The Invariant Syzygies of Lowest Degree for any number of Quartics. By A. YOUNG. Received and communicated June 14th, 1900.

In a previous paper* I found the irreducible system of types for any number of quartics. The reductions were effected by means of systems of equations between forms of the same degree and order; these were written $[abc\dots f] = R$, &c. If, instead of R , we write the sum of products of forms for which R stands, these systems of equations will give us the syzygies. This method has, as is shown in § 1, the merit of exhaustiveness.

So far as results are concerned, I have only dealt exhaustively with the invariant syzygies up to degree 7. There is no invariant syzygy of degree lower than 7; and all syzygies of this degree are linearly deducible from syzygies of a single form. From this syzygy it is possible to deduce, by means of differential operations, syzygies between covariants and syzygies of higher degree between invariants.

The system of syzygies for two quartics has been worked out by von Gall.† The syzygy degree 7 obtained here is identically zero when fewer than three quartics are under consideration, so that it does not appear in his paper. As some use is made of substitutions, it is convenient to use Roman letters for symbols indicating them.

The following abbreviations are used:—

$\{s\}$ denotes the sum of powers of the substitution s .

$\{s_1, s_2\}$ denotes the sum of the substitutions of the simplest group containing s_1 and s_2 ; and so on.

$\{a, b, c, \dots\}$ denotes the sum of the substitutions of the symmetric group of the letters $a, b, c, \dots$.

$\{a, b, c, \dots\}'$ denotes the sum of the substitutions of the alternating group, minus the sum of the substitutions which do not belong to the alternating group of the letters $a, b, c, \dots$.

The invariant types in my paper referred to above were written (aba) , $(abca)$; they are written here simply (ab) , (abc) , ...

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† *Math. Ann.*, Bd. xxxiii., p. 197, and Bd. xxxiv., p. 332.

1. All relations between quartic types, including, of course, syzygies, must depend ultimately on the fundamental identities found in § 2 of my previous paper. Hence, if we form from these all possible equations between quartic forms of a given degree and order, we shall obtain all the relations which exist between forms of that degree and order. To construct these equations it is only necessary to take those obtained by multiplying one or other of the identities by a single factor; all other equations being linearly dependent on these. Hence the relations $[abc\dots g] = R$, &c., already used to obtain the irreducible system of types, must also give all the syzygies.

Having obtained a relation between types of a certain degree, it is possible, by a certain operation, to obtain from it a relation between types of a higher degree. Thus the result of operating with $(kb) \left[k_0 \frac{\partial}{\partial b_0} + k_1 \frac{\partial}{\partial b_1} + k_2 \frac{\partial}{\partial b_2} \right]$ on the symbolical product $(abcd\dots)$ is $(abkcd\dots) + (akbcd\dots)$. Before, however, using this operator, it is necessary to have the relation written out in full in quadratic symbols. Now we shall find it convenient to ignore symbolical factors of the form $[au]$. When this is done, any single symbolical form represents a single invariant. In fact, the relations as written will be relations between invariants themselves, rather than between quadratic symbolical products. To pass from the actual coefficients of the quartics to quadratic symbols, and *vice versa*, we have the identity

$$(A_0, A_1, A_2, A_3, A_4 | x_1, x_2)^4 \equiv a_x^2 \equiv (a_0 x_1^2 + 2a_1 x_1 x_2 + a_2 x_2^2)^2.$$

$$\text{Therefore } A_0 = a_0^2, \quad A_1 = a_0 a_1, \quad A_2 = a_1 a_2, \quad A_3 = a_2^2,$$

$$6A_2 = 4a_1^2 + 2a_0 a_3 = 6a_1^2 + [aa] = 6a_0 a_3 - 2[aa].$$

Hence, if P be the type represented by the symbolical product $(abcd\dots)$,

$$P = (abcd\dots) + \Sigma [aa] P_a + \Sigma [aa][bb] P_{a,b} + \dots$$

To find P_b , consider the product

$$\begin{aligned} [ab][bc] &\equiv B_0 a_2 c_2 - 2B_1 (a_2 c_1 + a_1 c_2) + (B_2 + \tfrac{1}{3}[bb])(a_2 c_0 + a_0 c_2) \\ &\quad + 4(B_2 - \tfrac{1}{6}[bb]) a_1 c_1 - 2B_3 (a_1 c_0 + a_0 c_1) + B_4 a_0 c_0 \\ &= \text{a function of the coefficients } B + \tfrac{1}{3}[bb][ac]. \end{aligned}$$

Therefore

$$P_1 = -\frac{1}{3}(acd\dots).$$

The result, then, of operating with $[kb]\left[k_0\frac{\partial}{\partial b_0} + k_1\frac{\partial}{\partial b_1} + k_2\frac{\partial}{\partial b_2}\right]$ on the invariant type $(abcd\dots)$ is

$$(abkcd\dots) + (akbcd\dots) - \frac{2}{3}(bk)(acd\dots).$$

The effect of this operator, it is easy to see, is the same as that of substituting the coefficients of the covariant type 2 $[bk]b_{\alpha}k_{\alpha}$ for those of the quartic b_{α}^2 .

2. Before proceeding to the discussion of syzygies by the above method, we shall establish certain syzygies which exist for the types expressible as determinants.

First,

$$96 |ABCDE| |FGHKL| = \begin{vmatrix} (af) & (ag) & (ah) & (ak) & (al) \\ (bf) & (bg) & (bh) & (bk) & (bl) \\ (cf) & (cg) & (ch) & (ck) & (cl) \\ (df) & (dg) & (dh) & (dk) & (dl) \\ (ef) & (eg) & (eh) & (ek) & (el) \end{vmatrix}.$$

Similarly the product of $|ABCDE|$ and any form of the type I_6 , as also that of two forms I_6 , may be expressed as the sum of products of forms I_3 and I_3 .

Next, any function homogeneous and linear in the coefficients of each of the five quartics $a_{\alpha}^2, b_{\alpha}^2, c_{\alpha}^2, d_{\alpha}^2, e_{\alpha}^2$ is of the form

$$P = \sum \lambda A_r B_{r_1} C_{r_2} D_{r_3} E_{r_4},$$

where $r = 0, 1, 2, 3, 4$; and can therefore have only five distinct values. Hence

$$\{a, b, c, d, e\}' P = |ABCDE| Q. \quad (\text{I.})$$

Similarly,

$$\{a, b, c, d\}' P = |ABCDQ|,$$

and, if P be also homogeneous and linear in the coefficients of f_{α} ,

$$\{a, b, c, d, e, f\}' P = 0. \quad (\text{II.})$$

In (I.) put $P = |ABCD, EF|$; the right-hand side must be zero, otherwise Q would be an invariant of the first degree of f_{α}^2 ; this gives the identity for the forms $|ABCD, EF|$ (found before).

In (II.) put $P_2 = (fg) | ABCDE |$; the result is a relation which may be written

$$(fg) | ABCDE | = (ag) | FBCE | + (bg) | AFCE | + (cg) | ABFDE | \\ + (dg) | ABCFE | + (eg) | ABCDF |. \quad (\text{III.})$$

In this we may write for F_0, F_1, F_2, F_3, F_4 the coefficients of $F_4, -4F_3, 6F_2, -4F_1, F_0$ in any type $(f\dots)$, invariant or covariant. The result will be

$$(g\dots) | ABCDE | = \Sigma I_3 J,$$

where J is reducible, except when $(g\dots)$ belongs to one of the types I_2, I_3 . Operate with $[eh] \left[h_0 \frac{\partial}{\partial e_0} + h_1 \frac{\partial}{\partial e_1} + h_2 \frac{\partial}{\partial e_2} \right]$, and we obtain

$$J \cdot I_3 = \Sigma P_3,$$

where ΣP_3 represents a sum of products of forms, there being at least three forms in each product, and J is any type except I_2 or I_3 .

Operate on (III.) with

$$[eh] \left[h_0 \frac{\partial}{\partial e_0} + h_1 \frac{\partial}{\partial e_1} + h_2 \frac{\partial}{\partial e_2} \right] [fk] \left[k_0 \frac{\partial}{\partial f_0} + k_1 \frac{\partial}{\partial f_1} + k_2 \frac{\partial}{\partial f_2} \right], \\ 2(gfk) | ABCD, EH | = 2(geh) | ABCD, FK | + \Sigma P_3.$$

Similarly, $(geh) | ABCD, FK | = (fkh) | ABCD, GE | + \Sigma P_3$.

Therefore $(gfk) | ABCD, EH | = (hfk) | ABCD, EG | + \Sigma P_3$.

It is not possible to express $I_3 I_0$ in the form ΣP_3 ; for suppose

$$(abc) | DEFG, HK | = \Sigma P_3 = \Sigma \{ I_5 I_2 I_3 + I_4 I_3 I_2 + I_3 I_3 I_3 + I_2 I_2 I_3 \}.$$

Then the syzygy would still be true if we supposed the quartics represented by the letters a, b, c, h, k to be all the same—we may without confusion express this supposition by using only one letter for this quartic. Then

$$(aaa) | DEFG, AA | = \Sigma P_3 = \Sigma \{ I_5 I_2 I_3 + \dots \}.$$

Operate on this with $\{d, e, f, g\}'$; a little consideration will show that the result is

$$24(aaa) | DEFG, AA | = \lambda (aa) (aa) | DEF'GA |;$$

and therefore $(aaa) | DEFA, AA | = 0$.

Hence $(aua) = 0$ or $| DEFA, AA | = 0$;
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either result being absurd. In exactly the same way, it may be shown that there is no syzygy of the form

$$I_3 I_6 = \Sigma I_4 I_5 + \Sigma P_4$$

or of the form

$$I_3 I_6 = \Sigma I_5 I_3 + \Sigma P_5.$$

3. For the sake of finding syzygies, the equations of my previous paper must be written out in full. As regards degree 4, there is only one equation,

$$2(abcd) + 2(adb c) + 2(acdb) = (ab)(cd) + (ad)(bc) + (ac)(db);$$

which is the expanded form of

$$\begin{vmatrix} [ab] & [ac] & [ad] & [a\alpha] \\ [bb] & [bc] & [bd] & [ba] \\ [cb] & [cc] & [cd] & [ca] \\ [db] & [dc] & [dd] & [da] \end{vmatrix} = 0.$$

The only form of equation for degree 5 is

$$[ab\dots] = (abc)(de) + (abd)(ec) + (abe)(cd) + 2(ab)(cde). \quad (\text{IV.})$$

Now it was proved before that

$$60 |ABCDE| = -\{a, b, c, d, e\}'(abcde) + R.$$

In this case $R = \Sigma I_3 I_3$, but $\{a, b, c, d, e\}' \Sigma I_3 I_3 = 0$; hence R must here be zero. It will be convenient to use the form

$$(abcde)' = (abcde) - \frac{1}{2} \{ (ab)(cde) + (bc)(dea) + (cd)(eab) \\ + (de)(abc) + (ea)(bcd) \}.$$

Then $60 |ABCDE| = -\{a, b, c, d, e\}'(abcde)'.$

Further $[ab\dots]' = -\frac{1}{2} \{a, b, c, d, e\}'(ab)(cde).$

Hence $[ab\dots]' + [ae\dots]' - [ac\dots]' - [ad\dots]'$

$$= 2(abcde)' + 2(abdc e)' - 2(acbed)' - 2(acebd)' = 0;$$

and therefore

$$\begin{aligned} (abdc e)' - (acbed)' \\ &= (acebd)' - (abcde)' = (adecb)' - (acdbe)' \\ &= \dots = |ABCDE| \\ &= \frac{1}{2} \{ (abdc e)' + (acebd)' + (acdbe)' + (adecb)' + (adcb e)' + (abedc)' \} \\ &\quad + \frac{1}{24} \{a, b, c, d, e\}'(ab)(cde). \quad (\text{V.}) \end{aligned}$$

The equations of degree 5 give nothing in the way of syzygies.

For degree 6 there are two kinds of equation to be discussed—

$$[abc\dots] = (abcd)(ef) + (abce)(fd) + (abcf)(de) + 2(abc)(def). \quad (\text{VI.})$$

$$\text{and } [ab, cd, e, f] = (abcd)(ef) + (abef)(cd) + (abfe)(cd) + (ab)(cdef) \\ + (ab)(cdf e) + (abe)(cdf) + (abf)(cde) - (ab)(cd)(ef). \quad (\text{VII.})$$

As in the former problem, we will use, in addition, the equation

$$4[abc]^2[def]^2 \\ = 2[a-b-c-] + \{d, e, f\} [(ad)(be)(cf) - (ad)(becf) \\ - (bd)(aecf) - (cd)(aebf)] \\ = 4(abc)(def)$$

$$\text{or } [a-b-c-] \\ = 2(abc)(def) - \frac{1}{2}\{d, e, f\} [(ad)(be)(cf) - (ad)(becf) - (bd)(ceaf) \\ - (cd)(aebf)]. \quad (\text{VIII.})$$

Let S represent the sum of the sixty possible forms $(abcdef)$, and operate on (VIII.) with $\{a, b, c, d, e, f\}$;

$$72S = \{a, b, c, d, e, f\} [2(abc)(def) - 3(ad)(be)(cf) + 9(ad)(becf)] \\ = \{a, b, c, d, e, f\} [2(abc)(def) + \frac{3}{2}(ad)(be)(cf)]. \quad (\text{IX.})$$

Now we proved before that

$$6[adbecf] + [afbdce] + [aehfcd] \\ = \{(abc), (def)\} [ad, be, c, f] + \frac{1}{3}[abc\dots] - \frac{1}{9}S + \frac{1}{9}[a-b-c-] \\ = \{(abc), (def)\} [(cf)(adbe) + (ad)(becf) + (ad)(befc) + (be)(adcf) \\ + (be)(adfc) - (ad)(be)(cf) + (adc)(bef) + (adf)(bec) \\ + (ef)(abcd) + \frac{2}{3}(abc)(def) - \frac{2}{9}(abc)(def) \\ - 2(abd)(cef) - \frac{1}{3}(ad)(be)(cf) - \frac{1}{3}(ad)(bf)(ce) \\ - (ab)(de)(cf) + \frac{2}{9}(abc)(def) - \frac{1}{6}(ad)(be)(cf) \\ - \frac{1}{6}(ad)(bf)(ce) + (ad)(becf)] \\ = \{(abc), (def)\} [\frac{2}{3}(abc)(def) + (ad)\{4(hcf) + 2(befc)\} \\ - \frac{1}{2}(ab)(de)(cf) - \frac{1}{2}(ad)(bf)(ce) \\ - \frac{3}{2}(ad)(be)(cf)]. \quad (\text{X.})$$

We proceed to find the exact expression for $(abcdef)$ in terms of

the determinant forms degree 6 and products of types of lower degree. For this purpose, it is useful to note that $(abcdef)$ is unchanged by any substitution of the group $\{(abcdef), (ac)(df)\}$. Operate, then, with the sum of the substitutions of this group on the expression given by (X.) for

$$(abcfed) + (afcdeb) + (adcbe).$$

The result is

$$\begin{aligned} \{(abcdef), (ac)(df)\} [3(abcfed) - (ace)(bdf) - (ab)\{2(cfed) + (cedf)\} \\ - (ad)\{2(cbef) + (cefb)\} - (af)\{2(cdeb) + (cebd)\} \\ + \frac{1}{4}(ab)(ce)(df) + \frac{1}{4}(ad)(ce)(bf) + \frac{1}{4}(af)(ce)(bd) \\ + \frac{1}{4}(ab)(cd)(ef) + \frac{1}{4}(ad)(cf)(eb) + \frac{1}{4}(af)(cb)(ed) \\ + \frac{3}{4}(ab)(cf)(de) + \frac{3}{4}(ad)(cb)(ef) + \frac{3}{4}(af)(cd)(eb)] = 0 \end{aligned}$$

$$\begin{aligned} \text{or } \{(abcdef), (ac)(df)\} [3(abcfed) - (ace)(bdf) \\ - 2(ab)\{2(cdef) + (cedf)\} - (ad)\{2(bcef) + (bcef)\} \\ + \frac{1}{2}(ab)(ce)(df) + \frac{1}{2}(ab)(cd)(ef) + \frac{1}{4}(ad)(ce)(bf) \\ + \frac{1}{4}(ad)(cf)(be) + \frac{3}{4}(ab)(cf)(de)] = 0. \quad (\text{XI.}) \end{aligned}$$

Similarly, by operating on (X.) itself,

$$\begin{aligned} \{(abcdef), (ac)(df)\} [2(abcfed) + (abfdec) - (abc)(def) \\ - \frac{1}{3}(ab)\{4(cedf) + 2(cfed)\} - \frac{1}{3}(ad)\{5(bcef) + 4(befc)\} \\ + \frac{1}{6}(ab)(cd)(ef) + \frac{5}{12}(ab)(de)(cf) + \frac{1}{2}(ab)(ce)(df) \\ - \frac{1}{12}(ad)(ce)(bf) + \frac{3}{4}(ad)(be)(cf)] = 0. \quad (\text{XII.}) \end{aligned}$$

Operate on equations (V.) with $[fc] \left[f_0 \frac{\partial}{\partial c_0} + f_1 \frac{\partial}{\partial c_1} + f_2 \frac{\partial}{\partial c_2} \right]$; then
 $|DEAB, CF|$

$$\begin{aligned} = (acfebd) + (afcebd) - (abcfde) - (abfcde) \\ - \frac{2}{3}(cf)\{(uebd) - (abde)\} - (acf)(ebd) - (cfe)(bda) + (bcf)(dea) \\ + (cfd)(eab) - \frac{1}{2}(eb)\{(dacf) + (dafc)\} - \frac{1}{2}(bd)\{(acfe) + (afce)\} \\ - \frac{1}{2}(da)\{(cfed) + (fced)\} + \frac{1}{2}(de)\{(abcf) + (abfc)\} \\ + \frac{1}{2}(ea)\{(bcfd) + (bfcd)\} + \frac{1}{2}(ab)\{(cfde) + (fcde)\} \\ + \frac{1}{3}(cf)\{2(eb)(da) - 2(de)(ab)\} \end{aligned}$$

$$\begin{aligned}
&= (adecfb) + (adefcb) - (acfdbe) - (nfcdb) - \frac{2}{3}(cf) \{ (adeb) - (adbe) \} \\
&\quad - (ecf)(bad) - (cfb)(ade) + (acf)(dbe) + (cfd)(bea) \\
&\quad - \frac{1}{2}(ba) \{ (decf) + (dfc) \} - \frac{1}{2}(ad) \{ (ecfb) + (efcb) \} \\
&\quad - \frac{1}{2}(de) \{ (cfba) + (fcb) \} + \frac{1}{2}(db) \{ (eacf) + (eafc) \} \\
&\quad + \frac{1}{2}(be) \{ (acfd) + (afcd) \} + \frac{1}{2}(ea) \{ (cfdb) + (fdb) \} \\
&\quad + \frac{1}{3}(cf) \{ 2(ab)(de) - 2(bd)(ea) \}.
\end{aligned}$$

Operate on these with $\{ (abcdef), (ac)(df) \}$;

$$\begin{aligned}
&\{ (abcdef), (ac)(df) \} | DEAB, CF | \\
&= \{ (abcdef), (ac)(df) \} [2(abecfd) - 2(abcefd) \\
&\quad + (ab) \{ (cdef) + (cdf) \} - \frac{1}{3}(ad) \{ 5(becf) + 3(befc) - 2(bcef) \} \\
&\quad + \frac{2}{3}(ad)(be)(cf) - \frac{2}{3}(ab)(de)(cf)] \\
&= \{ (abcdef), (ac)(df) \} [(abcfed) - (acfdbe) \\
&\quad - (ab) \{ (cdef) + (cdf) \} + (ac) \{ (bedf) + (befd) \} \\
&\quad - \frac{2}{3}(ad) \{ (befc) - (becf) \} + \frac{2}{3}(ab)(de)(cf) - \frac{2}{3}(ad)(ce)(bf)].
\end{aligned}$$

Hence, using $[ad, be, c, f] = R$,

the last expression for $\{ (abcdef), (ac)(df) \} | DEAB, CF |$ becomes

$$\begin{aligned}
&\{ (abcdef), (ac)(df) \} [2(abcfed) + 4(abecfd) - 2(abd)(cef) \\
&\quad - (ab) \{ (cdef) + (cdf) \} + (ac) \{ (bedf) + (befd) \} \\
&\quad - \frac{1}{3}(ad) \{ 7(becf) + 8(befc) \} + \frac{2}{3}(ab)(de)(cf) \\
&\quad - \frac{2}{3}(ad)(ce)(bf) + (ad)(be)(cf)] \\
&= \{ (abcdef), (ac)(df) \} [4(abecfd) + \frac{2}{3}(ace)(bdf) - 2(abd)(cef) \\
&\quad + \frac{1}{3}(ab) \{ 5(cdef) + (cdf) \} + (ac) \{ (bedf) + (befd) \} \\
&\quad - \frac{1}{3}(ad) \{ 7(becf) + 4(befc) - 2(bcef) \} - \frac{1}{3}(ab)(ce)(df) \\
&\quad - \frac{1}{3}(ab)(cd)(ef) - \frac{5}{6}(ab)(de)(cf) - \frac{5}{6}(ad)(ce)(bf) \\
&\quad + \frac{5}{6}(ad)(be)(cf)],
\end{aligned}$$

using (XI.).

