ON COVARIANT TYPES OF BINARY n -ICS*

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1. The “finiteness” of the complete system of concomitants of a single binary form was first proved by Gordan† in the year 1868. Before that time it was believed that quantics of order higher than 4 did not possess a finite system of concomitants. Since then many different proofs of this central fact in the theory of forms have been given by various writers. Its truth has been established for ternary forms and forms having any greater number of variables; for forms possessing two or more different *sets* of variables, whether cogredient or otherwise; for any simultaneous system of forms; and for types of concomitants when the actual number of base forms becomes infinite, but the order of each is finite. Information, with references, as to the various proofs of the finiteness is to be found in any of the various editions of Meyer’s “Bericht über den gegenwärtigen Stand der Invariantentheorie.”‡

The majority of the proofs, especially of the later ones, keep the attention fixed on the collective idea of finiteness, and give little or no information as to the actual composition, formation, or extent of the complete system. There are two exceptions, which stand out pre-eminently from the above general rule: these are the proof given by Gordan and that due to Jordan.

Gordan proved the theorem by actually giving a process by means of which the complete system might be established, and by showing that this process yielded only a finite number of irreducible concomitants. In the later editions of his proof the process was considerably modified, but the principle remained the same.

* The paper as originally written consisted of §§ 2–8. The introduction (§ 1) and the table (§ 9) have been written at the suggestion of the referees.

† *Jour. für Math. (Crelle)*, Vol. LIX., pp. 323–354.

‡ The original edition appeared in 1892 in the *Jahresbericht der Deutschen Math.-Vereinigung*, Bd. I.; French revised edition by Fehr, 1897 (Gauthier-Villars, Paris); Italian revised edition by Vivanti, 1899 (Pellerano, Naples); and, finally, the article “Invariantentheorie” by Meyer in the *Encycl. Math. Wiss.*, Bd. I., p. 320 (I. B. 2), 1899.

Jordan's proof was published in 1876.* It is remarkable that it has not attracted more attention. The first half of the paper referred to is devoted to an introduction to the theory of symbolic algebra. The author then proceeded to examine the covariants of degree 3. His discussion was exhaustive and established the following theorem:—

If a_x^p, b_x^q, c_x^r be any three binary forms, the order of each being $\geq n$, then the various covariants furnished by the expression

$$(ab)^\lambda (bc)^\mu (ca)^\nu a_x^{p-\nu-\lambda} b_x^{q-\lambda-\mu} c_x^{r-\mu-\nu} \quad (\lambda + \mu + \nu = n)$$

may be linearly expressed in terms of the $n+1$ covariants C_{abc}^ρ ($\rho = 0, 1, \dots, n$),

where $C_{abc}^\rho = (ab)^{n-\rho} (bc)^\rho a_x^{p-n+\rho} b_x^{q-n} c_x^{r-\rho}$.

They may also be linearly expressed as a function of the covariants $C_{abc}^\rho, C_{bca}^\rho, C_{cab}^\rho$, where $\rho \geq \frac{1}{3}n$. If $n = 3k+2$, these last covariants will be all independent. If $n = 3k+1$, they will be connected by a relation which will permit us to express the sum $C_{abc}^k + C_{bca}^k + C_{cab}^k$ as a linear function of those C 's for which $\rho < k$. If $n = 3k$, there will be two relations permitting us to express two of $C_{abc}^k, C_{bca}^k, C_{cab}^k$ as a function of the third and of those covariants C for which ρ is less than k .†

In the next paragraph covariants of a system of quantics, the order of each being at least $2l$, were considered. It was proved that these belonged to three classes: (i.) covariants which have a factor $(ab)^{l+1}$; (ii.) covariants which have a factor $(ab)^l (bc)^l (ca)^\rho$ ($\rho > 0$); (iii.) covariants and products of covariants of the form $(ab)^\mu (bc)^\nu (cd)^{\mu'} (de)^{\nu'} \dots a_x^{p-\mu} \dots$; the symbols $a, b, c, \dots$ corresponding to any quantics of the system, and the exponents $\mu, \nu, \mu', \nu', \dots$ being different from zero and satisfying the inequalities $\nu \leq \frac{1}{2}\mu, \nu' \leq \frac{1}{2}\mu', \nu'' \leq \frac{1}{2}\mu'', \dots; \mu \leq l, \mu' \leq \mu - \nu - \epsilon, \mu'' \leq \mu' - \nu' - \epsilon', \dots$, where $\epsilon^{(i)}$ is equal to 0 or 1, according as $\mu^{(i)}$ is odd or even.

