This relation will help to reduce the unsymmetric equation (1) to the symmetrical form

$$4q \frac{dX_3}{dq} + \frac{d^2 X_3}{du dv} = - \frac{\{S'_1(0)\}^2 S'_1(u+v)}{S_2^2(u) S_2^2(v)} \dots\dots\dots (3).$$

Moreover, eliminating $\frac{d}{dv}$ by (2), we get

$$4q \frac{dX_3}{dq} - \frac{d^2 X_3}{du^2} = -2 \frac{\{S'_1(0)\}^2}{S_2(u) S_2(v)} \frac{d}{du} \frac{S_1(u+v)}{S_2(u)}.$$

When $v = 0$, this becomes

$$\left(4q \frac{d}{dq} - \frac{d^2}{du^2}\right) \frac{\Lambda_3(u)}{S_2(0) S_2(u)} = -2 \{S'_1(0)\}^2 \frac{S_1(u) S_2(u)}{S_2^3(u)}.$$

In a similar manner, we obtain

$$\left(4q \frac{d}{dq} - \frac{d^2}{du^2}\right) \frac{\Lambda_2(u)}{S_2(0) S_2(u)} = -2 \{S'_1(0)\}^2 \frac{S_1(u) S_2(u)}{S_2^3(u)}.$$

The Electrical Distribution on a Conductor bounded by Two Spherical Surfaces cutting at any Angle. By H. M. MACDONALD. Read January 10th, 1895. Received January 17th, 1895.

In Maxwell's *Electricity and Magnetism*, Vol. I., §§ 165, 166, the problem of the distribution of electricity induced by an electrified point placed between them on two planes cutting at an angle which is a submultiple of two right angles, and the inverse problem of the conductor formed by two spherical surfaces cutting at such an angle (the angle referring to the dielectric), are solved by the method of point images. This method is inapplicable when the (dielectric) angle is not a submultiple of two right angles, as has been shown by W. D. Niven, *Proc. Lond. Math. Soc.*, Vol. XXI., p. 27. The only other case which has been hitherto solved is, as far as I know, that of the spherical bowl (Lord Kelvin, *Papers on Electrostatics and Magnetism*, p. 178). In the paper by W. D. Niven mentioned above

an attempt is made to deduce the capacity of such a conductor from the solution of a functional equation for a particular value of one of the variables, but the result obtained does not seem in the case of the spherical bowl to agree with Lord Kelvin's. The results obtained hereafter also differ from those given by Niven.

The object of this paper is to obtain the solution in the general case. To effect this the functional image of a point placed between two planes intersecting at any angle is obtained in the form of a definite integral, § 1. In the next few paragraphs the reduction of this integral to known forms is effected in certain cases, and it is shown that the integration can be performed when the angle of intersection is any submultiple of four right angles; the case in which it is reducible to elliptic functions is also discussed.

In § 5 the functional image of a line of uniform density parallel to the intersection of the planes is deduced. In § 6 a certain definite integral is discussed shortly. In § 7 the potential due to a freely charged conductor bounded by two spherical surfaces cutting at any angle is obtained, and some particular cases discussed. The capacity of such a conductor is obtained in § 8 in finite terms, and some particular cases are discussed in § 9; one of the most interesting of these is the capacity of a hemisphere, which is found to be nearly seventeen-twentieths that of the complete sphere. Some cases are mentioned in the last paragraph which could be deduced from the results of the preceding ones.

1. *The Functional Image of a Point placed between Two Planes intersecting at any Angle.*

To obtain the potential produced by an electrified point situated in the space between two conducting planes intersecting at any angle. Consider a prism whose cross-section is a sector of a circle radius b and angle α , and having its ends perpendicular to the axis of the prism, distance h from each other. Take the origin of coordinates at the centre of the circle the sector of which forms one of the ends of the prism; then, using cylindrical coordinates, V the potential at any point inside the prism satisfies the equation

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} + \frac{1}{r^2} \frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} + 4\pi\rho = 0 \quad \dots\dots\dots (1),$$

where ρ is the density at the point r, θ, z , and $V = 0$ all over the

boundary. Assume

$$V = \Sigma W_n \sin \frac{n\pi\theta}{a},$$

where n is an integer, for $V = 0$ when $\theta = 0$ or $\theta = a$; then (1) becomes

$$\Sigma \left\{ \frac{\partial^2 W_n}{\partial r^2} + \frac{1}{r} \frac{\partial W_n}{\partial r} - \frac{n^2 \pi^2}{a^2 r^2} W_n + \frac{\partial^2 W_n}{\partial z^2} \right\} \sin \frac{n\pi\theta}{a} + \pi p = 0 \dots (2).$$

Put

$$W_n = \Sigma J_{n\pi/a}(\kappa r) U_n,$$

where the condition that $V = 0$, when $r = b$, makes κ a root of

$$J_{n\pi/a}(\kappa b) = 0,$$

and the summation is taken for all the roots of this equation; then (2) becomes

$$\Sigma \Sigma \left\{ \frac{\partial^2 U_n}{\partial z^2} - \kappa^2 U_n \right\} J_{n\pi/a}(\kappa r) \sin \frac{n\pi\theta}{a} + 4\pi p = 0 \dots \dots \dots (3).$$

Hence
$$4\pi p = \Sigma \Sigma \Sigma A_{\kappa \lambda n} (\lambda^2 + \kappa^2) J_{n\pi/a}(\kappa r) \sin \frac{n\pi\theta}{a} \sin \lambda z \dots \dots \dots (4),$$

where, since $V = 0$, when $z = 0$ and $z = h$, $\lambda = \frac{m\pi}{h}$, m being an integer. From (4), by an application of Fourier's theorem,

$$4\pi \int_0^h \rho' \sin \lambda z' dz' = \frac{h}{2} \Sigma \Sigma A_{\kappa \lambda n} (\lambda^2 + \kappa^2) J_{n\pi/a}(\kappa r) \sin \frac{n\pi\theta}{a},$$

whence

$$4\pi \int_0^a \int_0^h \rho' \sin \lambda z' \sin \frac{n\pi\theta'}{a} d\theta' dz' = \frac{ah}{4} \Sigma A_{\kappa \lambda n} (\lambda^2 + \kappa^2) J_{n\pi/a}(\kappa r).$$