From $[ab, de, c, f]$, we have

$$\begin{aligned}
&\{ (abcdef), (ac)(df) \} [(abcdef) + 4(abcefd) + (abfdec) - 2(abc)(def) \\
&\quad - 2(ab) \{ (decf) + (defc) \} - (ad)(bcef) + (ab)(de)(cf)] = 0.
\end{aligned}$$

Eliminate $(abfdec)$ between this equation and (XII.); then

$$\begin{aligned} \{ (abcdef), (ac)(df) \} [& (abcdef) + 4(abcefd) - 2(abecfd) - (abc)(def) \\ & - \frac{1}{3}(ab) \{ 6(cdef) + 2(cedf) - 2(cefd) \} \\ & + \frac{1}{3}(ad) \{ 5(becf) + 4(bcfe) - 3(bcef) \} - \frac{1}{6}(ab)(cd)(ef) \\ & + \frac{7}{12}(ab)(de)(cf) - \frac{1}{2}(ab)(ce)(df) + \frac{1}{12}(ad)(ce)(bf) \\ & - \frac{3}{4}(ad)(be)(cf)] = 0. \end{aligned}$$

Eliminate $(abcefd)$ and $(abecfd)$ between this equation and the two expressions we have found for $\{ (abcdef), (ac)(df) \} | DEAB, CF |$; then

$$\begin{aligned} & \frac{2}{3} \{ (abcdef), (ac)(df) \} | DEAB, CF | \\ = & \{ (abcdef), (ac)(df) \} [(abcdef) - (abc)(def) - \frac{1}{3}(ace)(bdf) \\ & + (abd)(cef) + \frac{1}{6}(ab) \{ -5(cdef) + 7(cedf) + 4(cefd) \} \\ & - \frac{1}{2}(ac) \{ (bedf) + (befd) \} + \frac{1}{6}(ad) \{ -3(becf) \} \\ & - \frac{1}{3}(ab)(de)(cf) - \frac{1}{3}(ab)(ce)(df) + \frac{1}{2}(ad)(ce)(bf) + \frac{1}{6}(ad)(be)(cf)]. \end{aligned}$$

Therefore

$$\begin{aligned} & 2(abcdef) \\ = & | ADEB, CF | + | EBCF, AD | + | CFAD, EB | \\ & + \{ (abcdef) \} [\frac{1}{3}(abc)(def) + \frac{1}{3}(ace)(bdf) - \frac{1}{3}(abd)(cef) \\ & + \frac{1}{2}(ab)(cdef) - \frac{1}{6}(ab)(cedf) - \frac{1}{6}(ac)(bdef) + \frac{1}{6}(ad)(becf) \\ & - \frac{1}{3}(ab)(cd)(ef) + \frac{1}{6}(ab)(ce)(df) - \frac{1}{12}(ad)(ce)(bf) \\ & - \frac{1}{12}(ad)(be)(cf)]. \end{aligned} \quad (XIII.)$$

This expression for $(abcdef)$ satisfies the equations $[abc\dots] = R$ and $[ab, cd, e, f] = R$ identically; hence no syzygies amongst invariants degree 6 can exist.

4. Degree 7.—From the invariant identity degree 6, we have

$$\{a, b, e\}' | FBAD, EC | = 2 | EBAD, FC | + 2 | FBAE, CD |.$$

Hence

$$\begin{aligned} & \{a, b, e\}' [(afdbec) + (cfdeba) + 2(efdcba) + (bfdace)] \\ = & \frac{1}{2} \{a, b, e\}' [| ABEF, DC | + | DCAB, EF | + 3 | ADFB, CE | \\ & + | CEAD, FB | + | FBCF, AD | \\ & + | ABFC, ED | + | EDAB, FC |] + R \\ = & 6 | ABEF, DC | + \{a, b, e\}' [(af)(dbec) + (bc)(aefd) - \frac{1}{2}(af)(bd)(ec)]. \end{aligned}$$

Operate on this with $[gf] \left[g_0 \frac{\partial}{\partial f_0} + g_1 \frac{\partial}{\partial f_1} + g_2 \frac{\partial}{\partial f_2} \right]$, and then with $\{g, d, f\}$; therefore, using the equations $[a...bec] = R$, &c.,

$$\begin{aligned} & 6 \{g, d, f\} | A, B, E, FG, DC | \\ &= \{g, d, f\} \{a, b, e\}' \left[\frac{1}{3} (fg) \{2 (adbec) + (abcde)\} - 2 (afg)(dbec) \right. \\ & \quad \left. + (afg)(bd)(ec) \right] \\ &= 12 | A, B, E, FG, DC | + 12 | A, B, E, DF, GC | + 12 | A, B, E, GD, FC |. \end{aligned}$$

Hence, operating with $\frac{1}{2} \{(cd)\}$,

$$\begin{aligned} & 12 | A, B, E, FG, DC | \\ &= \{(cd), (fg)\} \{a, b, e\}' \left[\frac{1}{3} (dg) | AFBECC | + (dg)(afbec) \right. \\ & \quad \left. - 2 (adg)(fbec) + (adg)(ec)(bf) \right]. \end{aligned}$$

and

$$\begin{aligned} & 12 | A, B, C, E, DFG | \\ &= -2 \{a, b, c, e\}' | A, B, E, FG, DC | \\ &= \{(fg)\} \{a, b, c, e\}' \left[\frac{1}{12} (dg) | AB'ECC | + \frac{1}{12} (fg) | ADBECC | \right. \\ & \quad \left. - (cg)(afbed) \right]. \end{aligned}$$

From these we proceed to find an expression for the reducible type $(abcdefg)$.

Operate then on (XIII.) with $[gf] \left[g_0 \frac{\partial}{\partial f_0} + g_1 \frac{\partial}{\partial f_1} + g_2 \frac{\partial}{\partial f_2} \right]$, and we obtain, after multiplying by 12 and reducing,

$$\begin{aligned} & 24 (abcdefg) + 24 (abcdegf) \\ &= \{(fg), (ae)(bd)\} \left[4 (fg)(abcde) + 6 \{ (ab)(cdefg) + (bc)(defga) \} \right. \\ & \quad - 2 \{ (ab)(cedfg) + (bc)(dfgea) \} \\ & \quad - \{ 2 (ac)(bdefg) + (bd)(acefg) + (ea)(bcdfg) \} + 4 (ad)(becfg) \\ & \quad + 4 (ace)(bdfg) + 4 (bcd)(aefg) + 8 (abc)(defg) \\ & \quad - 4 \{ (abd)(cefg) + (abe)(cdfg) + (acd)(befg) \} \\ & \quad + (afg) \{ 6 (bcde) - 2 (bdce) \} - 2 (bfg)(acde) + 2 (c'g)(adbe) \\ & \quad + 2 (afg) \{ (bd)(ce) - 2 (bc)(de) \} \\ & \quad + 2 (bfg) \{ (cd)(ae) + (ca)(ed) - (ad)(ce) \} \\ & \quad \left. - (cfg) \{ 2 (ad)(be) + (ae)(bd) \} \right] \end{aligned}$$

$$\begin{aligned}
& - (fg) \{4(ab)(cde) + (ae)(bcd) + (bd)(ace) - 2(ac)(bde)\} \\
& + 2(cg) |ABDEF| + 2(ag) \{bcdef - (bdcfe)\} \\
& + 2(hg) \{(adcef) + (dfeac) - (afcde) - (adfce)\} \\
& - (aeg) \{2(bcdf) - 2(bdcf)\} + (bdg) \{2(acef) - 2(aecf)\} \\
& - 2(abg) \{(fced) - (fcde)\} - 2(acg) \{(dbef) - (debfg)\} \\
& - 2(bcg) \{(adef) - (daef)\} - (aeg)(bd)(cf) + (bdg)(ae)(cf) \\
& - 2(abg)(cd)(ef) + 2(acg)(bd)(ef) + 2(abe)(cf)(dg) \\
& - 2(acd)(bf)(eg) \}.
\end{aligned}$$

Now,

$$\begin{aligned}
& 24(abcdefg) + 24(adcbe fg) \\
& = -24(abdcfeg) - 24(adlbcfeg) - 24(acbdefg) - 24(acdbefg) \\
& \quad + 24(bc)(adefg) + 24(cd)(abefg) + 24(db)(acefg) + 48(bcd)(aefg) \\
& = \{(bd), (ae)(fg)\} [-2(bd)(acefg) - (ac) \{4(bdefg) - 2(bdgef)\} \\
& \quad + 2(fc) \{(abdeg) - 2(abldge)\} \\
& \quad - (ef) \{6(bdcga) + 6(bdgac) - 4(bdacg)\} - 4(fg)(bdcea) \\
& \quad - 4(ae)(bdfcg) + 2(eg) \{(bdcfa) + (bdcaf)\} - 4(acf)(bdeg) \\
& \quad - 4(cef)(bdag) + 4(uce)(bdfg) + 4(cfg)(bdac) - 16(efg)(bdac) \\
& \quad + 8(ueg)(bdcf) - (abd) \{4(cefg) - 2(cfe g)\} \\
& \quad - (fbd) \{4(aegc) - 2(aceg)\} + (cbd) \{6(aefg) + 2(aegf)\} \\
& \quad + 2(abd) \{(ce)(fg) - (cg)(ef)\} + (fbd) \{6(ae)(cg) - 2(ec)(ag)\} \\
& \quad + (cbd) \{4(ag)(ef) - 2(af)(eg)\} \\
& \quad + (bd) \{6(ce)(afg) + 4(ag)(cef) - 2(cf)(aeg)\} \\
& \quad - 4(fd) |ABCEG| + (cd) \{10(abefg) + 2(abefg)\} \\
& \quad - 2(fd) \{(cbeag) + (abceg) - (abgce) - (ebgac)\} \\
& \quad - 2(ad) \{(ebcf g) + (fbecg) - (cbgef) - (gbfce)\} \\
& \quad + 2(acd) \{(befg) - (begf)\} + 2(egd) \{(acbf) - (abcf)\} \\
& \quad + 2(fcd) \{(aegb) - (eagb)\} + 2(agd) \{(befe) - (bcef)\} \\
& \quad + 2(acd) \{(fb)(eg) + (eb)(fg)\} - 2(egd) \{(fb)(ac) + (cb)(af)\} \\
& \quad - 2(fcd)(ab)(eg) + 2(agd)(cb)(ef) + 2(aef)(cb)(gd) \\
& \quad - 2(agc)(eb)(fd) - 2(ace)(fb)(gd) + 2(fgc)(ab)(ed) \}.
\end{aligned}$$

$$\begin{aligned}
& \text{Similarly,} \quad -24(adcbe\,fg) - 24(edcbaf\,g) \\
& = -24(ae)(dc\,bfg) - 24(fg)\{(adcbe) + (edcba)\} \\
& \quad -24(bcd)\{(f\,g\,a) + (a\,f\,g\,e)\} - 24(dcb\,a)(efg) - 24(dcbe)(a\,f\,g) \\
& \quad + 24(dcb)(ae)(fg) + 24(dcbe\,a\,f\,g) + 24(dcb\,a\,e\,f\,g) \\
& \quad + 24(dcb\,f\,g\,e\,a) + 24(dcb\,f\,g\,a\,e) \\
& = \{(ae), (bd)(fg)\} [2(ae)(dc\,bfg) \\
& \quad + (cd)\{4(f\,g\,b\,a\,e) + 6(b\,f\,g\,a\,e) - 2(g\,b\,f\,a\,e)\} \\
& \quad + (fg)\{-6(dc\,b\,a\,e) - 2(d\,b\,c\,a\,e)\} \\
& \quad + (gd)\{6(f\,c\,b\,a\,e) - 2(b\,f\,c\,a\,e) - 2(c\,b\,f\,a\,e)\} - 2(bd)(f\,g\,c\,a\,e) \\
& \quad + (fd)\{4(c\,g\,b\,a\,e) - 2(g\,c\,b\,a\,e)\} + 2(gc)(bd\,f\,a\,e) + 4(bdf)(c\,g\,a\,e) \\
& \quad + 4(f\,g\,d)(c\,b\,a\,e) + 4(g\,d\,c)(b\,f\,a\,e) - 4(bcd)(f\,g\,a\,e) - 8(dcf)(g\,b\,a\,e) \\
& \quad - 4(gcf)(bd\,a\,e) + (b\,a\,e)\{6(f\,g\,d\,c) + 2(f\,d\,g\,c)\} \\
& \quad + (f\,a\,e)\{4(d\,c\,b\,g) - 2(d\,c\,g\,b)\} - 2(c\,a\,e)(d\,b\,f\,g) \\
& \quad - (b\,a\,e)\{2(f\,d)(c\,g) + 2(f\,c)(g\,d) + 4(f\,g)(d\,c)\} \\
& \quad + (f\,a\,e)\{2(g\,b)(cd) - 2(g\,d)(cb)\} \\
& \quad + (c\,a\,e)\{2(bd)(fg) + 2(bf)(gd) - 2(bg)(df)\} \\
& \quad - (ae)\{4(cb)(f\,g\,d) - 2(fg)(bcd) + 2(bf)(cd\,g) + 2(cg)(bd\,f) \\
& \quad \quad - 2(bd)(cf\,g) - 2(df)(bc\,g)\} \\
& \quad + 4(da)|OBEFG| + 2(ca)\{(gef\,bd) - (bed\,gf)\} \\
& \quad + 2(ga)\{(bef\,cd) + (ceb\,fd) - (fed\,bc) - (dec\,fb)\} \\
& \quad + 2(ba)\{(cef\,gd) - (cefd\,g)\} + 4(bfa)\{(ec\,gd) - (ecd\,g)\} \\
& \quad - 2(bca)\{2(ed\,fg) - (ed\,gf) - (eg\,df)\} - (fga)\{2(ed\,bc) + 10(ebcd)\} \\
& \quad + 2(gca)\{2(ef\,db) - (ef\,bd) - (eb\,fd)\} - 2(bda)\{(eg\,cf) - (ec\,fg)\} \\
& \quad - 2(dfa)\{(eb\,gc) - (eg\,bc)\} - 2(bfa)(cg)(de) - 2(bca)(gd)(fe) \\
& \quad - 2(fga)(cd)(be) + 2(gca)(fb)(de) + 2(bda)(gc)(fe) \\
& \quad + 2(dfa)(cg)(be) - 2(dcf)(ba)(ge) - 2(bd\,g)(ca)(fe) \\
& \quad + 2(fbc)(da)(ge) + 2(fgb)(da)(ce)].
\end{aligned}$$

Add these three results, and we obtain, after some reduction,

$$\begin{aligned}
 & 336 (abcdefg) \\
 = & \{ (abcdefg), (bg)(cf)(de) \} \left[4 (ac) | BDEFG | + 16 (ad) | BCEFG | \right. \\
 & + (ab) \{ 40 (cdefg) - 8 (cdfge) - 8 (cedfg) \} \\
 & + (ac) \{ -8 (bdefg) - 4 (debfg) + 24 (bdgef) \} \\
 & + (ad) \{ -4 (bcefg) + 16 (cebf g) \} + (abc) \{ 36 (defg) - 4 (efdg) \} \\
 & + (abd) \{ 8 (cfeg) - 16 (cefg) \} + (abe) \{ 8 (cdfg) - 24 (cdfg) \} \\
 & + (ace) \{ 16 (bdfg) - 20 (bfldg) \} \\
 & + (abc) \{ -4 (cf)(dg) + 4 (cf)(eg) - 16 (de)(fg) \} \\
 & + (abd) \{ 28 (ce)(fg) - 12 (cf)(eg) \} \\
 & + (abe) \{ -2 (cd)(fg) - 8 (cg)(df) - 14 (cf)(dg) \} \\
 & \left. + (ace) \{ -12 (bd)(g) + 2 (bg)(df) + 6 (bf)(dg) \} \right].
 \end{aligned}$$

