ .064	29° 10'	31° 40'			minimum of x_0
40.0	1.0934	98° 0'	+ .226	308°	- 161°			
39.5	1.0650	229°	- ?			even	?	Figure-of-8 begins
39.0	1.0941	240°	- 1.06			even	0.5	
38.5	1.1164	258°	- ?			even	?	
38.0	1.1305	299°	- ?			even	?	
Satellite B, Plate, fig. 9.								
39.3	1.1575	87°40'	- .061			even	1.42	
39.0	1.1500	97° 0'	- .402			even	0.58	
38.5	1.1497	113°20'	- 1.82			even	0.31	
38.0	1.1470	131°50'	- 4.5			even	0.23	
Satellite C, Plate, fig. 10.								
39.3	1.1751	89°20'	+ .064	81° 0'	24° 30'			maximum of x_0
39.0	1.2338	114° 0'	+ .435	82° 40'	28° 30'			
38.75	1.2873	179°50'	+ 1.95			uneven	0.4	
38.5	1.2760	210°50'	> + 1			uneven	?	
38.0	1.2480	255°20'	> + 1			uneven	?	

Constant of Energy	Coord. of starting point	Synodic Period	Criterion of Stability	Apparent advance of pericentre in synodic period	Regression of pericentre in sid. period	Description of instability	Modulus of instability	Remarks
C	x_0	nT	$\delta \sin^2 \frac{1}{2} \alpha \sqrt{\Phi_0}$	$2\pi \left(\frac{1}{2} \epsilon - 1 \right)$	$2\pi \left(1 - \frac{1}{2} \frac{\epsilon}{1 + \frac{nT}{2\pi}} \right)$		$\frac{10\pi \sqrt{A}}{\log(D + \sqrt{D^2 + 1})}$	
Oscillating Satellite a.								
40.18	.7175		— ?			even	?	a point on SJ
40.0	.705	138°	— ?			even	?	
39.5	.693	?	— ?			even	?	
39.0	.687	146°	— 148			even	0.1	
38.5	.681	150°	— ?			even	?	
38.0	.676		— ?			even	?	
Oscillating Satellite b.								
38.88	1.3470		?			?	?	
38.5	1.2919	214°?	?			?	?	a point on SJ
38.0	1.2595	208°?	?			?	?	
Planet A, Plate, fig. 11.								
40.0	— .414	154°	+ .91	145°	5°30'			
39.5	— .424	165°	+ .98	162°	2°			
39.0	— .434	177°	+ 1.03			uneven	2.1	
38.5	— .444	191°	+ 1.08			uneven	1.25	
38.0	— .455	207°	+ 1.09			uneven	1.14	

Although the above table gives most of the facts, it will be well to draw attention to a few important points.

The passage of the family A of satellites into the figure-of-8 form is interesting. When my paper was published, I imagined that the oval A might advance until it touched the oval a , and that a figure-of-8 would then ensue. But Mr Hough has pointed out to me that two orbits can never touch — as should have been obvious to me. He has shown me arguments which lead us to believe that the genesis of the figure-of-8 is only explicable through the intervention of retrograde orbits*). There is undoubtedly at this point something wanting in my synopsis of orbits, and it will unfortunately need much computation to make good the omission. It seems probable that my lettering of the orbits by families will need to be amended.

I think it is certain that a more complex sort of figure-of-8 also exists, for we may imagine a body which describes two, three or more circuits round the point of zero force in an oval like a , before passing off into the branch round Jove. However these orbits can hardly be described as simply periodic, and I have not considered them in detail.

It appears from our table that the satellite orbit A is stable, but with only a very small margin of stability when $C = 40$. It is worthy of note that the criterion of stability after passing a minimum value of $\cdot 063$, is rapidly increasing, so that the orbit is tending towards uneven instability. I do not know whether or not that instability has set in before the formation of the figure-of-8 orbit A .

Mr Hill has drawn an interesting family of orbits of satellites, beginning with the orbit of the moon and ending with a cusped orbit. Now our moon undoubtedly belongs to the family A , whilst the cusped and looped orbits belong to the family C . He neglects the solar parallax, and this approximation has in fact led to the absorption of two families into one another. It appears now that it is not possible to comprehend the part played by this class of orbit without the inclusion of the solar parallax, for the asymmetry of the family C with regard to the line of quadratures is an essential feature in it.

Mr Hill draws attention to the minimum of distance at syzygies in the orbits of satellites, and this is observable in our family C , but we also find a maximum of distance in the family A at the superior syzygy.

*) M. Poincaré has, on other grounds, arrived at the conclusion that the A orbits and the figure-of-8 orbits are not algebraically continuous. See *Méc. Céle. T. III*, p. 354.

The periods of some of the satellites are extraordinarily long, that of the last figure-of-8 *A* being $\frac{299}{360}$ or $\frac{5}{6}$ of that of Jove, and that of the last looped orbit *C* being $\frac{235}{360}$ or nearly $\frac{2}{3}$ of that of Jove

§ 18.

On the probable forms of periodic orbits for values of *C* less than 38.

It is obvious from fig. 5 in the Plate that a portion of the figure-of-8 orbit *A* and the orbit *B* are tending to a closer and closer resemblance. The same difficulties that present themselves, in respect to the genesis of the figure-of-8, are here repeated. I believe however that the orbit *B* and the figure-of-8 *A* are both on the point of disappearance, but that the oval *a* will continue to exist and to expand.

The planetary orbit *A* will continue to expand, but the heliocentric distance at the conjunction remote from *J* will shortly reach a maximum and will then diminish, whilst the heliocentric distance at the other conjunction will increase rapidly. This will continue until the planetary orbit *A* has advanced to near the oval *a*; a new series of figure-of-8 planetary orbits will then arise, and the heliocentric distance at the remote conjunction will then increase.