Multiplying both sides by $r' J_{n\pi/a}(\kappa r')$, and integrating from 0 to b ,

$$\begin{aligned} 4\pi \int_0^b \int_0^a \int_0^h \rho' \sin \lambda z' \sin \frac{n\pi\theta'}{a} J_{n\pi/a}(\kappa r') r' dr' d\theta' dz' \\ = \frac{ahb^2}{8} J_{n\pi/a}^2(\kappa b) A_{\kappa \lambda n} (\lambda^2 + \kappa^2); \end{aligned}$$

therefore, for a charge q at the point r', θ', z' ,

$$\frac{ahb^2}{8} J_{n\pi/a}^2(\kappa b) A_{\kappa \lambda n} (\lambda^2 + \kappa^2) = 4\pi q \sin \lambda z' \sin \frac{n\pi\theta'}{a} J_{n\pi/a}(\kappa r'),$$

and
$$V = \frac{32\pi q}{ahb^2} \Sigma \Sigma \Sigma \frac{\sin \lambda z' \sin \frac{n\pi\theta'}{a} J_{n\pi/a}(\kappa r')}{(\lambda^2 + \kappa^2) J_{n\pi/a}^2(\kappa b)} \sin \lambda z \sin \frac{n\pi\theta}{a} J_{n\pi/a}(\kappa r).$$

Now,
$$\frac{1}{2\kappa} + \Sigma \frac{\kappa \cos \lambda \zeta}{\lambda^2 + \kappa^2} = \frac{h}{2} \frac{\cosh \kappa (h - \zeta)}{\sinh \kappa h},$$

if $2h > \zeta > 0$; therefore

$$2\Sigma \frac{\sin \lambda z \sin \lambda z'}{\lambda^2 + \kappa^2} = \frac{h}{2} \frac{\cosh \kappa (h - \overline{z - z'}) - \cosh \kappa (h - \overline{z + z'})}{\kappa \sinh \kappa h},$$

when $z > z'$. Hence

$$V = \frac{8\pi q}{ab^3} \Sigma \Sigma \frac{\sin \frac{n\pi\theta'}{a} J_{n\pi/a}(\kappa r')}{J_{n\pi/a}'(\kappa b)} \frac{\cosh \kappa (h - \overline{z - z'}) - \cosh \kappa (h - \overline{z + z'})}{\kappa \sinh \kappa h} \\ \times \sin \frac{n\pi\theta}{a} J_{n\pi/a}(\kappa r).$$

Change the origin to the point z' on the axis, and make both ends move off to an infinite distance; then the potential, due to an electrified point q at the point r' , θ' , of the plane $z = 0$, at any point of the space bounded by the infinitely long cylindrical surface, whose cross-section is a sector radius b and angle a , is given by

$$V = \frac{8\pi q}{ab^3} \Sigma \Sigma \frac{\sin \frac{n\pi\theta'}{a} J_{n\pi/a}(\kappa r')}{\kappa J_{n\pi/a}'(\kappa b)} e^{-\kappa z} \sin \frac{n\pi\theta}{a} J_{n\pi/a}(\kappa r),$$

when z is positive; when z is negative, $e^{-\kappa z}$ has to be written for $e^{+\kappa z}$.

To obtain V when b is made indefinitely great it is necessary to evaluate $b^3 J_{n\pi/a}'(\kappa b)$, subject to $J_{n\pi/a}(\kappa b) = 0$, when b is infinite;

$$J_{n\pi/a}(\kappa b) = \sqrt{\frac{2}{\pi \kappa b}} \cos \left\{ \frac{n\pi^2}{2a} + \frac{\pi}{4} - \kappa b \right\},$$

when b is very great;

$$J_{n\pi/a}'(\kappa b) = \sqrt{\frac{2}{\pi \kappa b}} \sin \left\{ \frac{n\pi^2}{2a} + \frac{\pi}{4} - \kappa b \right\},$$

when $J_{n\pi/a}(\kappa b) = 0$ and b is infinite.

If κ and $\kappa + \delta\kappa$ are consecutive roots of

$$J_{n\pi/a}(\kappa b) = 0,$$

$$b \delta\kappa = \pi;$$

therefore

$$b^3 J_{n\pi/a}'(\kappa b) = \frac{2b}{\pi \kappa} = \frac{2}{\kappa \delta\kappa}.$$

Hence the potential at the point r, θ, z of the dielectric bounded two conducting infinite planes intersecting at an angle α , due to charge q at the point $r', \theta', 0$, is given by

$$V = \frac{4\pi q}{\alpha} \sum_{n=1}^{\infty} \int_0^{\infty} e^{-nz} d\kappa \sin \frac{n\pi\theta'}{\alpha} \sin \frac{n\pi\theta}{\alpha} J_{n\pi/\alpha}(\kappa r') J_{n\pi/\alpha}(\kappa r),$$

when z is positive, and by

$$V = \frac{4\pi q}{\alpha} \sum_{n=1}^{\infty} \int_0^{\infty} e^{nz} d\kappa \sin \frac{n\pi\theta'}{\alpha} \sin \frac{n\pi\theta}{\alpha} J_{n\pi/\alpha}(\kappa r') J_{n\pi/\alpha}(\kappa r),$$

when z is negative.