Now, if we write

$$\begin{aligned}
 (cdefg)'' &= (cdefg) + \frac{1}{2} | CDEFG | \\
 &+ \{ (cdefg) \} \left[-\frac{1}{3} (cd)(efg) + \frac{1}{6} (ce)(dfg) \right],
 \end{aligned}$$

we shall have

$$(cdefg)'' = (cegd f)''$$

and

$$\{ e, f, g \} (cdefg) = 0,$$

so that the reductions will be much simplified. For the same reason we will use the form

$$(defg)' = (defg) - \frac{1}{6} \{ (de)(fg) + (df)(ge) + (dg)(ef) \}.$$

With this notation we obtain

$$\begin{aligned}
 & 336 (abcdefg) \\
 = & \{ (abcdefg), (bg)(cf)(de) \} \left[-20 (ab) | CDEFG | - 2 (ac) | BDEFG | \right. \\
 & + 10 (ad) | BCEFG | + (ab) \{ 40 (cdefg)'' - 8 (cdfge)'' - 8 (cedfg)'' \} \\
 & + (ac) \{ -8 (bdefg)'' - 4 (debfg)'' + 24 (bdgef)'' \} \\
 & + (ad) \{ -4 (bcefg)'' + 16 (cebf g)'' \} \\
 & + (abc) \{ 36 (defg)' - 4 (efdg)' \} + (abd) \{ 8 (cfeg)' - 16 (cefg)' \} \\
 & + (abe) \{ 8 (cfilg)' - 24 (cdfg)' \} + (ace) \{ 16 (bldfg)' - 20 (bfldg)' \} \\
 & + \frac{2}{3} (abc) \{ (de)(fg) + (df)(ge) + (dg)(ef) \} \\
 & - \frac{2}{3} (abd) \{ (ce)(fg) + (cf)(ge) + (cg)(ef) \} \\
 & + \frac{1}{3} (abe) \{ 5 (cd)(fg) - (cf)(gd) - (cg)(df) \} \\
 & \left. + \frac{2}{3} (ace) \{ -2 (bd)(fg) + (bf)(dg) + (bg)(df) \} \right]. \quad (\text{XIV.})
 \end{aligned}$$

If we take this value of $(abcdefg)$ and substitute in the three identities degree 7, viz.,

$$[abcd\dots] = R, \quad [abc, de, f, g,] = R, \quad [ab, cd, ef, g,] = R,$$

we shall obtain all the syzygies of this degree. Owing to the form of the expression obtained for $(abcdefg)$, it is only necessary to take one equation of each type. Take, then, the equation

$$[abc, de, f, g,] = R;$$

this may be written

$$\begin{aligned} \{ (fg), (ac)(de) \} & \left[2(abcdefg) + (abcfd eg) + \frac{1}{2}(fg) | ABCDE | \right. \\ & - \frac{1}{2}(fg)(abcde)'' - (de)(abcfg)'' - (abc)(defg)' - (deg)(abcf)' \\ & - \frac{1}{6}(abc) \{ (de)(fg) + 2(df)(eg) \} \\ & - \frac{1}{12}(fg) \{ 4(abe)(cd) - 2(abd)(ce) + 4(ade)(bc) - 2(acd)(be) \\ & \qquad \qquad \qquad - (bde)(ac) \} \\ & - \frac{1}{6}(de) \{ 2(abg)(cf) + 4(afg)(bc) - 2(acf)(bg) - (hfg)(ac) \} \\ & \left. - \frac{1}{6}(deg) \{ (ab)(cf) + (ac)(fb) + (af)(bc) \} \right] = 0. \end{aligned}$$

Hence, putting in the values of the first two forms,

$$\begin{aligned} \{ (fg), (ac)(de) \} & \left[(fg) \{ -3(abcde)'' - (acdbe)'' - 8(acbed)'' \} \right. \\ & + (bf) \{ 8(cdaeg)'' + 6(daceg)'' + 4(adecg)'' \} \\ & + (af) \{ -12(ecbdg)'' - 4(bdecg)'' + 2(debcg)'' + 2(ebdcg)'' \} \\ & + (ef) \{ 4(cbadg)'' + 6(cadbg)'' + 10(adcbg)'' - 8(bacd g)'' \} \\ & - 9(ac)(dbefg)'' - 9(de)(abcfg)'' + 18(cd)(abefg)'' \\ & + (ab) \{ 8(cdefg)'' - 2(cedfg)'' \} + (eb) \{ 8(acdfg)'' - 2(cadfg)'' \} \\ & - 3(bfg)(adce)' + 6(afg) \{ (cdbe)' - (cdeb)' \} \\ & + (efg) \{ 10(abcd)' + 2(acdb)' \} - 9(acf)(dbeg)' - 9(def)(abcg)' \\ & + 6(abf)(cd eg)' + 2(ebf) \{ (cadg)' - (cdag)' \} \\ & + 4(aef) \{ (cbd g)' - (cdbg)' \} + (adf) \{ -14(bceg)' - 10(becg)' \} \\ & + 6(abc)(dfeg)' + 6(dbe)(afcg)' - 3(abe)(cf dg)' \\ & \left. - 9(abd)(cfeg)' \right] = 0. \end{aligned} \tag{XV.}$$

This may be simplified thus:—Interchange a and e , c and d , and subtract the resulting from the original equation,

$$\begin{aligned} & \{(fg), (ac)(de)\} \left[(fg) \{ -2(abcde)'' + 2(acdbe)'' \} \right. \\ & \quad + (bf) \{ 2(daceg)'' - 2(adecg)'' \} + (af) \{ 4(becdg)'' - 4(debcg)'' \} \\ & \quad + (ef) \{ 4(cabdg)'' - 4(badcg)'' \} + (afg) \{ 4(cbde)' - 4(cdeb)' \} \\ & \quad + (efg) \{ 4(dcab)' - 4(dbca)' \} + (abf) \{ 4(cdeg)' - 4(cedg)' \} \\ & \quad + (ebf) \{ 4(dacg)' - 4(dacg)' \} + 4(aef) \{ (dcbg)' - (cdbg)' \} \\ & \quad \left. + (adf) \{ 4(becg)' - 4(bceg)' \} \right] = 0. \end{aligned} \quad (XVI.)$$

Now, let the expression

$$\begin{aligned} & (gf)(abcde)'' + (af)(gbdce)'' + (bf)(agced)'' + (cf)(adgbe)'' \\ & \quad + (df)(abegc)'' + (ef)(acbdg)'' - (agf)(bcd)'' - (bgf)(cdae)' \\ & \quad - (cgf)(deba)' - (dgf)(eacb)' - (egf)(abdc)' - (abf)(dcge)' \\ & \quad - (acf)(egdb)' - (adf)(cebg)' - (aef)(gbcd)' - (bcf)(edga)' \\ & \quad - (bdf)(agec)' - (bef)(dacg)' - (cdf)(gbae)' - (cef)(adbg)' \\ & \quad - (def)(gcba)' \equiv \phi(abcde, g, f); \end{aligned}$$

$$\text{then} \quad \phi(abcde, f, g) = \phi(bcdea, f, g) = \phi(edcba, f, g).$$

Then equation (XVI.) is equivalent to

$$\phi(abcde, g, f) + \phi(abcde, f, g) = \phi(acdbe, g, f) + \phi(acdbe, f, g).$$

Operate on this equation with the substitutions (bcd) , $(abcde)$, $(abcde)^3$, $(abcde)^4$ respectively; and we obtain

$$\begin{aligned} \phi(abcde, g, f) + \phi(abcde, f, g) &= \phi(acdbe, g, f) + \phi(acdbe, f, g) \\ &= \phi(adlce, g, f) + \phi(adlce, f, g) \\ &= \phi(acedb, g, f) + \phi(acedb, f, g) \\ &= \phi(abecd, g, f) + \phi(abecd, f, g) \\ &= \phi(acbed, g, f) + \phi(acbed, f, g) \\ &= 0, \end{aligned}$$

since the sum of the above six expressions is zero.

Consider then the expression $\phi(abcde, g, f)$; it may be written

$$\begin{aligned} & \{(abcde)\} \left[\frac{1}{5} (fg)(abcde)'' + (af)(gbdce)'' - (agf)(bcd)'' \right. \\ & \quad \left. - (abf)(dcge)' - (acf)(egdb)' \right]. \end{aligned}$$

But it might equally well be written

$$\{(gbdce)\} \left[\frac{1}{5} (af)(gbdce)'' + (gf)(abcde)'' - (gaf)(bcd)'\right. \\ \left. - (gbf)(cdae)' - (gdf)(eacb)'\right].$$

Hence $\phi(abcde, g, f) = \phi(gbdce, a, f) = \phi(aged, b, f) = \dots$

We have proved that

$$\{(fg)\} \phi(abcde, g, f) = 0;$$

hence also

$$\{(fa)\} \phi(abcde, g, f) = 0, \quad \{(fb)\} \phi(abcde, g, f) = 0, \quad \&c.$$

Suppose, now, we express $\phi(abcde, g, f)$ in linear symbols, the quartics being $\alpha_x, \beta_x^1, \gamma_x^1, \delta_x^1, \epsilon_x^1, \zeta_x^1, \eta_x^1$; then, if ξ and η be interchanged in ϕ and the result added to the original form, we obtain identically zero. Hence, if we put $\zeta_1/\zeta_2 = \eta_1/\eta_2$ in ϕ , the result is zero. Similarly, if we give ζ_1/ζ_2 any one of the six values $\alpha_1/\alpha_2, \beta_1/\beta_2, \&c.$, ϕ vanishes. Now, ϕ/ζ_2^4 is a rational integral function of ζ_1/ζ_2 , of degree 4, and it vanishes for six values of the variable; hence $\phi \equiv 0$.

The syzygy may be written symmetrically thus

$$\{(abcde), (gbdce)\} \left[(gf)(abcde)'' - \frac{5}{2} (agf)(bcd)' \right] = 0 \quad (\text{XVII.})$$

$$\text{or } \{(abcde), (gbdce)\} \left[(gf)(abcde) - \frac{5}{2} (agf)(bcd) - \frac{5}{3} (gf)(ab)(cde) \right. \\ \left. + \frac{5}{6} (gf)(ac)(bde) + \frac{5}{6} (agf)(bc)(ed) \right. \\ \left. + \frac{5}{12} (agf)(be)(cd) \right] = 0.$$

All other syzygies degree 7 are reducible to this one. In order to prove this, it is well to further abbreviate the notation. Thus, let

$$(gf)(abcde)_1 \equiv (gf)(abcde)'' - \{(abcde)\} \left[\frac{1}{2} (agf)(bcd)' \right].$$

Then the syzygy just obtained becomes

$$\{(abcde)\} \left[\frac{1}{5} (gf)(abcde)_1 + (af)(gbdce)_1 \right] = 0.$$

The syzygy

$$(fg) \{ (becd)_1 + (bedc)_1 \} + (fe) \{ (gbdac)_1 + (gbdca)_1 \} \\ + (fd) \{ (gcbca)_1 + (gcbca)_1 \} + (fc) \{ (egdba)_1 + (gedba)_1 \} \\ + (fa) \{ (gdecb)_1 + (geitcb)_1 \} + (eg) \{ (fbcad)_1 + (fbcd_a)_1 \} \\ + (gc) \{ (fdbea)_1 + (febda)_1 \} + (gd) \{ (efcba)_1 + (fecba)_1 \} \\ + (ga) \{ (fecdb)_1 + (fecdb)_1 \} + (cd) \{ (gabfe)_1 + (gfbac)_1 \} \\ + (ea) \{ (gfcdb)_1 + (fgcbd)_1 \} + (cc) \{ (gdjfab)_1 + (gfdab)_1 \} \\ + (dc) \{ (gbfca)_1 + (gbfca)_1 \} + (da) \{ (gfcdb)_1 + (gfcdb)_1 \} \\ + (ac) \{ (edgfb)_1 + (egdfb)_1 \} = 0 \quad (\text{XVIII.})$$

is also required. It is obtained by subtracting

$$\{(gedca)\} \left[\frac{1}{2} (bf)(gedca)_1 + (bg)(fecda)_1 \right]$$

from the sum of the syzygies

$$\begin{aligned} \{(gedca)\} \left[\frac{1}{2} (bf)(gedca)_1 + (fg)(becda)_1 \right] &= 0, \\ \{(fecda)\} \left[\frac{1}{2} (bg)(fecda)_1 + (bf)(gedca)_1 \right] &= 0, \text{ \&c.} \end{aligned}$$

First consider equation (XV.). We obtained (XVI.) from it by interchanging a and e , c and d , and subtracting the resulting from the original equation. If we now add, instead of subtracting, and discuss the result, we shall have completely discussed (XV.). Hence we need only consider the result of operating with $\{(ae)(cd)\}$ on (XV.), i.e.,

$$\begin{aligned} \{(fg), (ac)(de), (ae)(cd)\} \left[(fg) \{ -4 (bcdea)_1 - 8 (bedac)_1 \right. \\ + (bf) \{ 8 (cdacg)_1 + 10 (duceg)_1 \} \\ + (af) \{ 4 (bcdcg)_1 + 8 (ebdgc)_1 + 12 (debeg)_1 - 20 (ecbdg)_1 \\ \left. - 4 (bdecg)_1 \} \right. \\ + (ab) \{ 16 (cdefg)_1 - 4 (cedfg)_1 \} - 18 (ac)(dbefg)_1 \\ \left. + 18 (ae)(cbdfg)_1 \right] = 0. \end{aligned} \quad (XIX.)$$

From $\{(cdefg)\} \left[\frac{1}{2} (ab)(cdefg)_1 + (bc)(adefg)_1 \right] = 0$,
we obtain

$$\{(fg), (ac)(de), (ae)(cd)\} \left[4 (ab)(cdefg)_1 + 2 (bf)(cdacg)_1 \right] = 0.$$

Similarly,

$$\{(fg), (ac)(de), (ae)(cd)\} \left[4 (ab)(cedfg)_1 + 2 (bf)(ceadg)_1 \right] = 0.$$

$$\text{Also } \{(fg), (ac)(de), (ae)(cd)\} \left[(bf)(duceg)_1 + (fg)(bldae)_1 \right. \\ \left. + (fa) \{ 2 (edbcg)_1 + 2 (bdceg)_1 \} \right] = 0$$

$$\text{and } \{(fg), (ac)(de), (ae)(cd)\} \left[(bf)(ceadg)_1 + (fg)(deacb)_1 \right. \\ \left. + (fa) \{ 2 (dbecg)_1 + 2 (cedbg)_1 \} \right] = 0.$$

Using these four equations, (XIX.) becomes

$$\begin{aligned} \{(fg), (ac)(de), (ae)(cd)\} \left[(fg) \{ -2 (bedca)_1 - 14 (bcdea)_1 - 8 (bedca)_1 \} \right. \\ + (af) \{ 4 (ebdgc)_1 + 4 (bcdcg)_1 - 8 (cedbg)_1 - 8 (debeg)_1 \\ \left. - 20 (dcebg)_1 - 20 (ecbdg)_1 \} \right. \\ \left. - 18 (ac)(dbefg)_1 + 18 (ae)(cbdfg)_1 \right] = 0. \end{aligned} \quad (XX.)$$

But, from (XVIII.),

$$\{(fg), (ac)(de), (ae)(cd)\} [2(fg)(bedca)_1 - 4(af)\{(bedg)_1 + (dbceg)_1\} \\ - 2(ac)(dbefg)_1 - 2(ad)(cbefg)_1 + 4(ae)(cbdfg)_1] = 0.$$

Then (XX.) is merely the sum of numerical multiples of this, and two other equations similarly obtained, after interchanging c and e , and c and a , respectively in (XVIII.).

Take next the equation

$$[b, cd, ef, ga,] = R;$$

this may be written

$$\{(ce)(df), (ceg)(dfa)\} [6(abcdefg) - 6(ef)(abcdg) - 3(abg)(cdef) \\ + 3(abg)(cd)(ef)] = 0,$$

or, putting in the value of $(abcdefg)$,

$$\{(ce)(df), (ceg)(dfa), (cd)(ef)(ga)\} [(bc)\{20(defga)_1 + 16(deafg)_1 \\ + 12(dfgea)_1\} \\ + (cd)\{-20(befga)_1 + 8(beatg)_1\} + (ce)\{24(bgfda)_1 + 12(bfdga)_1\} \\ + (ef)\{4(bdega)_1 + 48(begda)_1 - 8(bglea)_1 + 4(bdgae)_1\}] = 0.$$

Eliminate the terms $(bc)(\dots)_1$, from this by means of (XVII.), and the equation reduces to

$$\{(ce)(df), (ceg)(dfa), (cd)(ef)(ga)\} [20(cd)\{(begaf)_1 - (befga)_1\} \\ + 20(ce)\{2(begda)_1 - (bglea)_1 - (bdgae)_1\}] = 0.$$

And this equation is merely the result of operating on (XVIII.) with $\{(ce)(df), (ceg)(dfa), (cd)(ef)(ga)\}$. The equation

$$[b, cd, ef, ga,] = R,$$

then, gives no fresh syzygy.

We may proceed in exactly the same way with the equation

$$[abcd\dots] = R,$$

with the same result. But the following is better. Suppose the quartics $e_{\rho}^2, f_{\rho}^2, g_{\rho}^2$, to be all the same; and in the invariant forms denote them all by the same letter e . (The form of the equation under consideration is practically equivalent to this supposition.)

$$\text{Then } [abc, de, e, e,] = 2(abdeee) + 2(abceee) + 2(abceee),$$

$$[abc, ed, e, e,] = 2(abceee) + 2(abceee) + 2(abceee),$$

$$[abc, ce, d, e,] = 2(abceee) + 2(abceee)$$

$$+ (abceee) + (abceee);$$

$$\text{therefore } 2[abc, ee, d, e,] + [abc, de, e, e,] - 2[abc, ed, e, e,] \\ = 6(abcddee) = [abcd...].$$

If the values of these expressions in terms of irreducible types be substituted, the equation is still identically true. Hence the equation

$$[abcd...] = R$$

can give us nothing more than is given by

$$[abc, de, f, g,] = R.$$

The above proof holds for types of any degree, and so we conclude that this form of equation never requires special consideration.

The expression for $(abcdefg)$ may be simplified with the help of (XVII.). Operating with $\{(abcdefg), (bg)(cf)(de)\}$ on the various syzygies of the form (XVII.), we obtain five independent relations; by means of these (XIV.) reduces to

$$\begin{aligned} & 36(abcdefg) \\ = & \{(abcdefg), (bg)(cf)(de)\} [6(ab)(cdefg)_1 - 3(ac)(bdefg)_1 \\ & - 3(ad)(cbefg)_1 - \frac{1}{7}^5(ab) | CDEFG | - \frac{3}{4}^2(ac) | BDEFG | \\ & + \frac{1}{4}^5(ad) | CBEFG | + (abc) \{6(defg)' - \frac{3}{2}(degf)'\} \\ & + (abd) \{6(cfeg)' - 3(cefg)'\} - 3(abe)(cdfg)' \\ & + (ace) \{3(bdfg)' - \frac{3}{2}(bfldg)'\} + (abc) \{(de)(fg) + (df)(ge) + (dg)(ef)\} \\ & - (abd) \{(ce)(fg) + (cf)(ge) + (cg)(ef)\} \\ & + (abe) \{\frac{5}{2}(cd)(fg) - \frac{1}{2}(cf)(dg) - \frac{1}{2}(cg)(df)\} \\ & + (ace) \{-2(bd)(fg) + (bf)(gd) + (bg)(df)\}] \\ = & \{(abcdefg), (bg)(cf)(de)\} [6(ab)(cdefg) - 3(ac)(bdefg) \\ & - 3(ad)(cbefg) + 6(abc)(defg) - 6(abd)(cefg) - 3(abe)(cdfg) \\ & - 3(ace)(bfldg) + (abc) \{- (dg)(ef) - 4(de)(fg) + 2(df)(eg)\} \\ & + (abd) \{4(ce)(fg) + 4(cg)(ef) - 2(cf)(ge)\} \\ & + (abe) \{- (cd)(fg) - (cg)(df) + 2(cf)(dg)\} \\ & + \frac{1}{2}(ace) \{(bd)(fg) + (bf)(gd) + (bg)(df)\} + \frac{6}{7}(ab) | CDEFG | \\ & - \frac{1}{7}^2(ac) | BDEFG | + \frac{1}{7}^2(ad) | BCEFG |]. \end{aligned}$$

5. From (XVII.), syzygies of higher degree may be deduced. Thus, if we operate on (XVII.) with

$$[gh][hk][kl]\left[l_0\frac{\partial}{\partial g_0}+l_1\frac{\partial}{\partial g_1}+l_2\frac{\partial}{\partial g_2}\right],$$

we obtain at once $(fghkl)(abcde) = \Sigma P_3$,

where ΣP_3 is used, as before, to denote a sum of products of forms, there being at least three forms in each product. Again, by means of syzygies deduced from (XVII.) it is possible to show that $I_4 I_5 = \Sigma P_4$.