In the next paragraph an upper limit to the degree in the coefficients of irreducible covariants of the third class was obtained.

In the last paragraph the finiteness was proved for any simultaneous system, the proof being equally applicable to the finiteness of types of concomitants, for the letters of a symbolical product were regarded as not interchangeable. In addition, an upper limit to the order (in the variables) of an irreducible concomitant of a system of binary forms, the order of

* *Liouville's Jour. de Math.*, 1876.

† This theorem is fundamental to the method used in the present paper. It is extended in Jordan's second paper, *Liouville's Jour. de Math.*, 1879. It was rediscovered by Stroh, *Math. Ann.*, Bd. xxxi., pp. 444-454, who proved it in a more general form than that given above, using an entirely different method.

each being equal to or less than n , was obtained. This limit, which for any value of n could be calculated by a definite arithmetical process, is the same as that given here for the first eleven values of n , but for $n = 12$, and for higher values of n , Jordan's upper limit is here shown to be too high.

In *Liouville's Journal de Mathématiques*, 1879, Jordan wrote a second paper to determine an upper limit for the degree of an irreducible concomitant.

Upper limits for the degree and order of an irreducible concomitant have also been given by Sylvester.* He assigned the limit

$$\Sigma \left(\frac{n^2+1}{2} \right) + \Sigma \left(\frac{\nu^2}{2} \right) - 2$$

as an upper limit to the order of a concomitant of a simultaneous system of binary forms of odd orders $n, n', n'', \dots$, and even orders $\nu, \nu', \nu'', \dots$; but he gave no proof. The limit is higher than that given by Jordan.

Our knowledge of complete systems of concomitants of binary forms is still limited to a few cases. Complete systems have been calculated for single forms of orders from 1 to 8 inclusive. For two quantics (h, k) the systems are known for the cases (2, 1), (2, 2), (2, 3), (3, 3), (3, 4), (2, 5), (2, 6), (4, 4). Systems of types of concomitants are known for the first four orders separately. The system for any number of quadratics and linear forms has been found. Also rules have been given by which the additional concomitants may be at once obtained when a linear or quadratic form is added to an already known complete system. References in nearly all the cases quoted are to be found in Meyer's *Bericht*.

2. The main object of the present paper is to obtain the complete irreducible system of types of covariants of binary forms of order n whose grade does not exceed $\frac{1}{2}n$.

The theorem to be proved is in reality an extension of the theorem proved for perpetuant types by Grace,† and follows his work closely both in statement and in demonstration.

The determinant factors of the symbolical products used here are alone written down, for, the orders of the various quantics considered being known, the remaining factors can at once be supplied.

* *Proc. Roy. Soc.*, Vol. xxvii., pp. 11, 12 (1878); *Comptes Rendus*, Vol. lxxxvi., pp. 1437-1441, 1491-1492, 1519-1522 (1878).

† *Proc. London Math. Soc.*, Vol. xxxv., pp. 107-111.

3. Any covariant which is of unit degree in the coefficients of each of the quantics $a_1^{n_1}, a_2^{n_2}, \dots, a_{\delta}^{n_{\delta}}$ can be expressed linearly in terms of (i.) covariants of the form $(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \dots (a_{\delta-1} a_{\delta})^{\lambda_{\delta-1}}$, where $\lambda_1 \leq 2^{\delta-2}$, $\lambda_2 \leq 2^{\delta-3}$, ..., $\lambda_{\delta-1} \leq 1$, and the arrangement of the letters $a_1, a_2, \dots, a_{\delta}$ is fixed; (ii.) covariants which have a factor of the form $(a_k a_l)^{\lambda} (a_k a_l)^{n_k - \lambda}$; (iii.) products of covariants of lower total degree.

Assuming that this theorem is true when the number of quantics concerned is less than δ , we shall first prove its truth when the number is equal to δ .

Any covariant of the kind under discussion may be expressed in terms of transvectants of the form $(a_{1_x}^{n_1}, C_{\delta-1})^{\mu}$, where $C_{\delta-1}$ is a covariant of unit degree in the coefficients of each of the quantics $a_2^{n_2}, a_3^{n_3}, \dots, a_{\delta}^{n_{\delta}}$. Hence, on the above assumption, the transvectant written down can be expressed linearly in terms of transvectants

$$[a_{1_x}^{n_1}, (a_2 a_3)^{\lambda_2} (a_3 a_4)^{\lambda_3} \dots (a_{\delta-1} a_{\delta})^{\lambda_{\delta-1}}]^{\mu},$$

and of covariants belonging to the second and third classes.*

Now all possible transvectants of this form are under consideration; hence each may be replaced by one of its terms.