It has been proved by Sonnine, *Math. Ann.*, xvi., that

$$\int_0^{\infty} J_{n\pi/\alpha}(\kappa r) J_{n\pi/\alpha}(\kappa r') d\kappa = \frac{2}{\pi (rr')^{n\pi/\alpha}} \int_0^{r'} \frac{x^{2n\pi/\alpha} dx}{\sqrt{(r^2 - x^2)(r'^2 - x^2)}},$$

where $r > r'$.

This latter integral can be transformed into

$$\frac{1}{\pi \sqrt{2rr'}} \int_0^{\infty} \frac{e^{-(n\pi\zeta)/\alpha} d\zeta}{\sqrt{\cosh \zeta - \cosh \eta}},$$

for all values of r and r' , where η is to be taken positive, and

$$2 \cosh \eta = \frac{r^2 + r'^2}{rr'}.$$

Hence, when $z = 0$, the potential is given by

$$\begin{aligned} V &= \frac{4q}{\alpha \sqrt{2rr'}} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{e^{-(n\pi\zeta)/\alpha} d\zeta}{\sqrt{\cosh \zeta - \cosh \eta}} \sin \frac{n\pi\theta}{\alpha} \sin \frac{n\pi\theta'}{\alpha}, \\ V &= \frac{2q}{\alpha \sqrt{2rr'}} \int_0^{\infty} \frac{d\zeta}{\sqrt{\cosh \zeta - \cosh \eta}} \\ &\quad \times \sum_{n=1}^{\infty} e^{-(n\pi\zeta)/\alpha} \left\{ \cos \frac{n\pi}{\alpha} (\theta - \theta') - \cos \frac{n\pi}{\alpha} (\theta + \theta') \right\} \end{aligned}$$

Therefore

$$\begin{aligned} V &= \frac{q}{\alpha \sqrt{2rr'}} \int_0^{\infty} \frac{d\zeta}{\sqrt{\cosh \zeta - \cosh \eta}} \\ &\quad \times \left\{ \frac{\sinh \frac{\pi\zeta}{\alpha}}{\cosh \frac{\pi\zeta}{\alpha} - \cos \frac{\pi(\theta - \theta')}{\alpha}} - \frac{\sinh \frac{\pi\zeta}{\alpha}}{\cosh \frac{\pi\zeta}{\alpha} - \cos \frac{\pi(\theta + \theta')}{\alpha}} \right\} \end{aligned}$$

This may be extended to any point by making η satisfy

$$2 \cosh \eta = \frac{r^2 + r'^2 + z^2}{rr'}.$$

2. *Form of the Function when a is a Submultiple of π .*

When π/a is an integer, the expression for V may be reduced to the known form and expressed as the sum of the reciprocals of the distances from a series of points. Let $\pi/a = p$; then, when p is an integer,

$$\frac{p \sinh p\zeta}{\cosh p\zeta - \cos p\phi} = \sum_{k=0}^{k=p-1} \frac{\sinh \zeta}{\cosh \zeta - \cos \left(\phi + \frac{2k\pi}{p} \right)},$$

and
$$\int_0^\infty \frac{d\zeta}{\sqrt{\cosh \zeta - \cosh \eta}} \frac{\sinh \zeta}{\cosh \zeta - \cos \phi} = \frac{\pi}{\sqrt{\cosh \eta - \cos \phi}};$$

hence

$$V = \frac{q}{\sqrt{2rr'}} \sum_{k=0}^{k=p-1} \left\{ \frac{1}{\sqrt{\cosh \eta - \cos \left(\theta - \theta' + \frac{2k\pi}{p} \right)}} - \frac{1}{\sqrt{\cosh \eta - \cos \left(\theta + \theta' + \frac{2k\pi}{p} \right)}} \right\}$$

that is,

$$V = q \sum_{k=0}^{k=p-1} \left\{ \frac{1}{\sqrt{r^2 + r'^2 + z^2 - 2rr' \cos \left(\theta - \theta' + \frac{2k\pi}{p} \right)}} - \frac{1}{\sqrt{r^2 + r'^2 + z^2 - 2rr' \cos \left(\theta + \theta' + \frac{2k\pi}{p} \right)}} \right\},$$

which is the expression obtained by the method of images.

3. *Form of the Function when a is an Odd Submultiple of 2π .*

When

$$a = \frac{2\pi}{2m+1},$$

m being a positive integer, the potential can be expressed as the sum

of a finite number of terms involving arctangents; for then

$$V = \frac{2m+1}{2\pi} \frac{q}{\sqrt{2rr'}} \int_{\eta}^{\infty} \frac{d\zeta}{\sqrt{\cosh \zeta - \cosh \eta}}$$

$$\times \left\{ \frac{\sinh \frac{2m+1}{2} \zeta}{\cosh \frac{2m+1}{2} \zeta - \cos \frac{2m+1}{2} (\theta - \theta')} - \frac{\sinh \frac{2m+1}{2} \zeta}{\cosh \frac{2m+1}{2} \zeta - \cos \frac{2m+1}{2} (\theta + \theta')} \right\},$$

that is,

$$V = \frac{q}{2\pi \sqrt{2rr'}} \sum_{k=0}^{k=2m} \int_{\eta}^{\infty} \frac{d\zeta}{\sqrt{\cosh \zeta - \cosh \eta}}$$

$$\times \left\{ \frac{\sinh \frac{\zeta}{2}}{\cosh \frac{\zeta}{2} - \cos \left(\frac{\theta - \theta'}{2} + \frac{2k\pi}{2m+1} \right)} - \frac{\sinh \frac{\zeta}{2}}{\cosh \frac{\zeta}{2} - \cos \left(\frac{\theta + \theta'}{2} + \frac{2k\pi}{2m+1} \right)} \right\},$$