Thus, operate on (XVII.) with $[hg]\left[h_0\frac{\partial}{\partial g_0}+h_1\frac{\partial}{\partial g_1}+h_2\frac{\partial}{\partial g_2}\right];$

$$2(fgh)(abcde) + \{(abcde)\}[(af)\{(hdcegh) + (hdcehg)\} + (agfh)(hced) - (abf)\{(edchg) + (edchg)\} - (acf)\{(dbegh) + (dbehg)\}]\} = \Sigma P_3.$$

Operate on this with $\{f, c, d, e\}'$; then

$$\{f, c, d, e\}'\{g, h\}[(fgh)(abcde) + \frac{1}{2}(af)\{|BEGD, CH| + |CHBE, GD|\} - \frac{1}{2}(bf)\{|CAGE, DH| + |DHCA, GE|\}] = \Sigma P_3.$$

Hence

$$\{f, c, d, e\}'\{a, b\}'\{g, h\}[(fgh)(abcde) + \frac{2}{3}(af)\{|CEGD, BH| + |BECD, GH|\}] = \Sigma P_3;$$

and therefore

$$\{f, c, d, e\}'\{a, b\}'\{g, h\}[(fgh)(abcde) + \frac{1}{6}(ag)|CEFD, BH| + \frac{1}{3}(agh)|BECD, FH|] = \Sigma P_3.$$

Operate now with $[fk]\left[k_0\frac{\partial}{\partial f_0}+k_1\frac{\partial}{\partial f_1}+k_2\frac{\partial}{\partial f_2}\right],$ and we obtain

$$\{c, d, e\}'\{a, b\}'\{g, h\}[-(fgh)(abcde) + (cgh)\{(abfkd) + (abhked) + (abdkfe) + (abdfke) + (abedfk) + (abedkf)\} + \frac{4}{3}(agh)|BECD, FK|] = \Sigma P_3.$$

Or, since $(cgh)|AKEB, DF| - (dgh)|AKEB, CF| = \Sigma P_3,$

$$\{c, d, e\}'\{a, b\}'\{f, k\}[-(fgh)(abcde) + \frac{4}{3}(agh)|BECD, FK| + (cgh)\{|BEDF, KA| + |BFED, KA| + |EDKA, BF| + |EKAD, FB|\}] = \Sigma P_3;$$

and therefore

$$\{c, d, e\}'\{a, b\}'\{f, k\}[-(fgh)(abcde)] = \Sigma P_3.$$

Hence $(fghk)\{(abcde)+(abdec)+(abecd)\} = \Sigma P_3.$

Similarly, $(fghk)\{(abcde)+(cabde)+(bcaed)\} = \Sigma P_3;$

therefore $(fghk)\{(aberd)-(abdce)\} = \Sigma P_3,$

and hence $(fghk)(abcde) = \Sigma P_3$

Thus the product of any pair of the invariant types I_4, I_5, I_6 —with the exception of I_4, I_4 —is expressible in the form ΣP_3 . But the product of I_2 or I_3 and any other invariant type is not so expressible.

Covariant syzygies may be obtained from (XVII.) by means of combinations of the operations just used with operations of the forms

$$\left[x_2 \frac{\partial}{\partial a_0} - x_1 x_2 \frac{\partial}{\partial a_1} + x_1^2 \frac{\partial}{\partial a_2} \right]^2$$

$$\text{and } \left[(-a_0 x_1 x_2 - a_1 x_2^2) \frac{\partial}{\partial b_0} + (a_0 x_1^2 - a_2 x_2^2) \frac{\partial}{\partial b_1} + (a_1 x_1^2 + a_2 x_1 x_2) \frac{\partial}{\partial b_2} \right]^2.$$

The search for covariant syzygies is, however, much simplified by noticing that each of the irreducible types, ${}_2C_4, {}_2C_5, {}_4C_3, {}_4C_4, {}_6C_2, {}_6C_3$, may be expressed in the form $R + \Sigma J$, where J denotes a Jacobian, and R a sum of products of concomitants; and therefore the product of any pair of these types may be expressed in the form ΣP_3 .

On the Differentiation of Single Theta-Functions. By Rev.

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1. The definition I adopt for the single theta-functions differs somewhat from that which is ordinarily used. Thus, taking the three equations

$$\left. \begin{aligned} \Theta_1(x) &= \Theta(x) \operatorname{sn} x \\ \Theta_2(x) &= \Theta(x) \operatorname{cn} x \\ \Theta_3(x) &= \Theta(x) \operatorname{dn} x \end{aligned} \right\}, \quad (1)$$

we at once obtain

$$\Theta \frac{d\Theta_1}{dx} - \Theta_1 \frac{d\Theta}{dx} = \Theta_2 \Theta_3,$$

and five others, which may be thus represented :

$$\left. \begin{aligned} M(\Theta\Theta_1) &= (d'-d)\Theta\Theta_1 = \Theta_3\Theta_3 \\ M(\Theta_3\Theta_1) &= (d'-d)\Theta_3\Theta_1 = \Theta_3\Theta \\ M(\Theta_3\Theta_2) &= (d'-d)\Theta_3\Theta_2 = \Theta\Theta_2 \\ M(\Theta_3\Theta_3) &= (d'-d)\Theta_3\Theta_3 = k^2\Theta\Theta_1 \\ M(\Theta_3\Theta) &= (d'-d)\Theta_3\Theta = k^2\Theta_2\Theta_1 \\ M(\Theta_2\Theta) &= (d'-d)\Theta_2\Theta = \Theta_3\Theta_1 \end{aligned} \right\} \quad (2)$$

The symbol M^n being defined by $M^n(XX')$

$$= (d'-d)^n XX'$$

$$= X \frac{d^n X'}{dx^n} - \dots + (-1)^r \frac{n!}{r!(n-r)!} \frac{d^r X}{dx^r} \frac{d^{n-r} X'}{dx^{n-r}} + \dots + (-1)^n \frac{d^n X}{dx^n} X', \quad (3)$$

we further obtain

$$\left. \begin{aligned} \frac{d^2}{dx^2}(\log \Theta_1 - \log \Theta) &= -\frac{1}{\operatorname{sn}^2 u} + k^2 \operatorname{sn}^2 u \\ \frac{d^2}{dx^2}(\log \Theta_2 - \log \Theta) &= -\frac{\operatorname{dn}^2 u}{\operatorname{cn}^2 u} + k^2 \operatorname{sn}^2 u \\ \frac{d^2}{dx^2}(\log \Theta_3 - \log \Theta) &= -\frac{k^2 \operatorname{cn}^2 u}{\operatorname{dn}^2 u} + k^2 \operatorname{sn}^2 u \end{aligned} \right\}, \quad (4)$$

$$\text{and, taking} \quad \frac{d^2}{dx^2} \log \Theta = C - k^2 \operatorname{sn}^2 u \quad (5)$$

and integrating twice, determining the constants so that $\Theta(x)$ is an even function and $\Theta 0$ has a constant value conveniently assigned to it, we have Θ , Θ_1 , Θ_2 , Θ_3 defined.

In Cayley's *Elementary Treatise* C has for value $1-E/K$, and $\Theta(x)$ is a function of x and k^2 . In this memoir $\Theta(x)$ is a function of x and of three constants $\Theta 0$, k^2 , and C .

We note the equations

$$\left. \begin{aligned} M^2(\Theta\Theta) &= 2(C\Theta^2 - k^2\Theta_1^2) \\ M^2(\Theta_1\Theta_1) &= 2(C\Theta_1^2 - \Theta^2) \\ M^2(\Theta_2\Theta_2) &= 2(C\Theta_2^2 - \Theta_3^2) \\ M^2(\Theta_3\Theta_3) &= 2(C\Theta_3^2 - k^2\Theta_2^2) \end{aligned} \right\}. \quad (6)$$

Taking the first of these, we find, successively,

$$\frac{1}{\Theta} \frac{d^3 \Theta}{dx^3} - \frac{3}{\Theta^2} \frac{d\Theta}{dx} \frac{d^2 \Theta}{dx^2} + \frac{2}{\Theta^3} \left(\frac{d\Theta}{dx} \right)^3 = -2k^2 \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x,$$

$$\begin{aligned} & \frac{1}{\Theta} \frac{d^4 \Theta}{dx^4} - \frac{4}{\Theta^2} \frac{d\Theta}{dx} \frac{d^3 \Theta}{dx^3} + \frac{3}{\Theta^3} \left(\frac{d^2 \Theta}{dx^2} \right)^2 \\ &= 6 \left\{ \frac{1}{\Theta} \frac{d^2 \Theta}{dx^2} - \frac{1}{\Theta^2} \left(\frac{d\Theta}{dx} \right)^2 \right\}^2 - 2k^2 (\operatorname{cn}^2 x \operatorname{dn}^2 x - \operatorname{sn}^2 x \operatorname{dn}^2 x - k^2 \operatorname{sn}^2 x \operatorname{cn}^2 x) \\ &= 6C^2 - 2k^2 - 4(3C - 1 - k^2) k^2 \operatorname{sn}^2 x, \end{aligned}$$

$$\text{and} \quad M^4(\Theta\Theta) = 2(6C^2 - 2k^2) \Theta^2 - 8(3C - 1 - k^2) k^2 \Theta_1^2, \quad (7)$$

$$M^4(\Theta\Theta) - 4(3C - 1 - k^2) M^2(\Theta\Theta) + 4\{3C^2 - 2(1 + k^2)C + k^2\} \Theta^2 = 0, \quad (8)$$

Θ , Θ_1 , Θ_2 , Θ_3 each satisfy this differential equation.

2. If we take

$$X = A \exp \left(\frac{1}{2} \mu m^2 x^2 + 2bm x \right) \Theta(mx + \alpha), \quad (9)$$

$$\text{so that} \quad \frac{1}{X} \frac{dX}{dx} = \mu m^2 x + 2bm + \frac{d}{dx} \log \Theta(mx + \alpha),$$

$$\frac{1}{X} \frac{d^2 X}{dx^2} - \frac{1}{X^2} \left(\frac{dX}{dx} \right)^2 = \mu m^2 + \frac{d^2}{dx^2} \log \Theta(mx + \alpha),$$

we shall obtain, as before,

$$\begin{aligned} M^4(XX) - 4m^2 \{3(C + \mu) - 1 - k^2\} M^2(XX) \\ + 4m^4 \{3(C + \mu)^2 - 2(1 + k^2)(C + \mu) + k^2\} X^2 = 0, \quad (10) \end{aligned}$$

the μ merging into $C + \mu$. So (9) is the complete integral of (10), four fresh constants, A , b , m , α , being introduced.

3. If we require to solve the equation

$$M^4(XX) = 0, \quad (11)$$

$$\text{taking} \quad k^2 = \kappa^2 = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \left. \vphantom{\begin{matrix} k^2 = \kappa^2 = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \\ 3(C + \mu) = 1 + k^2 \end{matrix}} \right\}, \quad (12)$$

and

$$3(C + \mu) = 1 + k^2$$

we have

$$1 - \kappa^2 + \kappa^4 = 0$$

and

$$3(C + \mu)^2 - 2(1 + k^2)(C + \mu) + k^2 = 0,$$

and

$$X = A e^{2mbx} \Theta(mx + \alpha) \quad (13)$$

will be the complete integral of (11), C , k^2 being determined by (12).

[(11) admits of three remarkable solutions, viz.,

$$(a) \quad X = x - \frac{A_7 x^7}{7!} + \frac{A_{13} x^{13}}{13!} - \dots, \quad (14)$$

$$(b) \quad X = 1 - \frac{B_3 x^3}{3!} + \frac{B_6 x^6}{6!} - \dots, \quad (15)$$

$$(c) \quad X = mx + c.$$

If we call the series (a) $\text{th}_1 x$, we shall have

$$\text{th}_1 \kappa^2 x = \kappa^2 \text{th}_1 x, \quad \text{th}_1 \kappa^4 x = \kappa^4 \text{th}_1 x; \quad (16)$$

$$\text{th} \kappa^2 x = \text{th}_2 x, \quad \text{th} \kappa^4 x = \text{th}_3 x; \quad (17)$$

th_2, th_3 , being what Θ_2, Θ_3 , become for the values of C, k^2 , given by (12). The further consideration of equation (11) must be postponed, the object of the present memoir being to establish the following propositions.]

4. We have, generally, corresponding to (2),

$$\left. \begin{aligned} M^{2n+1}(\Theta\Theta_1) &= W(2n+1) \Theta_2\Theta_3 \\ M^{2n+1}(\Theta_2\Theta_1) &= W_2(2n+1) \Theta_3\Theta \\ M^{2n+1}(\Theta_3\Theta_1) &= W_3(2n+1) \Theta\Theta_2 \\ M^{2n+1}(\Theta_2\Theta_3) &= W'(2n+1) \Theta\Theta_1 \\ M^{2n+1}(\Theta_3\Theta) &= W'_2(2n+1) \Theta_2\Theta_1 \\ M^{2n+1}(\Theta_1\Theta) &= W'_3(2n+1) \Theta_3\Theta_1 \end{aligned} \right\}, \quad (18)$$

and

$$\left. \begin{aligned} M^{2n}(\Theta\Theta_1) &= W(2n) \Theta\Theta_1 \\ M^{2n}(\Theta_2\Theta_1) &= W_2(2n) \Theta_2\Theta_1 \\ M^{2n}(\Theta_3\Theta_1) &= W_3(2n) \Theta_3\Theta_1 \\ M^{2n}(\Theta_2\Theta_3) &= W'(2n) \Theta_2\Theta \\ M^{2n}(\Theta_3\Theta) &= W'_2(2n) \Theta_3\Theta \\ M^{2n}(\Theta_1\Theta) &= W'_3(2n) \Theta_1\Theta \end{aligned} \right\}. \quad (19)$$

These W functions are all independent of x , being constants, functions of C and k^2 .

[Also we have, corresponding to (6),

$$\left. \begin{aligned} M^{2n}(\Theta\Theta) &= U(2n)\Theta^2 = 2V(2n)\Theta^2 - 2V'(2n)\Theta_1^2 \\ M^{2n}(\Theta_1\Theta_1) &= U_1(2n)\Theta_1^2 = 2V_1(2n)\Theta_1^2 - 2V'_1(2n)\Theta^2 \\ M^{2n}(\Theta_2\Theta_2) &= U_2(2n)\Theta_2^2 = 2V_2(2n)\Theta_2^2 + 2V'_2(2n)\Theta_3^2 \\ M^{2n}(\Theta_3\Theta_3) &= U_3(2n)\Theta_3^2 = 2V_3(2n)\Theta_3^2 - 2V'_3(2n)\Theta_2^2 \end{aligned} \right\}. \quad (20)$$

$M^{2n+1}(\Theta\Theta)$, &c., all, of course, vanish.

These V functions are, also, all independent of x , and functions of θ' and k^2 , to be determined.]

5. In what follows the symbols Z, Z_1, Z_2, Z_3 and L, L_1, L_2, L_3 are defined as follows:—

$$\left. \begin{aligned} \Theta Z &= \frac{d\Theta}{dx}, & \Theta_1 Z_1 &= \frac{d\Theta_1}{dx} \\ \Theta_2 Z_2 &= \frac{d\Theta_2}{dx}, & \Theta_3 Z_3 &= \frac{d\Theta_3}{dx} \end{aligned} \right\}, \quad (21)$$

$$\left. \begin{aligned} L_1 &= Z_1 + Z + Z_2 + Z_3 \\ L &= Z_1 + Z - Z_2 - Z_3 \\ L_2 &= Z_1 - Z - Z_2 + Z_3 \\ L_3 &= Z_1 - Z + Z_2 - Z_3 \end{aligned} \right\}; \quad (22)$$

$$\text{so that} \quad L \frac{dL}{dx} = L_2 \frac{dL_2}{dx} = L_3 \frac{dL_3}{dx} = -L_1 L_2 L_3 \quad (23)$$

$$\text{and} \quad \frac{dL_1}{dx} + L^2 = 4C, \quad L^2 - L_2^2 = 4, \quad L_1^2 - L_3^2 = 4k^2, \quad (24)$$

or we may define L, L_2, L_3 by the equations

$$L = \frac{2}{\text{sn } 2u}, \quad L_2 = \frac{2 \text{ cn } 2u}{\text{sn } 2u}, \quad L_3 = \frac{2 \text{ dn } 2u}{\text{sn } 2u}. \quad (25)$$

[By using L, L_2, L_3 , we can pass from one set of formulæ to another very easily, and the expressions obtained will be more elegant.

6. We might proceed by first proving that the V 's are constant, and, from that, that the W 's are. But, as the following theorem is

true for all functions which admit of successive differentiation, and so is generally true, and holds for all functions, and not theta-functions only, we shall first establish it, and then show that (18) (19), and (20) must be true, since theta-functions are holomorphic, and Θu always finite when u is finite.]