Unless $\lambda_2 + \mu < n_2$, the transvectant contains a term belonging to the second class. Hence all the covariants considered can be expressed in terms of covariants of the form $(a_1 a_2)^{\mu} (a_2 a_3)^{\lambda_2} \dots (a_{\delta-1} a_{\delta})^{\lambda_{\delta-1}}$, and of covariants belonging to the second and third classes.

4. Let

$$(a_4 a_5)^{\lambda_4} \dots (a_{\delta-1} a_{\delta})^{\lambda_{\delta-1}} a_{4_x}^{n_4 - \lambda_4} a_{5_y}^{n_5 - \lambda_4 - \lambda_5} a_{6_y}^{n_6 - \lambda_5 - \lambda_6} \dots a_{\delta_y}^{n_{\delta} - \lambda_{\delta-1}} \equiv a_4^{n_4 - \lambda_4} q_y^{\rho}.$$

Then $(a_1 a_2)^{\mu} (a_2 a_3)^{\lambda_2} \dots (a_{\delta-1} a_{\delta})^{\lambda_{\delta-1}}$ differs from any other term of

$$[(a_1 a_2)^{\mu} (a_2 a_3)^{\lambda_2}, a_{4_x}^{n_4 - \lambda_4} q_y^{\rho}]_{y=x}^{\lambda_3} \quad (\text{I.})$$

by covariants in which the number of factors involving a_1, a_2, a_3 only is greater than $\lambda_2 + \mu$.

Consider the transvectant $[(ab)^{\lambda} (bc)^{\kappa}, a_{4_x}^{n_4 - \lambda_4} q_y^{\rho}]_{y=x}^{\lambda_3}$, where $\kappa < 2^{\delta-3}$ and a, b, c are the letters a_1, a_2, a_3 in some order.

The systems of covariants $(ab)^{\lambda} (bc)^{\kappa}$ and $[(ab)^{\lambda}, c_x^{m_3}]^{\kappa}$, where $\lambda + \kappa$ has a constant value and κ remains less than $2^{\delta-3}$, are equivalent—in the

* It is easy to verify that, if $C_{\delta-1}$ is reducible, the transvectant $(a_{1_x}^{n_1}, C_{\delta-1})^{\mu}$ belongs, on the assumption made, to the second or third class.

sense that any member of one system can be expressed linearly in terms of members of the other.

Let us introduce a new symbol a where $a_x^{m_1+m_2-2\lambda}$ is the covariant $(ab)^\lambda$; m_1, m_2 being the orders of a and b . Then it will be sufficient to consider the transvectants $[(ac)^\kappa, a_{4,r}^{n_1-\lambda_1} q_y^\rho]_{y=x}^{\lambda_2}$, where $\kappa < 2^{\delta-3}$.

But on the assumption made, since this is a covariant of the $\delta-1$ quantities $a_x^{m_1+m_2-2\lambda}, c_{x,i}^{m_3}, a_{4,x}^{n_4}, \dots, a_{\delta,x}^{n_\delta}$, it can be expressed in terms of (i.) covariants $(ac)^{\mu_2} (ca_4)^{\mu_3} (a_4 a_5)^{\mu_4} \dots (a_{\delta-1} a_\delta)^{\mu_{\delta-1}}$, where $\mu_2 \leq 2^{\delta-3}, \mu_3 \leq 2^{\delta-4}, \dots, \mu_{\delta-1} \leq 1$; (ii.) covariants which have a factor of the form $(a_h a_k)^\nu (a_h a_i)^{n_k-\nu}$; (iii.) products of covariants of lower total degree.

Amongst the covariants here in the second class, there are included those which have a factor $(a_h a)^\nu (a a_k)^{m_1+m_2-2\lambda-\nu}$; but we may suppose that $m_1 \not\geq m_2$ and $\nu \leq m_1+m_2-2\lambda-\nu$, and hence that $\nu \leq m_1-\lambda$. Then every such covariant can be expressed in terms of such as have a factor of the form $(a_h a)^{m_1-\lambda} (ab)^\lambda$, and is therefore properly to be included in the second class.