whence

$$V = \frac{2q}{\pi \sqrt{2rr'}} \sum_{k=0}^{k=2m}$$

$$\times \left\{ \frac{1}{\sqrt{\cosh \eta - \cos \left(\theta - \theta' + \frac{4k\pi}{2m+1} \right)}} \tan^{-1} \sqrt{\frac{\cosh \frac{\eta}{2} + \cos \left(\frac{\theta - \theta'}{2} + \frac{2k\pi}{2m+1} \right)}{\cosh \frac{\eta}{2} - \cos \left(\frac{\theta - \theta'}{2} + \frac{2k\pi}{2m+1} \right)}} \right.$$

$$\left. - \frac{1}{\sqrt{\cosh \eta - \cos \left(\theta + \theta' + \frac{4k\pi}{2m+1} \right)}} \tan^{-1} \sqrt{\frac{\cosh \frac{\eta}{2} + \cos \left(\frac{\theta + \theta'}{2} + \frac{2k\pi}{2m+1} \right)}{\cosh \frac{\eta}{2} - \cos \left(\frac{\theta + \theta'}{2} + \frac{2k\pi}{2m+1} \right)}} \right\}.$$

This includes as a particular case half an infinite plane under the action of an electrified point; here $m = 0$, and therefore

$$V = \frac{2q}{\pi \sqrt{2rr'}} \left\{ \frac{1}{\sqrt{\cosh \eta - \cos (\theta - \theta')}} \tan^{-1} \sqrt{\frac{\cosh \frac{\eta}{2} + \cos \frac{\theta - \theta'}{2}}{\cosh \frac{\eta}{2} - \cos \frac{\theta - \theta'}{2}}} \right.$$

$$\left. - \frac{1}{\sqrt{\cosh \eta - \cos (\theta + \theta')}} \tan^{-1} \sqrt{\frac{\cosh \frac{\eta}{2} + \cos \frac{\theta + \theta'}{2}}{\cosh \frac{\eta}{2} - \cos \frac{\theta + \theta'}{2}}} \right\},$$

or, in terms of the cylindrical coordinates,

$$= \frac{2q}{\pi} \left\{ \frac{\tan^{-1} \sqrt{\left(\sqrt{r+r'^2+z^2} + 2\sqrt{rr'} \cos \frac{\theta-\theta'}{2} \right) / \left(\sqrt{r+r'^2+z^2} - 2\sqrt{rr'} \cos \frac{\theta-\theta'}{2} \right)}}{\sqrt{r^2+r'^2+z^2-2rr' \cos(\theta-\theta')}}} \right. \\ \left. - \frac{\tan^{-1} \sqrt{\left(\sqrt{r+r'^2+z^2} + 2\sqrt{rr'} \cos \frac{\theta+\theta'}{2} \right) / \left(\sqrt{r+r'^2+z^2} - 2\sqrt{rr'} \cos \frac{\theta+\theta'}{2} \right)}}{\sqrt{r^2+r'^2+z^2-2rr' \cos(\theta+\theta')}}} \right\}$$

Thus the potential can be expressed in terms of algebraic and circular functions when α is a submultiple of 2π .

4. Form of the Function in other Cases.

In all other cases the reduction of the expression for the potential involves transcendental functions: *e.g.*, when

$$\alpha = \frac{3\pi}{3m \pm 1},$$

it can be expressed in terms of complete elliptic integrals of the first and third kinds; for then, as above,

$$V = \frac{q}{3\pi\sqrt{2rr'}} \sum_{k=0}^{k=3m \text{ or } 3m-2} \int_{\eta}^{\infty} \frac{d\zeta}{\sqrt{\cosh \zeta - \cosh \eta}} \\ \times \left\{ \frac{\sinh \frac{\zeta}{3}}{\cosh \frac{\zeta}{3} - \cos \left(\frac{\theta-\theta'}{3} + \frac{2k\pi}{3m \pm 1} \right)} - \frac{\sinh \frac{\zeta}{3}}{\cosh \frac{\zeta}{3} - \cos \left(\frac{\theta+\theta'}{3} + \frac{2k\pi}{3m \pm 1} \right)} \right\}.$$

$$\text{Now,} \quad \frac{1}{3} \int_{\eta}^{\infty} \frac{d\zeta}{\sqrt{\cosh \zeta - \cosh \eta}} \frac{\sinh \frac{\zeta}{3}}{\cosh \frac{\zeta}{3} - \gamma} \\ = \frac{1}{3} \int_{x_0}^{\infty} \frac{dx}{(x-\gamma) \sqrt{(x-x_0)(x^2+xx_0+x_0^2-\frac{9}{4})}} \\ = 2\sqrt{\lambda} \frac{\lambda+x_0-\gamma}{\lambda-x_0+\gamma} \int_0^K \frac{du}{4\lambda(x_0-\gamma) + (\lambda-x_0+\gamma)^2 \sin^2 u} - \frac{K}{\sqrt{\lambda}(\lambda-x_0+\gamma)},$$

$$\text{where} \quad 4\lambda^2 = 12x_0^2 - 3, \quad 4\lambda k^2 = 2\lambda - 3x_0,$$

$$\text{and} \quad x_0 = \cosh \frac{\eta}{3}.$$

Hence

$$V = \frac{q}{\pi \sqrt{2rr'}} \sum_{k=0}^{\lambda=3m \text{ or } 3m-2} \left\{ 2\sqrt{\lambda} \frac{\lambda+x_0-\gamma_k}{\lambda-x_0+\gamma_k} \int_0^K \frac{du}{4\lambda(x_0-\gamma_k) + (\lambda-x_0+\gamma_k)^2 \operatorname{sn}^2 u} \right. \\ \left. - 2\sqrt{\lambda} \frac{\lambda+x_0-\gamma'_k}{\lambda-x_0+\gamma'_k} \int_0^K \frac{du}{4\lambda(x_0-\gamma'_k) + (\lambda-x_0+\gamma'_k)^2 \operatorname{sn}^2 u} \right. \\ \left. - \frac{K}{\sqrt{\lambda}} \left(\frac{1}{\lambda-x_0+\gamma_k} - \frac{1}{\lambda-x_0+\gamma'_k} \right) \right\},$$

where

$$\gamma_k = \cos \left(\frac{\theta - \theta'}{3} + \frac{2k\pi}{3m \pm 1} \right),$$

and

$$\gamma'_k = \cos \left(\frac{\theta + \theta'}{3} + \frac{2k\pi}{3m \pm 1} \right).$$

This includes as a particular case a conductor bounded by two planes intersecting at right angles, the dielectric angle being three right angles, under the influence of an electrified point.