Suppose that, X, X_1 being any functions of x capable of successive differentiation with respect to x ,

$$\frac{dX}{dx} = X(Y + \mu x), \quad \frac{dX_1}{dx} = X_1(Y_1 + \mu x), \quad (26)$$

and that R_n is generally defined by the equation

$$XX_1 R_n = M^n (XX_1); \quad (27)$$

then we have as follows:—

$$\left. \begin{aligned} R_1 &= Y_1 - Y \\ R_2 + \frac{dR_1}{dx} &= 2 \frac{dY_1}{dx} + R_1(Y_1 - Y) + 2\mu \\ R_3 - \frac{dR_1}{dx} &= 2 \frac{dY}{dx} + R_1(Y_1 - Y) + 2\mu \\ \dots &\dots \dots \dots \dots \dots \end{aligned} \right\} \quad (28)$$

$$\left. \begin{aligned} R_{n+1} + \frac{dR_n}{dx} &= 2 \frac{d^n Y_1}{dx^n} + 2n R_1 \frac{d^{n-1} Y_1}{dx^{n-1}} + \dots \\ &\dots + \frac{2(n)!}{r!(n-r)!} R_r \frac{d^{n-r} Y_1}{dx^{n-r}} + \dots + 2n R_{n-1} \frac{d(Y_1 + \mu x)}{dx} + R_n(Y_1 - Y) \\ (-1)^{n+1} \left(R_{n+1} - \frac{dR_n}{dx} \right) &= 2 \frac{d^n Y}{dx^n} - 2n R_1 \frac{d^{n-1} Y}{dx^{n-1}} + \dots \\ &\dots + (-1)^r \frac{2(n)!}{r!(n-r)!} R_r \frac{d^{n-r} Y}{dx^{n-r}} + \dots \\ &\dots + (-1)^{n-1} R_{n-1} \frac{d(Y + \mu x)}{dx} + R_n(Y_1 - Y) \end{aligned} \right\} \quad (29)$$

where we shall have

$$\frac{dR_n}{d\mu} = n(n-1) R_{n-2}. \quad (30)$$

7. To establish (29), we proceed as follows:—

$$\begin{aligned}
 M^{2n+1}(XX_1) &= X \frac{d^{2n+1}X_1}{dx^{2n+1}} - \frac{dX}{dx} \frac{d^{2n}X_1}{dx^{2n}} \\
 &\quad - 2n \left(\frac{dX}{dx} \frac{d^{2n}X_1}{dx^{2n}} - \frac{d^2X}{dx^2} \frac{d^{2n-1}X_1}{dx^{2n-1}} \right) + \dots \\
 &\quad \dots + (-1)^r \frac{(2n)!}{r!(2n-r)!} \left(\frac{d^rX}{dx^r} \frac{d^{2n-r+1}X_1}{dx^{2n-r+1}} - \frac{d^{r+1}X}{dx^{r+1}} \frac{d^{2n-r}X_1}{dx^{2n-r}} \right) + \dots \\
 &\quad \dots - 2n \left(\frac{d^{2n-1}X}{dx^{2n-1}} \frac{d^2X_1}{dx^2} - \frac{d^{2n}X}{dx^{2n}} \frac{dX_1}{dx} \right) \\
 &\quad + \frac{d^{2n}X}{dx^{2n}} \frac{dX_1}{dx} - \frac{d^{2n+1}X}{dx^{2n+1}} X_1.
 \end{aligned}$$

In each of these terms we develop the first term by applying Leibnitz's theorem to the second factor, obtaining, for a general term,

$$(-1)^r \frac{(2n)!}{r! p! (2n-2r-p)!} \frac{d^rX}{dx^r} \frac{d^pX_1}{dx^p} \frac{d^{2n-r-p}(Y_1+\mu x)}{dx^{2n-r-p}};$$

for

$$\frac{d^{2n-r}}{dx^{2n-r}} \{X_1(Y_1+\mu x)\} = \dots + \frac{(2n-r)!}{p!(2n-r-p)!} \frac{d^pX_1}{dx^p} \frac{d^{2n-r-p}(Y_1+\mu x)}{dx^{2n-r-p}} + \dots$$

In the second term of each of the above terms, we apply Leibnitz's theorem to the first factor, obtaining, for a general term,

$$-(-1)^r \frac{(2n)!}{(2n-r)! q! (r-q)!} \frac{d^qX}{dx^q} \frac{d^{r-q}(Y+\mu x)}{dx^{r-q}} \frac{d^{2n-r}X_1}{dx^{2n-r}}.$$

Now consider the series

$$\begin{aligned}
 &XX_1 \frac{d^{2n}(Y_1-Y)}{dx^{2n}} + 2n \left(X \frac{dX_1}{dx} - \frac{dX}{dx} X_1 \right) \frac{d^{2n-1}(Y_1+Y)}{dx^{2n-1}} + \dots \\
 &\quad \dots + \frac{(2n)!}{(2s)!(2n-2s)!} \left\{ X \frac{d^{2s}X_1}{dx^{2s}} - 2s \frac{dX}{dx} \frac{d^{2s-1}X_1}{dx^{2s-1}} + \dots + \frac{d^{2s}X}{dx^{2s}} X_1 \right\} \\
 &\quad \quad \quad \times \frac{d^{2n-2s}(Y_1-Y)}{dx^{2n-2s}} \\
 &\quad + \frac{(2n)!}{(2s+1)!(2n-2s-1)!} \left\{ X \frac{d^{2s+1}X_1}{dx^{2s+1}} - \dots - \frac{d^{2s+1}X}{dx^{2s+1}} X_1 \right\} \\
 &\quad \quad \quad \times \frac{d^{2n-2s-1}(Y_1+Y+2\mu x)}{dx^{2n-2s-1}} + \dots \\
 &\quad \dots + \left\{ X \frac{d^{2n}X_1}{dx^{2n}} - \dots + \frac{d^{2n}X}{dx^{2n}} X_1 \right\} (Y_1-Y);
 \end{aligned}$$

and the two developments are seen to be identical, and $M^{2n+2}(XX_1)$ can be developed in like manner.

[The identity of the two developments is readily seen by successively developing $M^3(XX_1)$, $M^4(XX_1)$, ... according to the directions.]

8. Hence we have, successively,

$$\begin{aligned}
 R_0 &= 1, \\
 R_1 &= Y_1 - Y, \\
 R_2 &= \frac{d(Y_1 + Y + 2\mu x)}{dx} + R_1(Y_1 - Y), \\
 R_3 &= \frac{d^2(Y_1 - Y)}{dx^2} + 2R_1 \frac{d(Y_1 + Y + 2\mu x)}{dx} + R_2(Y_1 - Y), \\
 &\dots \dots \dots \dots \dots \dots \dots \dots \dots \\
 R_{2n} &= \frac{d^{2n-1}(Y_1 + Y)}{dx^{2n-1}} + (2n-1) R_1 \frac{d^{2n-2}(Y_1 - Y)}{dx^{2n-2}} + \dots \\
 &\dots + \frac{(2n-1)!}{(2r+1)!(2n-2r-2)!} R_{2r+1} \frac{d^{2n-2r-2}(Y_1 - Y)}{dx^{2n-2r-2}} \\
 &\quad + \frac{(2n-1)!}{(2r+2)!(2n-2r-3)!} R_{2r+2} \frac{d^{2n-2r-3}(Y_1 + Y)}{dx^{2n-2r-3}} + \dots \\
 &\dots + (2n-1) R_{2n-2} \frac{d(Y_1 + Y + 2\mu x)}{dx} + R_{2n-1}(Y_1 - Y) \\
 R_{2n+1} &= \frac{d^{2n}(Y_1 - Y)}{dx^{2n}} + 2n R_1 \frac{d^{2n-1}(Y_1 + Y)}{dx^{2n-1}} + \dots \\
 &\dots + \frac{(2n)!}{(2r)!(2n-2r)!} R_{2r} \frac{d^{2n-2r}(Y_1 - Y)}{dx^{2n-2r}} \\
 &\quad + \frac{(2n)!}{(2r+1)!(2n-2r-1)!} R_{2r+1} \frac{d^{2n-2r-1}(Y_1 + Y)}{dx^{2n-2r-1}} + \dots \\
 &\dots + 2n R_{2n-1} \frac{d(Y_1 + Y + 2\mu x)}{dx} + R_{2n}(Y_1 - Y)
 \end{aligned} \tag{31}$$

9. Differentiating, we obtain,

$$\begin{aligned}
 \frac{dR_1}{dx} &= \frac{d(Y_1 - Y)}{dx}, \\
 \frac{dR_2}{dx} &= \frac{d^3(Y_1 + Y)}{dx^3} + 2R_1 \frac{d(Y_1 - Y)}{dx},
 \end{aligned}$$

and we observe the law

$$\begin{aligned}
 \frac{dR_{2r}}{dx} &= \frac{d^{2r}(Y_1+Y)}{dx^{2r}} + 2rR_1 \frac{d^{2r-1}(Y_1-Y)}{dx^{2r-1}} + \dots \\
 &\dots + \frac{(2r)!}{(2s)!(2r-2s)!} R_{2s} \frac{d^{2r-2s}(Y_1+Y)}{dx^{2r-2s}} \\
 &\quad + \frac{(2r)!}{(2s+1)!(2r-2s-1)!} R_{2s+1} \frac{d^{2r-2s-1}(Y_1-Y)}{dx^{2r-2s-1}} + \dots \\
 &\dots + 2rR_{2r-1} \frac{d(Y_1-Y)}{dx}
 \end{aligned}
 \left. \vphantom{\begin{aligned} \frac{dR_{2r}}{dx} = \dots \end{aligned}} \right\} \quad (32)$$

$$\begin{aligned}
 \frac{dR_{2r+1}}{dx} &= \frac{d^{2r+1}(Y_1-Y)}{dx^{2r+1}} + (2r+1)R_1 \frac{d^{2r}(Y_1+Y)}{dx^{2r}} + \dots \\
 &\dots + \frac{(2r+1)!}{(2s)!(2r-2s+1)!} R_{2s} \frac{d^{2r-2s+1}(Y_1-Y)}{dx^{2r-2s+1}} \\
 &\quad + \frac{(2r+1)!}{(2s+1)!(2r-2s)!} R_{2s+1} \frac{d^{2r-2s}(Y_1+Y)}{dx^{2r-2s}} + \dots \\
 &\dots + (2r+1)R_{2r} \frac{d(Y_1-Y)}{dx}
 \end{aligned}
 \left. \vphantom{\begin{aligned} \frac{dR_{2r+1}}{dx} = \dots \end{aligned}} \right\}$$

Now suppose this law to hold up to $\frac{dR_{2r-1}}{dx}$; then

$$\begin{aligned}
 \frac{dR_{2r}}{dx} &= \frac{d^{2r}(Y_1+Y)}{dx^{2r}} + (2r-1) \left\{ R_1 \frac{d^{2r-1}(Y_1-Y)}{dx^{2r-1}} + \frac{dR_1}{dx} \frac{d^{2r-2}(Y_1-Y)}{dx^{2r-2}} \right\} + \dots \\
 &\dots + \frac{(2r-1)!}{(2s+1)!(2r-2s-2)!} \left\{ R_{2s+1} \frac{d^{2r-2s-1}(Y_1-Y)}{dx^{2r-2s-1}} + \frac{dR_{2s+1}}{dx} \frac{d^{2r-2s-2}(Y_1-Y)}{dx^{2r-2s-2}} \right\} \\
 &\quad + \frac{(2r-1)!}{(2s+2)!(2r-2s-3)!} \left\{ R_{2s+2} \frac{d^{2r-2s-2}(Y_1+Y)}{dx^{2r-2s-2}} + \frac{dR_{2s+2}}{dx} \frac{d^{2r-2s-3}(Y_1+Y)}{dx^{2r-2s-3}} \right\} + \dots \\
 &\dots + R_{2r-1} \frac{d(Y_1-Y)}{dx} + \frac{dR_{2r-1}}{dx} (Y_1-Y).
 \end{aligned}$$

Substituting in this for ..., $\frac{dR_{2s}}{dx}$, $\frac{dR_{2s+1}}{dx}$, ..., and collecting together

all the terms containing $\frac{d^{2r-2s}(Y_1+Y)}{dx^{2r-2s}}$, we obtain

$$\begin{aligned} & \frac{(2r-1)!}{(2s)!(2r-2s-1)!} R_2 + \frac{(2r-1)!}{(2s-1)!(2r-2s)!} R_{2s-1} (Y_1-Y) \\ & + \frac{(2r-1)!}{1!(2r-2)!} \frac{(2r-2)!}{(2s-2)!(2r-2s)!} R_{2s-2} \frac{d(Y_1+Y+2\mu x)}{dx} + \dots \\ & \dots + \frac{(2r-1)!}{(2s-1)!(2r-2s)!} \frac{d^{2s-1}(Y_1+Y)}{dx^{2s-1}} \\ & = \frac{(2r-1)!}{(2s)!(2r-2s-1)!} R_{2s} + \frac{(2r-1)!}{(2s-1)!(2r-2s)!} R_2, \\ & = \frac{(2r)!}{(2s)!(2r-2s)!} R_{2s}, \end{aligned}$$

and similarly for the terms containing $\frac{d^{2r-2s-1}(Y_1-Y)}{dx^{2r-2s-1}}$; in like manner, the law can be shown to hold for $\frac{dR_{2r+1}}{dx}$. Thus (32) is established.

[The identity it expresses is readily seen by successively developing $\frac{dR_2}{dx}$, $\frac{dR_3}{dx}$, ... in the manner we here indicate.]

10. The formulæ (29) follow at once from (31) and (32). And (30) can be proved by induction in much the same way. We have, in succession,

$$\frac{dR_1}{d\mu} = 0, \quad \frac{dR_2}{d\mu} = 2,$$

$$\frac{dR_3}{d\mu} = 4R_1 + R_2 \frac{dR_2}{d\mu} = 6R_1,$$

and, assuming that, up till $r = n$,

$$\frac{dR_r}{d\mu} = r(r-1) R_{r-2},$$

we have $\frac{dR_{n+1}}{d\mu} + \frac{d^2 R_n}{dx d\mu} = \frac{dR_{n+1}}{d\mu} + n(n-1) \frac{dR_{n-2}}{dx},$

$$\begin{aligned} & \frac{dR_{n+1}}{d\mu} + n(n-1) \frac{dR_{n-2}}{dx} \\ &= 2 \frac{n(n-1)}{1 \cdot 2} \frac{d^{n-2} Y_1}{dx^{n-2}} + \dots + \frac{2(n)!}{r!(n-r)!} \frac{dR_r}{d\mu} \frac{d^{n-r} Y_1}{dx^{n-r}} + \dots \\ & \quad + 2n \frac{dR_{n-1}}{d\mu} \frac{d(Y_1 + \mu x)}{dx} + 2nR_{n-1} + \frac{dR'_n}{d\mu} (Y_1 - Y) \\ &= n(n-1) \left\{ 2 \frac{d^{n-2} Y_1}{dx^{n-2}} + \dots + \frac{2(n-2)!}{(r-2)!(n-r)!} R_{r-2} \frac{d^{n-2} Y_1}{dx^{n-2}} + \dots \right. \\ & \quad \left. + 2(n-2) R_{n-3} \frac{d(Y_1 + \mu x)}{dx} + R_{n-2} (Y_1 - Y) \right\} + 2nR_{n-1} \\ &= n(n-1) \left(R_{n-1} + \frac{dR_{n-2}}{dx} \right) + 2nR_{n-1}, \end{aligned}$$

so that $\frac{dR_{n+1}}{d\mu} = (n+1) nR_{n-1}.$

[11. Applying the general formulæ (29) to Θ functions, we observe that the Θ 's need not have the same argument, nor the same modulus. In fact, much more general formulæ than (18), (19), and (20) are obtainable.

To obtain (20), put $X = X_1 = \Theta(x),$

so that $Y = Y_1 = Z - \mu x;$

then we have, in succession,

$$\left. \begin{aligned} U(2) &= 2 \frac{dZ}{dx} = 2C - 2k^2 \operatorname{sn}^2 u \\ U(4) &= 2 \frac{d^3 Z}{dx^3} + 6U(2) \frac{dZ}{dx} \\ \dots & \dots \dots \dots \\ U(2n) &= 2 \frac{d^{2n-1} Z}{dx^{2n-1}} + \dots + \frac{2(2n-1)!}{(2r)!(2n-2r-1)!} U(2r) \frac{d^{2n-2r-1} Z}{dx^{2n-2r-1}} + \dots \\ \dots & \dots \dots \dots \dots \dots \dots \dots \dots \dots \end{aligned} \right\} \dots \dots \dots (33)$$

Now, Θx being *holomorphe dans toute l'étendue du plan*, it and its

derived functions are never infinite when x is finite. But, since

$$\frac{d^3}{dx^3} \operatorname{sn}^r x = r(r-1) \operatorname{sn}^{r-2} x - r^2(1+k^2) \operatorname{sn}^r x + r(r+1)k^2 \operatorname{sn}^{r+2} x,$$

$U(2n)$ develops into a finite series of even powers of $\operatorname{sn} x$. Now $U(2n)\Theta^2$ is never infinite when x is finite. So we have

$$U(2n) = 2V(2n) - 2V'(2n) \operatorname{sn}^2 x,$$

where $V(2n)$, $V'(2n)$ are constants, functions of C and k^2 , and the other equations (20) can be established in like manner.

I find that

$$U(6) = 8 \{ 15C^3 - 15k^2C + 2k^2(1+k^2) \} \\ - 8 \{ 45C^2 - 30(1+k^2)C + 4 + 11k^3 + 4k^4 \} k^3 \operatorname{sn}^4 x; \quad (34)$$

or, corresponding to (8), we may write

$$U(6) - 4(3C - 1 - k^2)U(4) - 12 \{ 3C^2 - 2(1+k^2)C + k^2 \} U(2) \\ + 96 \{ C^3 - C^2(1+k^2) + Ck^2 \} = 0. \quad (35)$$

The successive coefficients being symmetrical functions of L , L_2 , L_3 , the formulæ (8), (35) will be unaltered when Θ_1 , Θ_2 , Θ_3 take the place of Θ .]