At some stage a pair of new planetary orbits *B* and *C* will arise from a single orbit; of these *B* will be evenly unstable and *C* stable.

The orbit *B* will expand, approach to close resemblance with a portion of *A*, and then both will disappear.

Reverting now to the satellite *C*, we are able to conjecture its future course. The figures 4 and 5, or fig. 10, in the Plate, show the growth of the two loops from two cusps. In order to throw light on the future development of these curves, I have drawn Plate, fig. 2, which shows a non-periodic orbit for $C = 38.5^*$; in it we see that the upper loop has descended below the line of conjunction, and the lower loop has risen above. For some value of *C* a little less than 38 there must be a periodic orbit of this general form. We shall thus have a periodic orbit with five full moons in the month. In this sort of orbit the crossing point *P* will be at first a point of contact; the distance *JP* will then diminish to a minimum and afterwards increase. It seems possible that *P* may move outwards and *Q* inwards, so as to meet; the upper loop will perhaps then have spread so as to nearly coincide with the lower, and the lower with the upper, and both will closely resemble the oval *b*. I think that

*) It would have been better to have drawn the similar curve for $C = 38.0$, but this one suffices for the present purpose.

after this stage the orbit C will disappear, but the manner of the disappearance remains obscure; the oval b will continue to exist. It is probable that the discussion of retrograde orbits will be necessary for the elucidation of this point.

The form of this orbit C is interesting when taken in connection with the looped orbit to which M. Poincaré*) drew attention, and which has been traced by Lord Kelvin**). They both neglected the solar parallax, and with the degree of approximation adopted by them, the central space might be made as small and the loops as large as we like. But the inclusion of the solar parallax now appears to be essential to the proper consideration of these orbits.

It appears from fig. 1 that when $C = 34.91$, there is a new periodic orbit consisting of the point $x = -.9469$, $y = 0$. This point is the origin of a new family of oscillating planets, say c , which describe ovals with retrograde revolution round the point of zero force, for values of C less than 34.91.

Turning now to our conjectural planetary orbit C , we see that whilst initially it will be nearly circular, it will ultimately produce two excrescences near the ends of the oval c . These excrescences will become cusps, and then loops; the loops will cross one another, become nearly identical with one another and nearly coincident with the oval c , and the orbit C will probably then disappear.

The case of the superior planets has not yet been considered, and there is not much concerning them of which I feel confident***). It is obvious however that they are described with an apparently retrograde revolution, and that they contract as C falls in value. The orbits will be nearly circular, but will bulge inwards in the neighbourhood of Jove. At some stage the inward depression of the orbit will draw near to the oval b . At this stage there will probably arise a new family of orbits, having the form of a sort of inverted figure-of-8. If the old figure-of-8 be likened to two circles touching one another externally, the new figure may be compared with a small circle touching a large one internally. A similar series of changes must ultimately take place with the oval c , and probably we may have an orbit with loops at both ends of the line of conjunctions.

I will not hazard detailed conjectures as to the future of the three ovals a , b , c . I think however that it is probable that they will stretch out towards the vertices of the two equilateral triangles

*) *Méc. Céle.*, p. 109.

**) *Phil. Mag.*, Nov. 1892.

***) I have now (1898) traced some of them.

which may be erected on SJ as base. These vertices must be themselves the origins of a pair of similar ovals, and perhaps the extremities of a , b , c will stretch out towards this fourth system of ovals.

§ 19.

Classification of stable orbits of satellites.

We have seen that amongst satellites there are two classes of stable orbits, namely those of the A and C families. The Plate, fig. 7 exhibits the limits of the orbits which have been shown to be stable. The exact orbits which possess limiting stability would of course differ slightly from those drawn in this figure.

When C is large the stable orbits of the A family are approximate circles of small radius. As C decreases the orbits swell, but when C reaches 40.25 the radius vector at superior syzygy reaches a maximum. Hence the orbit

$$x_0 = 1.1150, \quad C = 40.25$$

gives one limit of the stable orbits of this family. The orbit

$$x_0 = 1.0334, \quad C = 40.0$$

gives approximately another limit as regards the inferior syzygy. The shaded space between these two orbits is filled with stable orbits.

The stable orbits of the C family begin when C is a little greater than 39.3, and the first one traced is that for which

$$x_0 = 1.1751, \quad \text{and } C = 39.3.$$

The stability of these orbits still subsists when $C = 39.0$, but this orbit is already very unstable when C has fallen to 38.75. Accordingly I take for the other limit of orbits of this kind

$$x_0 = 1.2338, \quad C = 39.0.$$

The shaded space between these two is filled with stable orbits.

It will be observed that there remains an unshaded tract within which no stable orbit can exist. I think moreover that it is probable that with a smaller mass for Jove we should have found a complete annulus within which stability is impossible.

This conclusion is interesting when viewed in connection with the distribution of the satellites and planets of our system, and it appears to me to be the first exact result, which throws any light on Bode's empirical law as to the mean distances of planets and satellites from their primaries.

It is as yet too soon to make a similar classification of stable planetary orbits, but this will follow in due course.

We have seen in an earlier section that unstable orbits are such as ultimately to lead to the absorption of bodies moving in them

into one or other of the perturbing centres. If there were a large number of perturbing centres, as in our planetary system, the problem would become incomparably more difficult, but I think that the present investigation affords evidence that if we were to have a system consisting of a large planet moving round the sun, and of a cloud of infinitesimal bodies circling about them, a system would ultimately be evolved where there would be inferior and superior planets and a pair of satellites moving in certain zones indicated by our figures.

In the paper in the *Acta Mathematica* the numerical results, which are exhibited here only in the form of figures, are given in an appendix.