5. *The Image of a Line of Uniform Density placed between Two Planes Intersecting at any Angle, the Line being Parallel to the Intersection of the Planes.*

Let ρ be the density per unit length of the line; then the potential due to it by § 1 is given by

$$V = \frac{\rho}{a \sqrt{2rr'}} \int_{-\infty}^{\infty} dz \int_0^{\infty} \frac{d\zeta}{\sqrt{\cosh \zeta - \cosh \eta}} \\ \times \left\{ \frac{\sinh \frac{\pi \zeta}{a}}{\cosh \frac{\pi \zeta}{a} - \cos \frac{\pi}{a} (\theta - \theta')} - \frac{\sinh \frac{\pi \zeta}{a}}{\cosh \frac{\pi \zeta}{a} - \cos \frac{\pi}{a} (\theta + \theta')} \right\},$$

where

$$\cosh \eta = \frac{r^2 + r'^2 + z^2}{2rr'} = \cosh \eta_0 + \frac{z^2}{2rr'}.$$

Changing the order of integration,

$$V = \frac{\rho}{a \sqrt{rr'}} \int_0^{\infty} d\zeta \int_{\frac{\sqrt{2rr'}(\cosh \zeta - \cosh \eta_0)}}^{\sqrt{2rr'}(\cosh \zeta - \cosh \eta_0)} \frac{dz}{\sqrt{\cosh \zeta - \cosh \eta_0 - \frac{z^2}{2rr'}}} \\ \times \left\{ \frac{\sinh \frac{\pi \zeta}{a}}{\cosh \frac{\pi \zeta}{a} - \cos \frac{\pi}{a} (\theta - \theta')} - \frac{\sinh \frac{\pi \zeta}{a}}{\cosh \frac{\pi \zeta}{a} - \cos \frac{\pi}{a} (\theta + \theta')} \right\},$$

$$= \frac{\rho\pi}{a} \int_{\eta_0}^{\infty} d\zeta \left\{ \frac{\sinh \frac{\pi\zeta}{a}}{\cosh \frac{\pi\zeta}{a} - \cos \frac{\pi}{a} (\theta - \theta')} - \frac{\sinh \frac{\pi\zeta}{a}}{\cosh \frac{\pi\zeta}{a} - \cos \frac{\pi}{a} (\theta + \theta')} \right\};$$

Hence
$$V = \rho \log \frac{\cosh \frac{\pi\eta_0}{a} - \cos \frac{\pi}{a} (\theta + \theta')}{\cosh \frac{\pi\eta_0}{a} - \cos \frac{\pi}{a} (\theta - \theta')},$$

or
$$V = \rho \log \frac{\gamma^{2\pi/a} + \gamma'^{2\pi/a} - 2\gamma^{\pi/a} \gamma'^{\pi/a} \cos \frac{\pi}{a} (\theta + \theta')}{\gamma^{2\pi/a} + \gamma'^{2\pi/a} - 2\gamma^{\pi/a} \gamma'^{\pi/a} \cos \frac{\pi}{a} (\theta - \theta')}.$$

This shows that the result obtained by Greenhill, *Quarterly Journal of Mathematics*, Vol. xv., is true when a is not restricted to be a sub-multiple of π . It could be deduced from the form for V in § 1, involving Bessel functions, by the use of the equality

$$\int_0^{\infty} J_{n\pi/a}(\kappa r) J_{n\pi/a}(\kappa r') \frac{d\kappa}{\kappa} = \frac{a}{2n\pi} \left(\frac{r'}{r} \right)^{n\pi/a},$$

where $r > r'$ and $n > 0$,

which is easy to establish.

6. *The Integral*
$$\int_0^{\infty} \frac{e^{-(n\pi\zeta/a)} d\zeta}{\sqrt{\cosh \zeta - \cosh \eta}}.$$

It is easy to show that this integral is equal to $\sqrt{2} Q_{(n\pi/a)-\frac{1}{2}}(\cosh \eta)$, where $Q_{(n\pi/a)-\frac{1}{2}}$ is the zonal harmonic of the second kind and order $\frac{n\pi}{a} - \frac{1}{2}$, that is, a toroidal function of the second kind. The corresponding expression for $P_{(n\pi/a)-\frac{1}{2}}$ is

$$\frac{1}{\pi\sqrt{2}} \int_{-\eta}^{\eta} \frac{e^{n\pi\zeta/a} d\zeta}{\sqrt{\cosh \eta - \cosh \zeta}}.$$

The expressions for $P_{(n\pi/a)-\frac{1}{2}}(\cos \theta)$ and $Q_{(n\pi/a)-\frac{1}{2}}(\cos \theta)$ can be easily written down and are complex. It will be observed that these forms for the toroidal functions correspond exactly with the forms for Mehler's functions $P_{n\pi/a-\frac{1}{2}}$, &c., Heine, Vol. II., § 62. From § 1, it follows that a zonal harmonic of the second kind can be expressed as

an integral of the product of two Bessel functions of order $\frac{1}{2}$ higher, the relation being

$$Q_{n-\frac{1}{2}}(\cosh \eta) = \pi \sqrt{rr'} \int_0^\infty J_n(\kappa r) J_n(\kappa r') d\kappa,$$

where
$$\cosh \eta = \frac{r^2 + r'^2}{2rr'},$$

and n is any positive real quantity. This result may be applied to show that the addition theorem for Bessels of order $\frac{2n+1}{2}$ is the differential form of the expansion of $(\cosh \eta - \cos \theta)^{-1}$ in zonal harmonics.