12. In (26), putting

$$X = \Theta(x), \quad X_1 = \Theta_1(x),$$

we have

$$Y + \mu x = Z, \quad Y_1 + \mu x = Z_1;$$

then, since

$$(Z_1 - Z) \Theta \Theta_1 = \Theta_2 \Theta_3,$$

we have, in succession,

$$W(1)(Z_1 - Z) = Z_1 - Z, \\ W(2) = \frac{d(Z_1 + Z)}{dx} + W(1)(Z_1 - Z)^2, \\ \frac{dW(2)}{dx} = \frac{d^2(Z_1 + Z)}{dx^2} + 2W(1)(Z_1 - Z) \frac{d(Z_1 - Z)}{dx}, \\ W(3)(Z_1 - Z) = \frac{d^2(Z_1 - Z)}{dx^2} + 2W(1)(Z_1 - Z) \frac{d(Z_1 + Z)}{dx} \\ + W(2)(Z_1 - Z), \\ W(3) \frac{d(Z_1 - Z)}{dx} = \frac{d^3(Z_1 - Z)}{dx^3} + 3W(1)(Z_1 - Z) \frac{d^2(Z_1 + Z)}{dx^2} \\ + 3W(2) \frac{d(Z_1 - Z)}{dx},$$

$$\begin{aligned}
 W(2n) &= \frac{d^{2n-1}(Z_1+Z)}{dx^{2n-1}} + (2n-1)W(1)(Z_1-Z) \frac{d^{2n-2}(Z_1-Z)}{dx^{2n-2}} + \dots \\
 &\quad \dots + \frac{(2n-1)(2n-2)}{1 \cdot 2} W(2n-3)(Z_1-Z) \frac{d^2(Z_1-Z)}{dx^2} \\
 &\quad + (2n-1)W(2n-2) \frac{d(Z_1+Z)}{dx} \\
 &\quad + W(2n-1)(Z_1-Z)^2 \\
 \frac{dW(2n)}{dx} &= \frac{d^{2n}(Z_1+Z)}{dx^{2n}} + 2nW(1)(Z_1-Z) \frac{d^{2n-1}(Z_1-Z)}{dx^{2n-1}} + \dots \\
 &\quad \dots + \frac{2n(2n-1)(2n-2)}{1 \cdot 2 \cdot 3} W(2n-3)(Z_1-Z) \frac{d^3(Z_1-Z)}{dx^3} \\
 &\quad + \frac{2n(2n-1)}{1 \cdot 2} W(2n-2) \frac{d^2(Z_1+Z)}{dx^2} \\
 &\quad + 2nW(2n-1)(Z_1-Z) \frac{d(Z_1-Z)}{dx} \\
 (W2n+1)(Z_1-Z) &= \frac{d^{2n}(Z_1-Z)}{dx^{2n}} + 2nW(1)(Z_1-Z) \frac{d^{2n-1}(Z_1+Z)}{dx^{2n-1}} + \dots \\
 &\quad \dots + \frac{2n(2n-1)}{1 \cdot 2} W(2n-2) \frac{d^2(Z_1-Z)}{dx^2} \\
 &\quad + 2nW(2n-1)(Z_1-Z) \frac{d(Z_1+Z)}{dx} \\
 &\quad + W(2n)(Z_1-Z) \\
 \frac{d}{dx} \{ W(2n+1)(Z_1-Z) \} &= \frac{d^{2n+1}(Z_1-Z)}{dx^{2n+1}} + (2n+1)W(1)(Z_1-Z) \frac{d^{2n}(Z_1+Z)}{dx^{2n}} + \dots \\
 &\quad \dots + \frac{(2n+1)2n(2n-1)}{1 \cdot 2 \cdot 3} W(2n-2) \frac{d^3(Z_1-Z)}{dx^3} \\
 &\quad + \frac{(2n+1)2n}{1 \cdot 2} W(2n-1)(Z_1-Z) \frac{d^2(Z_1+Z)}{dx^2} \\
 &\quad + (2n+1)W(2n) \frac{d(Z_1-Z)}{dx}
 \end{aligned}$$

Here we observe that, since

$$\frac{d(Z_1 + Z)}{dx} = -\frac{1}{\operatorname{sn}^2 x} + 2C - k^2 \operatorname{sn}^2 x, \quad (37)$$

$W(2n)$ develops into a finite series of even powers, positive and negative, of $\operatorname{sn} x$. Also, since $W(2n+1)(Z_1 - Z)$ develops into such a series multiplied by $\operatorname{sn} x \operatorname{cn} x \operatorname{dn} x$, $W(2n+1)$ does so too.

Now, Θx , $\Theta_1 x$ being holomorphic throughout, and they and their derived functions never infinite for finite values of x , $M^{2n}(\Theta\Theta_1)$, $M^{2n+1}(\Theta\Theta_1)$ are likewise never infinite.

$$\text{So, if } W(2n)\Theta\Theta_1 = \left(\frac{B'}{\operatorname{sn}^2 x} + C' + D' \operatorname{sn}^2 x \right) \Theta\Theta_1,$$

we must have $B' = D' = 0$.

So $W(2n)$ is constant. In like manner, $W(2n+1)$ is constant.

Of course, then, $\frac{dW(2n)}{dx}$, $\frac{dW(2n+1)}{dx}$ vanish. So, too, the other W 's in (18), (19) are all constant.

13. If we express W , W' in terms of the L 's, we can at once find W_2 , W_3 , &c., by permutation.

For small values of n we have as follows:—

$$\left. \begin{aligned} W(1) &= 1, & W'(1) &= k^2 \\ W_2(1) &= 1, & W'_2(1) &= k^2 \\ W_3(1) &= 1, & W'_3(1) &= 1 \end{aligned} \right\}; \quad (38)$$

$$\left. \begin{aligned} W(2) &= W'(2) = 2C - 1 - k^2 \\ W_2(2) &= W'_2(2) = 2C - k^2 \\ W_3(2) &= W'_3(2) = 2C - 1 \end{aligned} \right\}, \quad (39)$$

$$\left. \begin{aligned} W(3) &= 6C - 1 - k^2 \\ W_2(3) &= 6C - 4 - k^2 \\ W_3(3) &= 6C - 1 - 4k^2 \\ W'(3) &= k^2 W(3) \\ W'_2(3) &= k^2 W_2(3) \\ W'_3(3) &= W_3(3) \end{aligned} \right\}, \quad (40)$$

$$\left. \begin{aligned} W(4) &= 12C^2 - 12(1+k^2)C + 1 + 10k^2 + k^4 \\ W_2(4) &= 12C^2 - 12k^2C + k^4 \\ W_3(4) &= 12C^2 - 12C + 1 \end{aligned} \right\}, \quad (41)$$

$$\left. \begin{aligned} W(5) &= 60C^3 - 20(1+k^2)C + 1 - 6k^2 + k^4 \\ &= 16C^3 + 24C(C-1) + 24C(C-k^2) + (C-1)^3 \\ &\quad - 6(C-1)(C-k^2) + (C-k^2)^2 \\ W_2(5) &= 16(C-1)^2 + 24(C-1)(C-k^2) + 24(C-1)C \\ &\quad + (C-k^2)^2 - 6(C-k^2)C + C^2 \\ &= 60C^3 - 20(1+k^2)C + 16 + 24k^2 + k^4 \\ W_3(5) &= 60C^3 - 20(1+k^2)C + 1 + 24k^2 + 16k^4 \end{aligned} \right\}, \quad (42)$$

$$\left. \begin{aligned} W(6) &= 120C^3 - 180(1+k^2)C^2 + 30(1+10k^2+k^4) \\ &\quad - 1 - 59k^2 - 59k^4 - k^6 \\ &= 15(2C-1-k^2)^3 - 14(2C-1-k^2)(1-k^2)^2 - 32C(1-k^2)^2 \\ W_2(6) &= 15(2C-k^2)^3 - 14(2C-k^2)k^4 - 32(C-1)k^4 \\ &= 120C^3 - 180k^2C^2 + 30k^4C + 32k^4 - k^6 \\ W_3(6) &= 120C^3 - 180C^2 + 30C - 1 + 32k^2 \end{aligned} \right\}. \quad (43)$$

Extensions of the Riemann-Roch Theorem in Plane Geometry.

By F. S. MACAULAY. Received and communicated June 14th, 1900. Received, in revised form, October 22nd, 1900.

§ 1. INTRODUCTION.

The ordinary form of the theorem known as the Riemann-Roch theorem in the geometry of plane algebraic curves is capable of generalization in more than one respect. In the following paper I have stated and proved what I believe to be the most general form of the theorem (§ 4); and have given a corresponding, but less

definite, theorem for a more extended case (§ 6). The paper forms an addition to an earlier one on "The Theorem of Residuation, &c." (*Proc. Lond. Math. Soc.*, Vol. xxxi., pp. 381-423), to which several references are made in the course of the proof (§§ 2-4). Apart, however, from any external reference, I have endeavoured in the first instance to explain what the generalized Riemann-Roch theorem is:

Suppose that two given algebraic curves C_1, C_2 , having no common factor, intersect in any manner in the plane, and that the origin is taken at one of their points of intersection. In the most general case C_1, C_2 will have multiple points of any kind at the origin, with any kind of contact. The intersection of C_1 and C_2 at the origin is called a *base-point*, whose *degree* α is the number of ordinary points to which it is equivalent. This base-point α , and the conditions which it supplies for a curve, are determined by finding its α independent equations, which are the linear equations satisfied identically by the coefficients of both C_1P_1 and C_2P_2 , where P_1, P_2 are arbitrary power series in x, y .

More generally, the condition that a given system of linear equations may be the equations of a base-point is that the equations are satisfied by the coefficients of UP as a consequence of their being satisfied by the coefficients of U , where, as before, P is arbitrary. Without entering into further details as to the character of the equations, we may add that the particular kind of base-point which is the whole intersection of two given curves at a point is called a *one-set point*, and its set of equations a *one-set system*.

Base-points include all kinds of points, viz., the ordinary point, of degree 1 and order 1, the equation of which is $z_0^0 = 0$ (z_q^p being the coefficient of $x^p y^q$ in the general polynomial of any order, so that $p \geq q \geq 0$); the ordinary i -point, of degree $\frac{1}{2}i(i+1)$ and order i , which supplies an ordinary i -fold point for a curve and no further conditions, whose equations are $z_q^p = 0$ for all values of p, q , such that $i > p \geq q$; the one-set point, which is the whole base-point common to any two curves at a point; and, in general, the t -set point, which is the whole base-point common to $t+1$ fixed curves at a point, of any order $i \geq t$ and any (finite) degree $q \geq \frac{1}{2}i(i+1)$.

A base-point is said to form part of and to be *contained in* another if both are situated at the same point and if the equations of the first are deducible from those of the second. Given any base-point, we can always find an ordinary i -point, and also a one-set point, which contains it.

A *point-base* is any group of base-points (including any number of ordinary points), all finitely separated and in the finite region of the plane; and its degree N is the sum of the degrees of its base-points. A point-base forms part of and is contained in another if each of its base-points is contained in a base-point of the other.

If S is the general algebraic curve through any given part q of a one-set point a , then the *remaining part* r of a is that base-point through which S' must pass in order that the composite curve $S+S'$ (analytically SS') may pass through a . The base-points q, r are said to be residual with respect to a , and the sum of their degrees $q+r$ is equal to a .

Similarly, if $N+N'$ is any given point-base made up of one-set points (including ordinary points), and N any given part of it, and S the general algebraic curve through N , then the remaining part N' is that point-base through which S' must pass in order that the composite curve $S+S'$ may pass through $N+N'$. The point-bases N, N' are said to be residual with respect to $N+N'$.

With this explanation it is hoped that the enunciation of the generalized Riemann-Roch theorem (§ 4) may be understood. This theorem gives all the defects and excesses of N in terms of those of N when $N+N'$ forms the whole intersection of any two given curves. The n -ic defect of N is the degree of freedom of an n -ic through N ; and the n -ic excess of N is the excess of the number N over the number of independent conditions supplied by N for an n -ic.

The generalized Riemann-Roch theorem may be thus contrasted with the ordinary theorem:—The ordinary theorem refers only to the point-base formed by the whole intersection of any given curve C_m with an adjoined curve C_{m-3} ; not to the whole intersection of any two given curves. Again, in the ordinary theorem, the point-base N , instead of being *any* part of the whole intersection $N+N'$ of C_m and C_{m-3} , has an ordinary $(i-1)$ -point at each multiple point of order i on C_m and $i-1$ on C_{m-3} ; the corresponding residual base-point of N' being then also an ordinary $(i-1)$ -point. Lastly, the ordinary theorem gives only the values of d'_{m-3} and c'_{m-3} , viz., $d'_{m-3} = c_{m-3}$ and $c'_{m-3} = d_{m-3}$.

§ 2. RESOLUTION OF BASE-POINTS INTO SIMPLE ELEMENTS.

It is easy, in practically endless ways, to resolve any given base-point q into simple elements, that is, to arrange its equations in such a way that any the first q_1 of them shall be the equations of a base-

point q_1 .* It will be sufficient to explain one way of doing this. Let $E = 0$ be any prime equation of the base-point q , and let n be the highest value of p in any z_q^p appearing in E . Arrange the derived equations in the following order:—

$$\begin{aligned} (E)_{-n}^{-n} &= 0, \\ (E)_{-n+1}^{-n} &= 0, \quad (E)_{-n+1}^{-n+1} = 0, \\ (E)_{-n+2}^{-n} &= 0, \quad (E)_{-n+2}^{-n+1} = 0, \quad (E)_{-n+2}^{-n+2} = 0, \\ \dots &\quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ (E)_{-0}^{-n} &= 0, \quad (E)_{-0}^{-n+1} = 0, \quad (E)_{-0}^{-n+2} = 0, \quad \dots, \quad (E)_{-0}^0 = 0. \end{aligned}$$

Any one of these equations has all its derivatives occurring before it. Thus the equation $(E)_{-m+1}^{-m+1} = 0$ is preceded immediately by its derivative $(E)_{-m+1}^{-m} = 0$, and at a greater distance by its derivative $(E)_{-m}^{-m} = 0$. Hence, if we take the equations in this order, and reject all those which are deducible from equations previously written, the first q_1 which remain are always the equations of a base-point q_1 . These equations may be combined in any way with the derivatives of all the other prime equations of q ; it being only necessary that the derivatives of the same prime equation preserve the right order among themselves. The reasoning is not affected if two different prime equations have several of their derivatives the same.

In the case of a one-set point $q+r$ a further property can be proved, viz., that its $q+r$ independent equations can be so arranged that the first q of them are the equations of any given base-point q contained in $q+r$, while any the first q_1 of them are the equations of a base-point q_1 . If $E = 0$ is the prime equation, and

$$E_1 = E_2 = \dots = E_{q+r} = 0$$

any set of independent equations of the one-set point, and if S and S' stand for $\sum z_q^p x^{p-q} y^q$ and $\sum z_{q'}^p x^{p-q} y^q$ respectively, then the value of E for SS' is identically

$$E_1 E_{q+r} + E_2 E_{q+r-1} + \dots + E_q E_{r+1} + E_{q+1} E_r + \dots + E_{q+r} E_1,$$

* A base-point differs in character from a point-group in respect to the fact that the simple elements from which it is built up must be regarded as having an order of arrangement.

The notation employed is the same as that in the paper of Vol. xxxi. The E 's here used have no connexion with the E_n which stands for n -ic excess (§§ 4-6).

where $F_1, F_2, \dots, F_{q+r}$ are linear homogeneous functions of the coefficients z_q^p of S' (*loc. cit.*, p. 402). The equations

$$F_1 = F_2 = \dots = F_{q+r} = 0$$

express the condition that E vanishes for SS' when S is arbitrary, i.e., they are a set of equations of the one-set point $q+r$, but with z_q^p written in the place of z_r^p . Now choose

$$E_1 = E_2 = \dots = E_q = 0$$

to be the equations of the given base-point q ; then

$$F_1 = F_2 = \dots = F_r = 0$$

are the equations of the corresponding residual base-point r (*loc. cit.*, p. 402). Further, choose $E_1, E_2, \dots, E_q$ so that any the first q_1 of them correspond to a base-point q_1 . Also choose $F_1, F_2, \dots, F_r$ so that any the first r_1 of them correspond to a base-point r_1 . This last is effected by a suitable linear transformation of $F_1, F_2, \dots, F_r$, and a corresponding linear transformation of $E_{q+1}, E_{q+2}, \dots, E_{q+r}$, which leaves the value of $E_{q+1}F_r + E_{q+2}F_{r-1} + \dots + E_{q+r}F_1$ identically the same as before. Then, since

$$F_1 = F_2 = \dots = F_{r_1} = 0 \quad (r_1 = 1, 2, \dots, r)$$

are the equations of a base-point r_1 ,

$$E_1 = E_2 = \dots = E_{q_1} = 0 \quad (q_1 = q+r-1, q+r-2, \dots, q)$$

are the equations of the corresponding residual base-point q_1 (*loc. cit.*, p. 402). Hence any the first q_1 of the equations

$$E_1 = E_2 = \dots = E_{q+r} = 0$$

are the equations of a base-point q_1 .

Also, if S passes through the base-point q_1 whose equations are $E_1 = E_2 = \dots = E_{q_1} = 0$, and S' through the base-point r_1-1 whose equations are $F_1 = F_2 = \dots = F_{r_1-1} = 0$, where $q_1+r_1 = q+r$, then SS' passes through a base-point $q+r-1$ contained in $q+r$. The value of E for SS' is in this case $E_{q_1+1}F_{r_1}$, which does not in general vanish; but all the derivatives of E vanish for SS' . For the value of $(E)^{-t}_m$ for SS' is the value of E for $x^{t-m}y^m SS'$, which is equal to $(E_{q_1+1})^{-t}_m F_{r_1}$ and to $E_{q_1+1}(F_{r_1})^{-t}_m$. Each of these vanishes, for if $r_1 \leq r$, then $(F_{r_1})^{-t}_m = 0$; and if $r_1 > r$, then $q_1+1 \leq q$ and $(E_{q_1+1})^{-t}_m = 0$. The curve SS' does not, however, pass through the whole one-set point $q+r$ unless E_{q_1+1} vanishes for S , or F_{r_1} for S' .

From this it follows that, if $N+N'$ is any point-base made up of one-set points, and if N is any given part of it, and N' the remaining part, then the conditions which $N+N'$ supplies for a curve may be resolved into simple elements $C_1, C_2, \dots, C_{N+N'},^*$ which can be chosen definitely in such a way that the first N of them are the conditions which the given point-base N supplies for a curve, and any the first N_1 of them the conditions which a point-base N_1 supplies. To these will correspond a unique residual set of conditions $C'_1, C'_2, \dots, C'_{N+N'}$, such that any the first N'_1 of them are the conditions supplied for a curve by the point base N'_1 which is residual to N_1 with respect to $N+N'$, where $N_1+N'_1 = N+N'$. Moreover, if the conditions $C_1, C_2, \dots, C_{N_1}$ are satisfied for S , and $C'_1, C'_2, \dots, C'_{N'_1-1}$ for S' , then SS' passes through a point-base $N+N'-1$ contained in $N+N'$. And SS' does not pass through the whole of $N+N'$ unless C_{N_1+1} is satisfied for S , or $C'_{N'_1}$ for S' .

§3. POINT-BASE FORMING THE WHOLE INTERSECTION OF TWO CURVES.

The $(l+m-3)$ -ic excess of the point-base lm formed by the whole intersection of two curves C_l, C_m is 1. In other words, the point-base lm supplies $lm-1$ independent conditions for a curve of order $l+m-3$. It does not necessarily follow that any point-base N forming part of the point-base lm supplies N independent conditions for a curve of order $l+m-3$. Hence the following theorem, which states this to be the fact, is of importance:—

Any curve of order $l+m-3$ through any point-base $lm-1$, forming part of the intersection of two curves C_l, C_m , passes necessarily through the whole intersection of C_l, C_m .

Any point-base $lm-1$ forming part of the point-base lm has a residual point-base of degree 1, which must therefore be an ordinary point. Take the origin at this point. Let C_{l+m-3} be any curve of order $l+m-3$ through the point-base $lm-1$. Then, since x and y are two curves through the residual point-base, we have

$$x C_{l+m-3} \equiv C_l A_{m-2} + C_m A_{l-2},$$

and

$$y C_{l+m-3} \equiv C_l B_{m-2} + C_m B_{l-2},$$

* These conditions are that certain homogeneous linear functions of the coefficients of the curve S , expressed by means of the values of S and the partial differential coefficients of S at the various base-points of $N+N'$, should vanish.

where A_{l-2} , A_{m-2} , B_{l-2} , B_{m-2} are polynomials whose orders are indicated by the suffixes. From these we have

$$x(C_l B_{m-2} + C_m B_{l-2}) \equiv y(C_l A_{m-2} + C_m A_{l-2}),$$

or
$$C_l(xB_{m-2} - yA_{m-2}) \equiv C_m(yA_{l-2} - xB_{l-2}).$$

Hence, since C_l , C_m have no common factor, and $xB_{m-2} - yA_{m-2}$ is of less order than C_m , we have

$$xB_{m-2} - yA_{m-2} \equiv 0 \quad \text{and} \quad yA_{l-2} - xB_{l-2} \equiv 0.$$

Hence A_{m-2} and A_{l-2} are both divisible by x ; and, writing xA_{m-3} , xA_{l-3} for them, we have

$$C_{l+m-3} \equiv C_l A_{m-3} + C_m A_{l-3},$$

which proves the theorem.