5. It follows from what we have just proved that, if either μ or λ_2 is less than $2^{\delta-3}$, the transvectant (I.) can be expressed in terms of covariants which contain a greater number of factors involving a_1, a_2, a_3 only. We may then suppose that $\lambda_2+\mu \leq 2 \cdot 2^{\delta-3}$. Again, the covariant $(a_1 a_2)^\mu (a_2 a_3)^{\lambda_2}$ belongs to the second class unless n_1, n_2, n_3 are each greater than $\lambda_2+\mu$. For, if $n_1 \leq \lambda_2+\mu$,

$$\begin{aligned} (a_1 a_2)^\mu (a_2 a_3)^{\lambda_2} &= (a_1 a_2)^\mu [(a_1 a_3) - (a_1 a_2)]^{n_1-\mu} (a_2 a_3)^{\lambda_2+\mu-n_1} \\ &= \sum A_r (a_1 a_3)^r (a_1 a_2)^{n_1-r} (a_2 a_3)^{\lambda_2+\mu-n_1}. \end{aligned}$$

In this case the transvectant (I.), and every one of its terms, belongs to the second class.

When n_1, n_2, n_3 are all greater than $\lambda_2+\mu$, and $\lambda_2+\mu$ is not less than $2^{\delta-2}$, we may express the covariant $(a_1 a_2)^\mu (a_2 a_3)^{\lambda_2}$ linearly in terms of the covariants*

$$\begin{aligned} &(a_2 a_3)^{\lambda_2+\mu}, \quad (a_2 a_3)^{\lambda_2+\mu-1} (a_3 a_1), \quad \dots, \quad (a_2 a_3)^{\lambda_2+\mu-2^{\delta-3}+1} (a_3 a_1)^{2^{\delta-3}-1}; \\ &(a_3 a_1)^{\lambda_2+\mu}, \quad (a_3 a_1)^{\lambda_2+\mu-1} (a_1 a_2), \quad \dots, \quad (a_3 a_1)^{\lambda_2+\mu-2^{\delta-3}+1} (a_1 a_2)^{2^{\delta-3}-1}; \\ &(a_1 a_2)^{\lambda_2+\mu}, \quad (a_1 a_2)^{\lambda_2+\mu-1} (a_2 a_3), \quad \dots, \quad (a_1 a_2)^{2^{\delta-2}} (a_2 a_3)^{\lambda_2+\mu-2^{\delta-2}}. \end{aligned}$$

Now, by § 3, if C be any one of the covariants in the first two rows just written down, the number of factors involving a_1, a_2, a_3 only in the transvectant $(C, a_{4,r}^{n_1-\lambda_1} q_y^\rho)_{y=x}^{\lambda_2}$ can be increased.

* Stroh, *Math. Ann.*, Bd. xxxi., pp. 444-454; Jordan, *Liouville's Jour. de Math.*, 1876, 1879.

Hence these covariants can ultimately be expressed in terms of covariants of the second and third classes, and of covariants which contain the factor $(a_1 a_2)^{\lambda_1}$, where $\lambda_1 \leq 2^{\delta-2}$.

If, now, we write $a_x^{n_1+n_2-2\lambda_1} \equiv (a_1 a_2)^{\lambda_1}$, and use the assumption for covariants of total degree $\delta-1$, we see, as in § 3, that all covariants can be expressed linearly in terms of covariants of the second and third classes, and of covariants of the form $(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \dots (a_{\delta-1} a_\delta)^{\lambda_{\delta-1}}$, where

$$\lambda_1 \leq 2^{\delta-2}, \quad \lambda_2 \leq 2^{\delta-3}, \quad \dots, \quad \lambda_\delta \leq 1.$$

Thus the theorem is true when there are δ quantics involved, provided that it is true when any $\delta-1$ quantics are involved.

6. When only three quantics are involved we have to consider covariants of the form $(a_1 a_2)^\lambda (a_2 a_3)^\mu (a_3 a_1)^\nu$.

If $\lambda + \mu = n_2$, this covariant belongs to the second class.

If $\lambda + \mu < n_2$, the index of $(a_3 a_1)$ can be diminished by means of the identity

$$(a_3 a_1) = -(a_1 a_2) - (a_2 a_3).$$

Hence we have only to consider covariants $(a_1 a_2)^\lambda (a_2 a_3)^\mu$, where $\lambda + \mu < n_2$.

If $\mu = n_3$, this covariant belongs to the second class.