7. Potential due to a Conductor formed by Two Intersecting Spheres.

The potential at any point, due to a conductor, charged to potential V_0 , whose bounding surface consists of two segments of spheres, standing on the same circle and intersecting at any angle α , can be obtained from the above by making q equal to $-V_0 r'$, and inverting with respect to the point r', θ' , the radius of inversion being r' . Take as coordinates in the inverted system η, ξ , where e^η is the ratio (greater than unity) of the distances of any point from the points in which the plane through the point and the centre of the circle of intersection of the segments perpendicular to its plane meets it, and ξ is the internal angle between these distances. Then

$$\xi = \theta - \theta',$$

the η of the new system is the η of the former, and, if R is the distance of any point in the plane $z = 0$ of the old system from r', θ' ,

$$R/\sqrt{2rr'} = \sqrt{\cosh \eta - \cos \xi}.$$

Hence the potential V at any point in the new system is given by

$$V = V_0 - \frac{V_0 \sqrt{\cosh \eta - \cos \xi}}{a} \int_0^\infty \frac{d\zeta}{\sqrt{\cosh \zeta - \cosh \eta}} \times \left\{ \frac{\sinh \frac{\pi \zeta}{a}}{\cosh \frac{\pi \zeta}{a} - \cos \frac{\pi \xi}{a}} - \frac{\sinh \frac{\pi \zeta}{a}}{\cosh \frac{\pi \zeta}{a} - \cos \frac{\pi}{a} (\xi + 2\beta)} \right\},$$

where the bounding surfaces of the conductor are

$$\xi = -\beta, \quad \xi = \alpha - \beta,$$

α being the dielectric angle. When π/α is an integer p , V is given by

$$V = V_0 - V_0 \sqrt{\cosh \eta - \cos \xi} \sum_{k=0}^{p-1} \left\{ \frac{1}{\sqrt{\cosh \eta - \cos \left(\xi + \frac{2k\pi}{p} \right)}} - \frac{1}{\sqrt{\cosh \eta - \cos \left(\xi + 2\beta + \frac{2k\pi}{p} \right)}} \right\},$$

or $V = V_0 \sqrt{\cosh \eta - \cos \xi} \sum_{k=0}^{p-1} \frac{1}{\sqrt{\cosh \eta - \cos \left(\xi + 2\beta + \frac{2k\pi}{p} \right)}} - V_0 \sqrt{\cosh \eta - \cos \xi} \sum_{k=1}^{p-1} \frac{1}{\sqrt{\cosh \eta - \cos \left(\xi + \frac{2k\pi}{p} \right)}},$

which is the potential due to a series of electrified points arranged along the line of centres of the spheres as in Maxwell, Vol. I., Art. 166.

When $\alpha = \frac{2\pi}{2m+1},$

using § 3, V is given by

$$V = V_0 - \frac{2V_0}{\pi} \times \sum_{k=0}^{m-1} \left\{ \frac{\sqrt{\cosh \eta - \cos \xi}}{\sqrt{\cosh \eta - \cos \left(\xi + \frac{4k\pi}{2m+1} \right)}} \tan^{-1} \sqrt{\frac{\cosh \frac{\eta}{2} + \cos \left(\frac{\xi}{2} + \frac{2k\pi}{2m+1} \right)}{\cosh \frac{\eta}{2} - \cos \left(\frac{\xi}{2} + \frac{2k\pi}{2m+1} \right)}} \right. \\ \left. - \frac{\sqrt{\cosh \eta - \cos \xi}}{\sqrt{\cosh \eta - \cos \left(\xi + 2\beta + \frac{4k\pi}{2m+1} \right)}} \tan^{-1} \sqrt{\frac{\cosh \frac{\eta}{2} + \cos \left(\frac{\xi}{2} + \beta + \frac{2k\pi}{2m+1} \right)}{\cosh \frac{\eta}{2} - \cos \left(\frac{\xi}{2} + \beta + \frac{2k\pi}{2m+1} \right)}} \right\}.$$

Making $m = 0$, we have the case of the spherical bowl charged to

potential V_0 , giving

$$V = V_0 - \frac{2V_0}{\pi} \left\{ \tan^{-1} \sqrt{\frac{\cosh \frac{\eta}{2} + \cos \frac{\xi}{2}}{\cosh \frac{\eta}{2} - \cos \frac{\xi}{2}}} - \sqrt{\frac{\cosh \eta - \cos \xi}{\cosh \eta - \cos (\xi + 2\beta)}} \tan^{-1} \sqrt{\frac{\cosh \frac{\eta}{2} + \cos \left(\frac{\xi}{2} + \beta \right)}{\cosh \frac{\eta}{2} - \cos \left(\frac{\xi}{2} + \beta \right)}} \right\},$$

that is,

$$V = \frac{2V_0}{\pi} \left\{ \tan^{-1} \sqrt{\frac{\cosh \frac{\eta}{2} - \cos \frac{\xi}{2}}{\cosh \frac{\eta}{2} + \cos \frac{\xi}{2}}} + \sqrt{\frac{\cosh \eta - \cos \xi}{\cosh \eta - \cos (\xi + 2\beta)}} \tan^{-1} \sqrt{\frac{\cosh \frac{\eta}{2} + \cos \left(\frac{\xi}{2} + \beta \right)}{\cosh \frac{\eta}{2} - \cos \left(\frac{\xi}{2} + \beta \right)}} \right\},$$

the equation to the bowl being

$$\xi + \beta = 0.$$

The case of the circular disc is obtained by putting $\beta = \pi$, which gives

$$V = \frac{4V_0}{\pi} \tan^{-1} \sqrt{\frac{\cosh \frac{\eta}{2} - \cos \frac{\xi}{2}}{\cosh \frac{\eta}{2} + \cos \frac{\xi}{2}}}.$$

By means of § 4, the potential when

$$\alpha = 3\pi/(3m \pm 1)$$

could be expressed in terms of elliptic integrals of the first and third kinds, which would include as a particular case the conductor formed by two orthogonal spheres, the dielectric angle being three right angles, but, as it will be shown afterwards that the capacity can always be expressed in a finite form, it is unnecessary to do so. The expression for the density at any point of the conductor could be written down by differentiating V with respect to ξ , and multiplying by the necessary coefficient $\frac{2}{2\pi r} (\cosh \eta - \cos \xi)$.