§ 4. THE GENERALIZED RIEMANN-ROCH THEOREM.

If the point-base forming the whole intersection of two curves C_l , C_m , which have no common factor and no intersection at infinity, is divided into any two residual point-bases N , N' , and if d_n , d'_n , D_n are the n -ic defects, and e_n , e'_n , E_n the n -ic excesses, of N , N' , $N+N'$ respectively, then $d'_n = e_n + D_n$ and $e'_n = d_n - D_n$, where $n+n' = l+m-3$.

We shall prove first that $e'_n \geq d_n - D_n$. Since d_n+1 and D_n+1 are the numbers of linearly independent curves of order n through N and $N+N'$ respectively, it is clear that $d_n - D_n \geq 0$. Also, if $d_n - D_n = 0$, then $e'_n \geq d_n - D_n$, since $e'_n \geq 0$. This shows that the inequality holds in all cases in which N does not lie on an n -ic, for then $d_n = -1 = D_n$, and in all cases in which an n -ic through N passes of necessity through the whole of $N+N'$.

If $d_n - D_n > 0$, we have $d_n \geq 0$, since $D_n \geq -1$. Also $d_n - D_n$ is the number of (extra) independent conditions which the whole point-base $N+N'$ supplies for an n -ic drawn through N . Hence an n -ic can be drawn through N , and can be made to satisfy $d_n - D_n - 1$ further independent conditions supplied by the point-base $N+N'$, without vanishing identically and without passing through whole of $N+N'$. Let the conditions that $N+N'$ supplies for a curve, viz., $C_1, C_2, \dots, C_{N+N'}$, be arranged definitely as in § 2; and suppose them applied successively to a curve of order n . Among the conditions $C_{N+1}, C_{N+2}, \dots, C_{N+N'}$ there will be $d_n - D_n$, each one of which is

not a consequence of all the preceding (including $C_1, C_2, \dots, C_N$). Let these be $C_{N_1+1}, C_{N_2+1}, \dots, C_{N_p+1}$, where $N_p > N_{p-1} > \dots > N_1 \geq N$ and $p = d_n - D_n$. The last condition C_{N_p+1} will require that the n -ic vanishes identically if $D_n = -1$, otherwise not; but an n -ic can in any case satisfy all the conditions up to C_{N_p} . Now apply the residual set of conditions $C'_1, C'_2, \dots, C'_{N'}$ to a curve of order n' . Then the condition $C'_{N'_p}$, where $N_p + N'_p = N + N'$, will be satisfied as a consequence of the conditions $C'_1, C'_2, \dots, C'_{N'_p-1}$. For an n -ic which satisfies the conditions $C_1, C_2, \dots, C_{N_p}$ and an n' -ic which satisfies the conditions $C'_1, C'_2, \dots, C'_{N'_p-1}$ make up an $(n+n')$ -ic which passes through a point-base $N+N'-1$ contained in $N+N'$ (§ 2). This must pass through the whole of $N+N'$ (§ 3). Hence one of the conditions $C_{N_p+1}, C'_{N'_p}$ must be satisfied (§ 2); and since, by hypothesis, the condition C_{N_p+1} is not necessarily satisfied by the n -ic, the condition $C'_{N'_p}$ is necessarily satisfied by the n' -ic. In the same way it is shown that each of the conditions $C'_{N'_p}, C'_{N'_p-1}, \dots, C'_{N'_1}$ is a consequence of those which precede ($N'_p < N'_{p-1} < \dots < N'_1 \leq N'$ since $N_p > N_{p-1} > \dots > N_1 \geq N$). If at any time the conditions applied to the n' -ic make it vanish identically, then all the remaining conditions up to $C'_{N'}$ are consequences of those already applied. Hence there are at least $p = d_n - D_n$ of the conditions $C'_1, C'_2, \dots, C'_{N'}$ which are consequences of the rest, when applied to an n' -ic. Hence $e'_n \geq d_n - D_n$. Similarly $e_n \geq d'_n - D'_n$.

These two inequalities can be converted into equalities by means of the known formulæ

$$N - e_n + d_n + 1 = \frac{1}{2} (n+1)(n+2), \quad (1)$$

$$\text{and } D_n + 1 = \left[\frac{1}{2} (n-l+1)(n-l+2) \right] + \left[\frac{1}{2} (n-m+1)(n-m+2) \right], \quad (2)$$

the square brackets in (2) indicating that the products enclosed are to be retained only when the individual factors are positive. By formula (2), since $n+n' = l+m-3$, we have

$$D_n + 1 + D'_n + 1 = \frac{1}{2} (n-l+1)(n-l+2) + \frac{1}{2} (n'-l+1)(n'-l+2).$$

$$\text{Also, by (1),} \quad e'_n - d'_n = N' + 1 - \frac{1}{2} (n'+1)(n'+2),$$

$$\text{and} \quad e - d_n = N + 1 - \frac{1}{2} (n+1)(n+2).$$

$$\text{Also} \quad N + N' = lm.$$

Adding, we have $(e'_n - d_n + D_n) + (e_n - d'_n + D_n) = 0$.

Therefore $e'_n = d_n - D_n$ and $d'_n = e_n + D_n$. (3)

The two formulæ (3) are equivalent by virtue of (1) and (2).

§ 5. CONSEQUENCES OF THE THEOREM.

The equations $e'_n = d_n - D_n$ and $d'_n = e_n + D_n$,

where

$$n' = l + m - n - 3,$$

besides giving all the excesses and defects of N' in terms of those of N , yield the following important results, which are different aspects of a single theorem. We assume as heretofore that the whole intersection of C_l, C_m is in the finite region of the plane.

(i.) If $e'_n = 0$, then $d_n = D_n$, i.e., an n -ic through N passes through the whole of $N + N'$; and, if $e'_n > 0$, then $d_n > D_n$. Hence an n -ic drawn through any point-base N , forming part of the intersection of two curves C_l, C_m , is or is not necessarily of the form $C_l S_{n-l} + C_m S_{n-m}$, according as the $(l + m - n - 3)$ -ic excess of the point-base N' , forming the remaining intersection of C_l, C_m , is or is not zero.

(ii.) If $e_n = 0$, then $d'_n = D_n$. Hence, if the n -ic excess of the point-base N is zero, i.e., if the point-base N supplies N independent conditions for an n -ic, or in other words is regular for an n -ic, then an $(l + m - n - 3)$ -ic through N' is of the form $C_l S + C_m S'$. If $l + m - n - 3$ is less than m , but not less than l , the $(l + m - n - 3)$ -ic is of the form $C_l S$; and, if $l + m - n - 3$ is less than both l and m , there is no $(l + m - n - 3)$ -ic through N' .

(iii.) An important particular case of (ii.) is obtained by putting $n = l$. If the point-base N , forming part of the intersection of two curves C_l, C_m , is regular for an l -ic, then any $(m - 3)$ -ic through the point-base N' , forming the remaining intersection of C_l, C_m , is of the form $C_l S$; and there is no $(m - 3)$ -ic through N' if $l > m - 3$. (Cf. loc. cit., p. 417, footnote.)

§ 6. POINT-BASE MADE UP OF ONE-SET POINTS.

The reasoning by which the theorem of § 4 is established is applicable, except as regards the use made of Cayley's theorem in

formula (2), to any point-base $N+N'$ which consists entirely of one-set points. We thus have the following theorem :—

If a point-base $N+N'$, made up of one-set points, is divided into any two residual point-bases N, N' , and if d_n, d'_n, D_n are the n -ic defects, and e_n, e'_n, E_n the n -ic excesses, of $N, N', N+N'$ respectively, then $e'_n \geq d_n - D_n$ and $d'_n \leq e_n + D_n$, provided any $(n+n')$ -ic through any point-base $N+N'-1$ contained in $N+N'$ passes necessarily through the whole of $N+N'$ or vanishes identically.

The greatest value of $n+n'$, consistent with the conditions, gives the best results.

These two inequalities are not equivalent, and cannot be converted into equalities, because Cayley's theorem is no longer applicable. The first gives an inferior limit, and the second a superior limit, for the degree of freedom of a curve of any order through N' , if the degrees of freedom of curves of all orders through $N+N'$ and through N are known. By means of formula (1) the inequalities may be written

$$D'_n - N' \geq e_n - e'_n \geq N - E_n.$$

If $N+N'$ is a point-group two other inequalities of some interest can be proved. Those already given may be written

$$D_n \geq d_n - e'_n \quad \text{and} \quad D_n \geq d'_n - e_n,$$

provided an $(n+n')$ -ic drawn through all but any one of the $N+N'$ points passes necessarily through the last or vanishes identically. The other inequalities may be written

$$D'_n \leq e_n + e'_n \quad \text{and} \quad E'_n \leq e'_n + e_n,$$

the first with the proviso that an $(n-n')$ -ic can be drawn through the N points without passing through any of the N' points, and the second with the proviso that an $(n-n')$ -ic can be drawn through the N' points without passing through any of the N points. It is to be observed that, if n is the same in all four inequalities, n' is different in all except the first two. The greatest value of n' , consistent with the conditions, gives in every case the best results.

The N' points supply $d_n - D_n$ independent conditions for an n -ic through the N points. Hence $d_n - D_n$ points can be chosen out of the N' points such that, if an n -ic can be drawn through the N and the $d_n - D_n$ points, it must pass through all the N' points. Now, if an

n' -ic can be drawn through the $d_n - D_n$ points, and an $(n - n')$ -ic through the N points, without passing through any of the N' points, then the n' -ic and $(n - n')$ -ic make up an n -ic which must pass through all the N' points. The n' -ic must therefore pass through all the N' points, *i.e.*, the N' points do not supply more than $d_n - D_n$ independent conditions for an n' -ic. Also, if no n' -ic can be drawn through the $d_n - D_n$ points, then again the N' points do not supply more than $d_n - D_n$ conditions for an n' -ic. Hence, in all cases, if the proviso respecting the $(n - n')$ -ic holds, we have $N' - e'_n \leq d_n - D_n$, *i.e.*, $E \leq e_n + e'_n$, by formula (1). The inequality $E_n \leq e'_n + e_n$ is proved in a similar way. The reasoning is not applicable if $N + N'$ is a point-base made up of one-set points unless the residual point-bases N, N' into which it is divided are entirely separate.

§ 7. GENERAL REMARKS.

The following notes will be of some use if they prove suggestive of other possible modes of investigation of the properties of point-bases.

(i.) The most general property of point-bases proved above is that expressed in § 6 by the inequality $e'_n \geq d_n - D_n$, which may be interpreted analytically as follows:—Choose the point-base $N + N'$ arbitrarily by giving to each of its one-set points a any situation and any prime equation. The value of a for each one-set point may be found from its prime equation, and the value of $N + N'$ is Σa . The conditions supplied by the point-base $N + N'$ for a curve of any order n are known, and the value of E_n is to be found from the fact that $N + N' - E_n$ is the number of independent conditional equations which the point-base $N + N'$ supplies for an n -ic. Let $n + n'$ be the highest value of n for which E_n does not vanish,* *i.e.*, let $n + n'$ be such that $E_{n+n'} > 0$, $E_{n+n'+1} = 0$. Choose arbitrarily a base-point q out of each one-set point a , *i.e.*, choose any of the equations of a as the prime equations of q . Then the point-base N made up of the

* As n increases E_n diminishes, never remaining stationary. Hence E_n must vanish if n has a sufficiently high finite value. Or, again, e_n must vanish for a given point-base N if e_n vanishes for any point-base which contains N ; and a point-base can always be found which contains N and consists entirely of ordinary i -points. For this latter kind of point-base e_n vanishes when n has a sufficiently high value. (C. A. Scott, "A Proof of Noether's Fundamental Theorem," *Mathematische Annalen*, Vol. LII., 1899, pp. 593-597, § 3.)

base-points q is any part of $N+N'$, and the values of N (viz., Σq) and e_n may be found in the same way as those of $N+N'$ and E_n . Let S be the general algebraic curve of an order $>n+n'$ which passes through N , and let S'_n be the general algebraic curve of order n' . Then the number of independent conditional equations (not involving the parameters of S) which must be satisfied by the coefficients of S'_n in order that the composite curve SS'_n may satisfy all the conditions supplied by the point-base $N+N'$ is $N'-e'_n$; and the inequality $e'_n \geq d_n - D_n$ states that the number $N'-e'_n \leq E_n - e_n$.

The validity of the theorem is dependent on the proviso that any $(n+n')$ -ic through any point-base $N+N'-1$ contained in $N+N'$ passes of necessity through the whole of $N+N'$, or vanishes identically. This is expressed analytically by saying that, if S is any $(n+n')$ -ic such that both $(x-x_1)S$ and $(y-y_1)S$ satisfy all the conditions supplied by the point-base $N+N'$, then S itself must as a consequence satisfy all the same conditions, or vanish identically, no matter what x_1 and y_1 are. The only pairs of values of x_1, y_1 which need be tried are the coordinates of the points of the point-base.

(ii.) The limitation of a point-base, by definition (§ 1), to the finite region of the plane is important. Base-points which go to infinity disappear, and must be regarded as non-existent, unless some method is devised for taking them into account. If C_l, C_m are any two given curves, having no common factor, the number of conditions satisfied identically by the curve $C_l S_1 + C_m S_2$ (S_1, S_2 being arbitrary polynomials) is $lm-d$, where d is the number of (ordinary) points of intersection of the two curves C_l, C_m which have disappeared at infinity. The number $lm-d$ is also the degree of the whole common point-base of the two curves; and the general curve of order n through the whole common point-base is of the form $C_l S_1 + C_m S_2$, where, however, S_1, S_2 must be of higher orders than $n-l, n-m$ respectively, if n exceeds a certain value.

This suggests the question whether, having given any point-base made up of one-set points, two curves C_l, C_m can always be found which pass through the point-base and have no further intersection (either at the base-points or at any other points) in the finite region of the plane. Any curve through the point-base would then be of the form $C_l S_1 + C_m S_2$; but even if, as supposed, two fixed curves C_l, C_m should suffice to determine the point-base uniquely, it does not follow that the fact could be used to much advantage.

(iii.) Since a base-point is the intersection of two or more curves at a point, it is natural to seek a method of obtaining the equations of a base-point at a given point on a given curve.

First, transfer the origin to any ordinary point on the curve. Expand y in powers of x (or x in powers of y if $x = 0$ is the tangent to the curve at the origin), and substitute the expansion for y in the power series $\Sigma z_q^p x^{n-q} y^q$, or the expansion for η in $\Sigma z_q^p x^n \eta^q$. We thus obtain a power series in x whose coefficients are linear homogeneous functions of the z 's. The coefficient of x^n equated to zero is then the prime equation of a base-point lying on the curve at the origin; and this is a one-set point of degree $n+1$. The independent derived equations are those obtained by equating the coefficients of $x^{n-1}, x^{n-2}, \dots, x^0$ to zero. There is no definite limit to the degree, since n is unrestricted; but there is only one base-point of assigned degree on the curve at the origin. It should be added that the prime equation $\Sigma \lambda_q^p z_q^p = 0$ is of a simple and special type, all its derivatives $\Sigma \lambda_q^p z_q^{p-l} = 0$ being deducible from those alone for which $m = 0, l = 1, 2, \dots, n$.

Next, transfer the origin to a multiple point of order i on the curve. By the same process as before, we obtain one prime equation corresponding to each linear branch through the origin. For a super-linear branch of order k (the tangent not being $x = 0$) we expand η in integral powers of $x^{1/k}$ and substitute in $\Sigma z_q^p x^n \eta^q$. The coefficients of $x^{n/k}, x^{n-1/k}, \dots, x^{n-(k-1)/k}$ equated to zero give us k more prime equations, which are not of the same simple type as those corresponding to linear branches. We thus obtain, in all, i independent prime equations, which may be called the fundamental prime equations corresponding to the given curve at the origin. Any equation linearly deducible from these and their derivatives is the prime equation of a one-set point on the curve; and any number of such equations (after omitting those which are deducible from the rest and their derivatives) are the prime equations of a base-point on the curve.

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APPENDIX (b).

(SESSION 1899–1900.)

In connexion with the recent issue of "A List of Members from the Foundation of the Society," it may be of interest to note that the meetings of the Society have been as follow:—From January 16th, 1865, at University College, W.C.; from November 8th, 1866, at the Rooms of the Chemical Society, in Burlington House; and from November 10th, 1870, to the present date, at 22 Albemarle Street, W.

With reference to Mr. Macdonald's communication (pp. 152–157), the following letter has been received from Dr. L. Gegenbauer:—

Soeben erhalte ich das neueste Heft der *Proceedings of the London Mathematical Society*, in welchem Ihre interessante Arbeit, "The Addition Theorem for the Bessel Functions," enthalten ist. Sie beweisen daselbst das von mir in meiner im lxxix. Bande der *Sitzungsberichte der mathematisch-naturwissenschaftlichen Classe der Kais. Akademie der Wissenschaften in Wien* veröffentlichten Abhandlung, "Über die Bessel'schen Functionen," zuerst aufgestellte Additionstheorem der Bessel'schen Functionen erster Art in der Weise, dass Sie zunächst die Function $\frac{J^n(\sqrt{a^2+b^2-2ab\cos\vartheta})}{\{\sqrt{a^2+b^2-2ab\cos\vartheta}\}^n}$ durch ein bestimmtes Integral ausdrücken, sodann einen Factor unter dem Integralzeichen in eine nach den Functionen $C_n^x(\cos\vartheta)$ fortschreitende Reihe entwickeln und endlich die Integration unter Benützung des von

Ihnen in Ihrer Arbeit ausgewerteten bemerkenswerten Integrale

$$\int_{c-i\infty}^{c+i\infty} e^{bt-(a^2+b^2)t/2} J_n\left(\frac{ab}{t}\right) \frac{dt}{t^{n+1}} = 2\pi i J_{(n)}'' J_{(b)}''$$

gliedweise ausführen.

Genau auf den gleichen Wege lässt sich ein anderer, meines Wissens bisher noch nicht veröffentlichter Beweis des genannten Theoremes führen, wie ich Ihnen in den folgenden Zeilen auseinandersetzen werde.