If $\mu < n_3$ and $\lambda = 1$, we can, by means of the relation

$$\begin{aligned} (a_1 a_2)^{\mu+1} &= \{(a_1 a_2) + (a_2 a_3)\}^{\mu+1} \\ &= (a_2 a_3)^{\mu+1} + \binom{\mu+1}{1} (a_1 a_2) (a_2 a_3)^\mu + \binom{\mu+1}{2} (a_1 a_2)^2 (a_2 a_3)^{\mu-2} + \dots, \end{aligned}$$

express the covariant in terms of members of the first and third classes.

Hence the theorem is true when three quantics are involved; it is therefore true in general.

7. Let us suppose that all the quantics considered in the theorem just proved are of the same order n . Then the covariants of the second class all contain a factor of the form $(ab)^\lambda (bc)^{n-\lambda}$. These can be expressed in terms of transvectants, such as $[(ab)^\lambda (bc)^{n-\lambda} (ca)^r, C]^\varpi$, where r is positive or zero.

Now Jordan* has proved that the covariant $(ab)^\lambda (bc)^{n-\lambda} (ca)^r$ can be expressed in terms of covariants $(ab)^\mu (bc)^{n-\mu} (ca)^r$, where $\mu \geq \frac{1}{2}n$, and $\mu - r \geq 2(n - \mu - r)$, i.e., $r \leq 2n - 3\mu$.

Hence all covariant types of binary forms of order n can be expressed in terms of (i.) covariant types of the form $(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \dots (a_{\delta-1} a_\delta)^{\lambda_{\delta-1}}$,

* *Loc cit.*

where $\lambda_1 \nless 2^{\delta-2}$, $\lambda_2 \nless 2^{\delta-3}$, ..., $\lambda_{\delta-1} \nless 1$; (ii.) covariant types which have a factor $(ab)^\lambda (bc)^{n-\lambda} (ca)^r$, where $\lambda \nless \frac{1}{2}n$, $r \nless 2n-3\lambda$; (3) products of covariants of lower degree. This theorem at once gives a system of types in terms of which all irreducible types of grade not exceeding $\frac{1}{2}n$ may be expressed.

8. It appears practically certain that the theorem for perpetuant types is exact, *i.e.*, that the covariant types for binary forms of infinite order $(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \dots (a_{\delta-1} a_\delta)^{\lambda_\delta}$, where $\lambda_1 \nless 2^{\delta-2}$, $\lambda_2 \nless 2^{\delta-3}$, ..., $\lambda_\delta \nless 1$, and the order of the letters is fixed beforehand, are both independent and irreducible (this system being equivalent to that used by Grace). If this is the case, covariants of this form must be independent and irreducible for quantics of finite order. It does not follow, however, that they cannot be expressed in terms of covariants of higher grade, or else in terms of covariants belonging to the second class. In fact, as it is easy to verify, if $\lambda_1 + \lambda_r \geq n$, such a covariant type of a system of binary n -ics can be expressed in terms of members of the second class, and of covariants $(a_1 a_2)^{\lambda'_1} (a_2 a_3)^{\lambda'_2} \dots (a_{\delta-1} a_\delta)^{\lambda'_{\delta-1}}$, where one or more of the differences $\lambda'_1 - \lambda_1$, $\lambda'_2 - \lambda_2$, ..., $\lambda'_{r-1} - \lambda_{r-1}$ is positive, and the rest are zero.

NOTE ON THE FOREGOING PAPER.

By J. H. GRACE.

Consider any number of forms of order not exceeding n .

Following Jordan, we add to the system every covariant of the second degree whose order does not exceed n .

Thus, if a_x^p and b_x^q be two forms, we add the covariant $(ab)^\lambda a_x^{p-\lambda} b_x^{q-\lambda}$ to the system whenever $p+q-2\lambda \leq n$, and therefore it is always added to the original system when $\lambda \geq \frac{1}{2}p$ or $\frac{1}{2}q$.

Hence by the elementary theory of transvectants, if $\lambda \geq \frac{1}{2}n$, a covariant involving the factor $(ab)^\lambda$ can be expressed as an aggregate of covariants in each of which the letters a and b are replaced by a single symbol also belonging to a form of order not greater than n .

It follows at once that in seeking for the covariant of highest order such a covariant can be neglected because the same order would occur for a lower degree.

The covariants of Class II. in the preceding paper can therefore be neglected because (see § 1) either λ or $n_k - \lambda$ must be at least equal to $\frac{1}{2}n_k$.

Hence a covariant of the highest order will appear in the first class.

Now the highest order for degree i is $n_i - 2i + 2$, corresponding to the form $(a_1 a_2)^{2^{i-2}} (a_2 a_3)^{2^{i-3}} \dots (a_{i-1} a_i)$.