8. *The Capacity of the Conductor.*

The capacity of the conductor formed by the intersection of two spheres at any angle α can be determined from the above. It can be easily proved that the capacity of any conductor is equal to the potential at any point, of the electricity induced on the inverse conducting surface with respect to this point, due to a charge $-r'$ at the point, where r' is the radius of inversion.

It follows that the capacity of the conductor under consideration is given by

$$O = \frac{r'}{\pi\sqrt{2}} \int_0^\infty \frac{d\zeta}{\sqrt{\cosh \zeta - 1}} \left\{ \frac{\frac{\pi}{a} \sinh \frac{\pi\zeta}{a}}{\cosh \frac{\pi\zeta}{a} - \cos \frac{2\pi\beta}{a}} - \frac{\frac{\pi}{a} \sinh \frac{\pi\zeta}{a}}{\cosh \frac{\pi\zeta}{a} - 1} + \frac{\sinh \zeta}{\cosh \zeta - 1} \right\}.$$

To evaluate this integral, let

$$\pi/a = p/q,$$

where p and q are integers; then

$$O = \frac{r'}{2\pi} \int_0^\infty \frac{d\zeta}{\sinh \frac{q\zeta}{2}} \left\{ \frac{p \sinh p\zeta}{\cosh p\zeta - \cos 2p\gamma} - \frac{p \sinh p\zeta}{\cosh p\zeta - 1} + \frac{q \sinh q\zeta}{\cosh q\zeta - 1} \right\},$$

where $\gamma = \beta/q$. Putting $e^{-\zeta} = x$, this becomes

$$O = \frac{2r'}{\pi} \int_0^1 \frac{x^{q-1} dx}{1-x^{2q}} \left\{ \frac{p(1-x^{4p})}{x^{4p} - 2x^{2p} \cos 2p\gamma + 1} - \frac{p(1+x^{2p})}{1-x^{2p}} + \frac{q(1+x^{2q})}{1-x^{2q}} \right\} dx.$$

Now,
$$\frac{px^{q-1}(1-x^{4p})}{(1-x^{2q})(x^{4p} - 2x^{2p} \cos 2p\gamma + 1)}$$

$$= \sum_{k=1}^{k=q-1} (-)^k \frac{p}{q} \frac{\sin \frac{2kp\pi}{q} \sin \frac{k\pi}{q}}{\left(\cos 2p\gamma - \cos \frac{2kp\pi}{q} \right) \left(x^2 - 2x \cos \frac{k\pi}{q} + 1 \right)} \\ + \sum_{k=0}^{k=2p-1} \frac{1}{2 \sin \left(q\gamma + \frac{kq\pi}{p} \right)} \frac{\sin \left(\gamma + \frac{k\pi}{p} \right)}{x^2 - 2x \cos \left(\gamma + \frac{k\pi}{p} \right) + 1},$$

and

$$\begin{aligned}
 & \frac{q(1+x^{2q})x^{q-1}}{(1-x^{2q})^2} - \frac{p(1+x^{2p})x^{p-1}}{(1-x^{2q})(1-x^{2p})} \\
 &= -\frac{\partial}{\partial x} \frac{x^q}{x^{2q}-1} - \frac{p(1+x^{2p})x^{q-1}}{(1-x^{2q})(1-x^{2p})} \\
 &= -\frac{\partial}{\partial x} \sum_{k=1}^{k=q-1} \frac{(-1)^k}{q} \left\{ \frac{x \cos \frac{k\pi}{q} - 1}{x^2 - 2x \cos \frac{k\pi}{q} - 1} \right\} \\
 &\quad - \sum_{k=1}^{k=q-1} (-1)^k \frac{p}{q} \cot \frac{kp\pi}{q} \frac{\sin \frac{k\pi}{q}}{x^2 - 2x \cos \frac{k\pi}{q} + 1} \\
 &\quad - \sum_{k=1}^{k=p-1} \operatorname{cosec} \frac{kq\pi}{p} \frac{\sin \frac{k\pi}{p}}{x^2 - 2x \cos \frac{k\pi}{p} + 1}.
 \end{aligned}$$

Hence

$$\begin{aligned}
 \frac{\pi C}{2r'} &= \sum_{k=1}^{k=q-1} (-1)^k \frac{\sin \frac{2kp\pi}{q}}{\cos 2p\gamma - \cos \frac{2kp\pi}{q}} \frac{p\pi}{2q} \left(1 - \frac{k}{q}\right) \\
 &\quad + \frac{\frac{\pi}{2} + \tan^{-1} \cot \left(\gamma + \frac{k\pi}{p}\right)}{4 \sin q \left(\gamma + \frac{k\pi}{p}\right)} - \sum_{k=1}^{k=q-1} \frac{(-1)^k}{2q} \\
 &\quad - \sum_{k=1}^{k=q-1} (-1)^k \cot \frac{kp\pi}{q} \frac{p\pi}{2q} \left(1 - \frac{k}{q}\right) - \sum_{k=0}^{k=p-1} \frac{\pi \left(1 - \frac{k}{p}\right)}{2 \sin \frac{kq\pi}{p}}.
 \end{aligned}$$