In meiner im LXXXV. Bande der citierten *Sitzungsberichte* erschienenen Abhandlung, "Das Additionstheorem derjenigen Functionen, welche bei der Entwicklung von e^{axi} nach den Näherungsnennern regulärer Kettenbrüche auftreten," habe ich behufs neuer Begründung meines Additionstheorems folgende Gleichung ohne Beweis mitgeteilt

$$(1) \int_0^\pi e^{xi \cos \phi \cos \psi} J^{(2\nu-1)}(a \sin \phi \sin \psi) C_n''(\cos \phi) \sin^{(2\nu+1)} \phi d\phi \\ = i^n \sqrt{\frac{2}{a\pi}} \left(\frac{2^{2\nu} \Pi\left(\frac{2\nu-1}{2}\right) \Pi(\nu)}{\Pi(2\nu)} \right)^2 \sin^{(2\nu+1)} \psi C_n''(\cos \psi) J^{n+\nu}(a),$$

aus der speciell folgt:

$$(2) \int_0^\pi e^{xi \cos \phi} J^{(2\nu-1)}(y \sin \phi) \sin^{(2\nu+1)} \phi d\phi \\ = \sqrt{\frac{2}{\pi y}} \left(\frac{2^{2\nu} \Pi\left(\frac{2\nu-1}{2}\right) \Pi(\nu)}{\Pi(2\nu)} \right)^2 \left(\frac{y}{\sqrt{x^2+y^2}} \right)^\nu J^\nu(\sqrt{x^2+y^2}).$$

Einen Beweis dieser Relation habe ich in meiner im 10 Jahrgange der *Monatshefte für Mathematik und Physik*, publicierten Note, "Notiz über die Bessel'schen Functionen erster Art," mitgeteilt und zugleich hervorgehoben, dass dieselbe zu der Entwicklung

$$e^{xi \cos \phi \cos \psi} J^{(2\nu-1)}(a \sin \phi \sin \psi) \\ = 2^{2\nu-1} [\Pi(\nu-1)]^3 (\sin \phi \sin \psi)^{(2\nu-1)} \sqrt{\frac{2}{a\pi}} \sum_{\lambda=0}^{\infty} i^\lambda \frac{(\lambda+\nu) \Pi(\lambda)}{\Pi(\lambda+2\nu-1)} \\ \times J_{(\lambda)}^{\nu+\lambda} C_\lambda''(\cos \phi) C_\lambda''(\cos \psi)$$

führt, in welcher mehrere Relationen, die Herr E. W. Hobson in seiner im XXV. Bande der *Proceedings of the London Mathematical Society*, enthaltenen Arbeit "On Bessel's Functions and Relations connecting them with Hyper-Spherical and Spherical Harmonics," abgeleitet hat, als specielle Fälle enthalten sind.

Setzt man in der Gleichung (2),

$$x = -a + b \cos \vartheta,$$

$$y = b \sin \vartheta,$$

so verwandelt sich dieselbe in

$$\begin{aligned} & \frac{J^\nu (\sqrt{a^2 + b^2 - 2ab \cos \vartheta})}{\{\sqrt{a^2 + b^2 - 2ab \cos \vartheta}\}^\nu} \\ &= \sqrt{\frac{\pi}{2}} \left(\frac{\Pi(2\nu)}{2^{2\nu} \Pi\left(\frac{2\nu-1}{2}\right) \Pi(\nu)} \right)^2 (b \sin \vartheta)^{-1(2\nu-1)} \int_0^\pi e^{(-a + b \cos \vartheta) i \cos \phi} \\ & \quad \times J^{1(2\nu-1)}(b \sin \vartheta \sin \phi) \sin^{1(2\nu+1)} \phi \, d\phi. \end{aligned}$$

Entwickelt man nun die Function $e^{bi \cos \vartheta \cos \phi} J^{1(2\nu-1)}(b \sin \vartheta \sin \phi)$ mit Hilfe der Gleichung (3) in eine nach den Functionen $C_\lambda^*(\cos \phi)$ fortschreitende Reihe, so erhält man weiter

$$\begin{aligned} & \frac{J^\nu (\sqrt{a^2 + b^2 - 2ab \cos \vartheta})}{\{\sqrt{a^2 + b^2 - 2ab \cos \vartheta}\}^\nu} \\ &= \frac{2^{1-2\nu}}{b^\nu} \left(\frac{\Pi(2\nu-1)}{\Pi\left(\frac{2\nu-1}{2}\right)} \right)^2 \sum_{\lambda=0}^{\lambda=\infty} i^\lambda \frac{(\lambda+\nu) \Pi(\lambda)}{\Pi(\lambda+2\nu-1)} J^{\nu+\lambda}(b) C_\lambda^*(\cos \vartheta) \\ & \quad \times \int_0^\pi e^{-ai \cos \phi} C_\lambda^*(\cos \phi) \sin^{2\nu} \phi \, d\phi, \end{aligned}$$

woraus sofort das Additionstheorem folgt, weil, wie ich in meiner im LXXV. Bande der citierten *Sitzungsberichte* veröffentlichten Mittheilung "Über die Functionen $C_\lambda^*(x)$ " gezeigt habe

$$2^\nu i^\nu \Pi(\nu-1)(\lambda+\nu) y^{-\nu} J_{(\nu)}^{\lambda+\nu},$$

der Coefficient von $C_\lambda^*(x)$ in der Entwicklung von e^{axi} nach den eben genannten Functionen ist.

Gestatten Sie mir noch, darauf hinzuweisen, dass man unter Benutzung des eben erwähnten Satzes aus Ihrer Gleichung (A) sofort das Integral

$$\begin{aligned} & \int_0^\pi \frac{J^\nu (\sqrt{a^2 + b^2 - 2ab \cos \vartheta})}{\{\sqrt{a^2 + b^2 - 2ab \cos \vartheta}\}^\nu} C_n^*(\cos \phi) \sin^{2\nu} \phi \, d\phi \\ &= \frac{2^{\nu+1} \nu \Pi(\nu) \Pi(n+2\nu-1)}{\Pi(n)} \left(\frac{2^\nu \Pi\left(\frac{2\nu-1}{2}\right)}{\Pi(2\nu)} \right)^2 (ab)^{-\nu} J_{(a)}^{n+\nu} J_{(b)}^{n+\nu}, \end{aligned}$$

2 F 2

ableiten kann, aus welchem sich, wie ich im LXXXV. Band der oft genannten *Sitzungsberichte* bemerkte, unmittelbar das Additions-theorem ergibt.

Sollten Sie glauben, dass die obigen Auseinandersetzungen die Mitglieder der London Mathematical Society interessieren, so ersuche ich Sie, dieselben ihnen im Anschlusse an Ihre Arbeit vorzulegen.

For the accompanying obituary notice of the late Signor Beltrami, the Council are indebted to Prof. G. H. Bryan.

Eugenio Beltrami was born at Cremona on November 16th, 1835. His grandfather, Giovanni Beltrami, was a famous sculptor on precious stones, and his father, Eugenio Beltrami, senior, was also an artist, especially in miniatures, his mother, Elisa, of the Venetian family Barozzi, being still alive. Beltrami inherited the love of fine arts, and was brought up in his youth by his mother and grandfather. He attended school at Cremona, except in 1848-49, when, owing to the war in Lombardy, he went with his mother to Venice. From 1853 to 1856 he studied mathematics under Brioschi at Pavia, where he held a scholarship in the Collegio Ghisleri. His University career was cut short in consequence of an agitation of the students against their Rector, and he was obliged to accept an appointment at Verona as private secretary to Diday, engineer to the Lombardy Railways. For political reasons he lost this post in January, 1859, but was reappointed after the battle of Magenta, when Diday transferred his office to Milan, taking Beltrami with him.

At Milan Beltrami renewed his mathematical studies, devoting much time to reading the works of Bordoni, Brioschi, Monge, Jacobi, Abel, Gauss, Lagrange, and Riemann, and an impetus was given to these studies by his making the acquaintance of Cremona, when that mathematician came to Milan to deliver a course of lectures at the Liceo S. Alessandro.

Beltrami rapidly developed as a mathematician, and in 1862 we find him publishing two papers in the *Annali di Matematica*, edited by Tortolini, of Rome. These proofs of his power resulted in his appointment as Professor Extraordinarius of Algebra and Analytical Geometry in the University of Bologna. His tenure of this chair was of short duration, for in the following year he was appointed Professor Ordinarius of Geodesy at Pisa. Here he remained three years and made the acquaintance of Betti, which developed into a

life-long friendship. He also enjoyed the society of Riemann, who was staying in Pisa on account of his health. But the climate did not suit his mother, and, accordingly, Beltrami sought an early opportunity of returning to Bologna, where he lectured on rational mechanics. In February, 1868, he married Amalia Pedrocchi, of Venice, and about the same time published his famous memoir on non-Euclidean geometry, entitled, "Saggio d'interpretazione della geometria non euclidea." In February, 1870, Rome was restored to Italy and became the seat of Government. On the appointment of Antonio Scialoja as Minister of Public Instruction, the University of Rome was reconstituted, and in October, 1873, Beltrami was offered, and accepted, the post of Professor Ordinarius in the subject of Rational Mechanics, with the added duty of conducting a class in Higher Analysis. On entering into office, however, Beltrami found that the scheme of reconstitution was hardly accomplishing its ends in the way that had been predicted; he was alarmed at the difficulties attaching to his new duties, which interfered with his private studies, and he lost no time in seeking a more congenial post. Accordingly, in 1876 he accepted an appointment at Pavia as Professor of Mathematical Physics and Lecturer in Higher Mechanics. His departure from Rome was a source of much regret to his colleagues there, who hoped that he might soon return to them; but this hope was not realized until 1891. At Pavia, if the climate was not healthier, he, at any rate, found greater quiet and new friends, among whom Felice Casorati became his close companion. The premature death of Casorati in 1890 was a source of considerable grief to Beltrami, and was, probably, the factor which determined his return to Rome in the following year. Here he held the office of Councillor of Public Instruction; two years ago he was elected President of the Reale Accademia dei Lincei, and he was elected to the Senatorial rank shortly before his death.

Of Beltrami's works, the catalogue in the special number of the *Atti dei Lincei* for June, 1900, enumerates one hundred and thirty-four. Prof. Cremona, in his obituary address to the Academy, describes Beltrami as a "self-made man"; but by this he means rather a "self-taught man"—a man of original ideas, in fact. His duties at Pisa directed his attention to Gauss's theory of surfaces and the mathematical theory of maps. He was thus led on to the subject of conformable representation, to transformations of complex variables on surfaces, and, finally, to non-Euclidean geometry; in which latter subject we are largely indebted to Beltrami for the fundamental

properties of the pseudosphere, and for many developments of the doctrines of Gauss and Lobatchewsky.

In his obituary address on Brioschi Beltrami distinguished two schools of mathematicians—the classical school, represented by Euler and Jacobi; and another school, founded by Lagrange, and developed by Gauss and Dirichlet, and still further developed by Cauchy and Riemann. Beltrami claimed that Brioschi was one of the first school, and Cremona now places Beltrami himself in the second.

From 1871 to 1874 Beltrami published an extended series of articles on the kinematics of fluids in the *Memorie* of the Bologna Academy, and, subsequently, he devoted considerable attention to the theory of potential, elasticity, electricity and magnetism, conduction of heat, and optics. In the study of elasticity he was led to the conclusion that Lamé's equations were intimately connected with the euclideanity of space, and he sought to extend them to spaces of constant curvature. The discoveries of Maxwell, leading to the substitution of stress in a medium for action at a distance, afforded a new field for Beltrami's scientific work, and Hertz's experiments, elaborated in Italy by Righi, lent an additional interest to this study. Instead of starting with Hamilton's generalized equations of dynamics, Beltrami sought to base the equations of the electromagnetic field directly on D'Alembert's principle—a method which, without introducing any substantial modification in Maxwell's theory, led to the required result in a simpler manner. The last period of Beltrami's life was devoted almost entirely to this branch of applied mathematics; yet we find him also writing on thermodynamic potentials, on spherical functions, on Huygens's principle, and as late as 1895 on Lagrange's equations. Beltrami's writings were all characterized by a certain perfection of style and neatness of expression which enabled him to render signal service to mathematicians in clearing up obscure points in the previously existing literature on the branches of mathematics which he made his especial study.

That a certain connexion exists between music and mathematics has been frequently stated. Such an idea was put forward by Sylvester in his paper "On the Real and Imaginary Roots of Equations" (*Phil. Trans.*, 1864). Beltrami was an ardent musician, and in his leisure hours he derived great pleasure from performing on the pianoforte selections from the works of Bach, Mendelssohn, Schumann, and other composers. Prof. Cremona gives extracts from two of Beltrami's letters, one to Dr. Gustav Wolff, Professor at the

Conservatorium of Leipzig, embodying the idea of a relation between music and mathematics, and the other referring to Sylvester's note and to the fact that the composer Meyerbeer had commenced his career with the study of mathematics.

Beltrami sought no distinctions, but these were freely conferred on him. The Academy of Bologna, the Italian Society, the Accademia dei Lincei, the Istituto Lombardo, the Turin Academy, the Royal Society of Naples, the Academies of Berlin, Paris, Munich, Göttingen, and our Society, were glad to include him on their membership lists. The Universities of Kazan and Halle conferred honorary degrees on him; he was also one of the Knights of Savoy (*Cavaliere del merito civile di Savoia*) from 1879; and it is interesting to learn that Beltrami, in conjunction with Hermite and Bjerknes, were appointed as a commission to report on the claims of Sofia Kowalewsky for the chair which she subsequently held in the University of Stockholm till her death, two years later.

John James Walker, the son of John Walker and Ann (*née* Fricker) his wife, was born at Kennington, Surrey, on October 2nd, 1825.

On the paternal side he was Irish: his great grandfather Matthias, his grandfather John (F.T.C.D.), and his father were all graduates of Trinity College, Dublin, and were remarkable for their combined mathematical and classical tastes. John Walker, junior, appears to have been for a short time (1831) the Headmaster of "University of London School"*—a fact which did not come to J. J. Walker's knowledge apparently until the summer of 1899.† The connexion with this school soon terminated, and a "London High School" was started at Tavistock House, where, subsequently, Charles Dickens had his residence. Later on Mr. Walker was appointed Headmaster of Plymouth New Grammar School, and here our Mr. Walker entered upon his earliest training, which was carried on entirely under his father's eye.

A short time previous to his father's death, which took place in 1845, Mr. Walker entered Trinity College, Dublin, in 1843. Here he took his degree in 1849, as a Senior Moderator in Mathematics and Physics, having been also an Honourman in Classics frequently

* Cf. *Alphabetical and Chronological Register for 1831-1891*, p. 9 (University College School, London, by Temple Orme).

† As I learn from Mr. W. F. Walker. [R. T.]

during his undergraduate course. Being a Nonconformist, he was debarred from competing for either University scholarship or fellowship. In the year subsequent to taking his degree he was elected a "Bishop Law's Mathematical Prizeman," one of the relatively highest mathematical distinctions of the College. He took his M.A. degree in 1857. For about fifteen years his time was wholly occupied with private pupils, and for about nine years he had exclusive charge of the present Lords Ardilaun and Iveagh. When this last charge was completed Mr. Walker removed to London, and shortly after he commenced his relations with University College School. Mr. Walker and myself became colleagues in September, 1865, he taking afternoon work as Lecturer in Applied Mathematics and Physics, and I as morning (Pure) Mathematical Master. By this arrangement we very rarely met at school, but, as we were both elected members of the Mathematical Society, on October 16th, 1865, we had frequent opportunities of meeting one another. From 1868 to 1882 Mr. Walker was Vice-Principal of University Hall, and in this period (in 1874) he married Emma, daughter of the late Mr. William Turner, of Newcastle, who survives him. His connexion with the School terminated in 1888.* Mr. Walker took great interest in the work of our Society, being a frequent contributor to the *Proceedings* and a referee. He was a member of the Council from November, 1869, to November, 1874, and then again from November, 1876, to November, 1894; he was Vice-President for two periods of two years and was President from November, 1888, to November, 1890.† Mr. Walker was also a member of the Physical Society, and in 1883 he was elected a Fellow of the Royal Society.

His range of mathematical reading was extensive, and a manuscript memorandum of his states that he contributed various mathematical papers to the *Philosophical Magazine* ("Iris seen in Water," 1853), *Cambridge and Dublin Mathematical Journal*, *Quarterly Journal of Mathematics*, *Messenger of Mathematics*, *Philosophical Transactions* (1888), and *Reports of British Association* (1859-1863), and numerous articles and reviews, besides questions and solutions furnished to the *Educational Times*. To our *Proceedings* he contributed about twenty-four papers or notes.

* From 1871 to 1883 Mr. Walker was Examiner in Mathematics and Natural Philosophy for the Hibbert Trust Scholarships.

† In *Who's Who and Men of the Time* (1900) he is said to have been President from 1880 to 1890!

His earlier work, done when M'Cullagh, Salmon, Graves, and Townsend ruled over Trinity, had a strong bias in the direction of "pure geometry," and his later work tended in the direction of the algebraic methods and subjects. The valedictory address, "Of the Influence of Applied on the Progress of Pure Mathematics" (*Proc.*, xxii., 4-18), is said by Prof. Greenhill, in his valedictory address (*Proc.*, xxiv.), to show "how many of the most abstruse theorems of pure analysis owe their origin to ideas which arose in connexion with concrete and even practical requirements." The subject dealt with in this address was one fertile in interest and importance in its scope, and, doubtless, had Mr. Walker had *carte blanche*, he would, from the stores acquired by a fairly extensive range of reading, have made it even a finer piece of work than it is. The pains evidently taken with it are characteristic of his usual patient care and habit of looking all round his subject.

The following picture of him as a teacher comes from an old pupil:—"My recollections of him and of his 'Natural Philosophy' lectures are among the pleasantest and most vivid remembrances of the old days at University College School. His refinements of explanation, his precision of expression, his grave courtesy of manner—how they all come back! . . . We had excellent opportunities, all unsuspected, of learning science and manners from a master of both. His reproaches were pointed and eloquent, but too nice for the thick-skinned, . . . an excellent teacher for a willing ear."

Mr. W. F. Walker* writes that the general tenour of his brother's life was, throughout, simple and devoid of all exciting events; a life of quiet, industrious work, scholastic tastes, of calm and genial intercourse with family and friends. Underlying a somewhat reserved temperament, not only were his family affections as a son and brother and relative always keen, loyal, and self-sacrificing, but his attachments to, and esteem and confidence from, the early and later friends of his private life always kept up and treasured to the last.

On February 14th, 1900, Mr. Walker appeared to be quite in his usual health; on the 15th he was found insensible in his bath-room,

* This gentleman has placed a quantity of interesting matter at my disposal, which I have freely used. He was only ten years of age when the father died, and henceforward our Mr. Walker, then a student, was *in loco parentis* to Mr. W. F. Walker.

and appeared to have passed away without a struggle or a symptom of pain.

CORRIGENDA.

On p. 309 of Vol. ~~xxx~~., strike out the passage beginning "Suppose now," and ending "arbitrary zero."

Pp. 370-384. Date at head of pages should be March 8, 1900.