There are two cases, viz., q an odd integer and q an even integer.(1) $q = 2m + 1$ an odd integer. Then

$$\begin{aligned}
 \frac{\pi C}{2r'} &= \sum_{k=1}^{k=m} (-1)^k \frac{p\pi}{2q} \left\{ \frac{\sin \frac{2kp\pi}{q}}{\cos 2p\gamma - \cos \frac{2kp\pi}{q}} - \cot \frac{kp\pi}{q} \right\} \\
 &\quad + \sum_{k=1}^{k=p-1} \frac{\pi}{4} \left\{ \operatorname{cosec} q \left(\gamma + \frac{k\pi}{p}\right) - \operatorname{cosec} \frac{kq\pi}{p} \right\} + \frac{\pi}{4} \cot
 \end{aligned}$$

$$\text{or } \frac{O}{r'} = \sum_{k=1}^{h-m} (-1)^k \frac{\pi}{a} \left\{ \frac{\sin \frac{2k\pi^2}{a}}{\cos \frac{2\pi\beta}{a} - \cos \frac{2k\pi^2}{a}} - \cot \frac{k\pi^2}{a} \right\} \\ + \sum_{k=1}^{h-p-1} \frac{1}{2} \left\{ \operatorname{cosec} (\beta + ka) - \operatorname{cosec} ka \right\} + \frac{1}{2} \operatorname{cosec} \beta.$$

(2) $q = 2m$ an even integer. Then

$$\frac{O}{r'} = \sum_{k=1}^{h-q-1} (-1)^k \frac{\pi}{a} \left(1 - \frac{k}{q}\right) \left\{ \frac{\sin \frac{2k\pi^2}{a}}{\cos \frac{2\pi\beta}{a} - \cos \frac{2k\pi^2}{a}} - \cot \frac{k\pi^2}{a} \right\} \\ + \sum_{k=0}^{h-p-1} \frac{1 - \frac{4}{\pi} \tan^{-1} \tan \frac{1}{2} \left(\gamma + \frac{k\pi}{p} \right)}{2 \sin (\beta + ka)} + \frac{1}{\pi q} - \sum_{k=1}^{h-p-1} \frac{\left(1 - \frac{k}{p}\right)}{\sin ka}.$$

In the above r' is the diameter of the circle of intersection, $\xi = -\beta$ one of the bounding surfaces, $\xi = \alpha - \beta$ the other.

9. Particular Cases.

1. When $q = 1$, this is the case when the problem is solvable by the method of point images and the capacity is given by

$$O = \sum_{k=1}^{h-p-1} \frac{r'}{2} \left\{ \operatorname{cosec} (\beta + ka) - \operatorname{cosec} ka \right\} + \frac{r'}{2} \operatorname{cosec} \beta,$$

which gives for the sphere [$p = 1$]

$$O = \frac{r'}{2} \operatorname{cosec} \beta = \text{the radius.}$$

2. When $q = 2$, the capacity is given by

$$O = \sum_{k=0}^{h-p-1} r' \frac{\pi - 4 \tan^{-1} \tan \frac{1}{2} \left(\frac{\beta}{2} + \frac{k\pi}{p} \right)}{2\pi \sin (\beta + ka)} + \frac{r'}{2\pi} - \sum_{k=1}^{h-p-1} \frac{r' (1 - k/p)}{\sin ka}.$$

which gives for the spherical bowl ($p = 1$)

$$O = \frac{r'}{2\pi} + \frac{(\pi - \beta) r'}{2\pi \sin \beta},$$

agreeing with Lord Kelvin's result.

3. When $q = 3$, the capacity is given by

$$C = \frac{r'}{3} \left\{ \cot \frac{p\pi}{3} - \frac{\sin \frac{2p\pi}{3}}{\cos \frac{2p\beta}{3} - \cos \frac{2p\pi}{3}} \right\} + \sum_{k=1}^{\infty} \frac{r'}{2} \{ \operatorname{cosec} (\beta + k\alpha) - \operatorname{cosec} k\alpha \} + \frac{r'}{2} \operatorname{cosec} \beta.$$

A particular case of this result is the conductor formed by two spheres cutting at right angles, the dielectric angle α being $3\pi/2$. In this case $p = 2$, and the capacity is given by

$$C = \frac{4r'}{3\sqrt{3}} \frac{1 - \cos \frac{4\beta}{3}}{1 + 2 \cos \frac{4\beta}{3}} + \frac{r'}{2} (1 + \operatorname{cosec} \beta - \sec \beta).$$

From this the capacity of a hemisphere can be found by putting $\beta = \pi$, which gives

$$C = r' \left(1 - \frac{1}{\sqrt{3}} \right) = r' \cdot 42265, \text{ nearly,}$$

where r' is the diameter, that is, the capacity is .8453 of the radius, approximately.

10. *Solvable Cases which can be obtained from the preceding.*

The case of the conductor formed by two intersecting spheres under the influence of an electrified point can be obtained by an inversion from § 8, and therefore the distribution on such a conductor under the influence of any fixed distribution.

If a sphere is described with its centre on the line of intersection of the two planes in § 1, the functional image of an electrified point for the space enclosed by the planes and the sphere can be obtained by adding to the expression there given the functional image of an electrified point placed at the inverse with respect to the sphere of the first point; whence, by inversion, can be obtained the potential due to a conductor formed by three intersecting spheres, two of which intersect at any angle whilst the third intersects them at right angles (the latter two dielectric angles being $\pi/2$). The capacity of this latter conductor could then be expressed by the method of § 8 in finite terms, and the distribution on it due to an electrified point could be obtained by another inversion. This last includes some interesting particular cases, *e.g.*, a complete sphere with a circular which passes through its centre) projecting from it.