XLVIII. On Bernoulli's Solution of the problem of shortest twilight

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Thirlwall Castle a bed of encrinal limestone reposes on the basalt, and, as is usually the case when in contact with that rock, has become crystalline in texture, and phosphorescent if laid on red-hot iron: its colour is also changed from dark blue to yellowish white.

Newcastle-upon-Tyne, Dec. 18, 1832.

XLVIII. *On Bernoulli's Solution of the Problem of Shortest Twilight.* By T. S. DAVIES, Esq. F.R.S. L. & E., F.R.A.S., &c.

[Concluded from p. 185.]

BEFORE we proceed, however, to these supplementary theorems, we may make one or two other remarks upon the result just obtained.

By the theory of equations, if ρ' and ρ'' denote the values of ρ in the last equation, ρ' referring to the minimum, and ρ'' to the maximum cases, we have

$$\cos \rho' \cos \rho'' = \cos^2 \lambda \dots\dots\dots (10)$$

which is independent of the almancantars between which the star moves. Also taking the difference,

$$\begin{aligned} \cos \rho' - \cos \rho'' &= \cos \lambda \frac{\cos^2 \frac{1}{2} \rho' \cos \rho'' - \cos^2 \frac{1}{2} \rho_1 + \rho_{11}}{\cos \frac{1}{2} \rho' \cos \rho_{11} \cos \frac{1}{2} \rho_1 + \rho_{11}} \\ &= \cos \lambda \frac{\frac{1}{2} \{1 + \cos \rho' \cos \rho_{11} - 1 - \cos \rho_1 + \rho_{11}\}}{\frac{1}{2} (\cos \rho' \cos \rho_{11} + \cos \rho_1 + \rho_{11})} \\ &= \cos \lambda \frac{\cos \rho' \cos \rho_{11} - \cos \rho_1 + \rho_{11}}{\cos \rho' \cos \rho_{11} + \cos \rho_1 + \rho_{11}} \\ &= \cos \lambda \tan \rho_1 \tan \rho_{11} \dots\dots\dots (11) \end{aligned}$$

which, when ρ_1 and ρ_{11} (the zenith distances of the almancantars) are taken the complements of each other, the difference of the cosines of the polar distances of the sun for the maximum and minimum time is constant.

Again, from the same equation (9), we have

$$\cos \rho' + \cos \rho'' = 2 \cos \lambda \frac{1 + \cos \rho_1 \cos \rho_{11}}{\cos \rho_1 + \cos \rho_{11}} \dots\dots\dots (12)$$

and by division, we have

$$\frac{\cos \rho'}{\cos \rho''} = \frac{\cos^2 \frac{1}{2} \rho' \cos \rho_{11}}{\cos^2 \frac{1}{2} \rho_1 + \rho_{11}} \dots\dots\dots (13)$$

The question actually resolved is,—what declination of the

sun would cause the time of passing from one given almacantar to another to be a maximum or a minimum? The actual path of the sun is no part of the data; and hence D'Alembert's difficulty here arose from *tacitly* introducing a new condition into the *solution* of the problem, which had not been introduced into the process for finding the *equation* of the problem. It is true that under the physical circumstances which were known to exist (but which were unknown in the problem in the aspect under which he viewed it as a mathematician,) there was an obstacle to the maximum answer taking place actually in certain latitudes; but in other given latitudes the maximum might, with the present position of the ecliptic, be actually attained. His showing that the maximum could not be attained in certain particular cases, did not show that it was unattainable in all: and hence his inference, as a general one, that the second answer did not refer to the maximum case of the mathematical problem which he had solved, was in all respects inaccurate. It is important to keep in mind the fact, that the *mathematical* problem which we have solved is not always identical with the *physical* problem which we had *proposed*.

Delambre has given as a remarkable property of the azimuths at the beginning and end of twilight, in the case of the minimum, that they are supplementary; but this is not confined to the case in which the horizon is one of the almacantars, as the following investigations show:

1. Let us take $\rho_I + \rho_{II}$. Then

$$\begin{aligned} \cos PZN &= \frac{\cos \rho - \cos \lambda \cos \rho_I}{\sin \lambda \sin \rho_I} \\ &= \cot \lambda \frac{1 + \cos \rho_I \cos \rho_{II} - \sin \rho_I \sin \rho_{II} - \cos^3 \rho_I - \cos \rho_I \cos \rho_{II}}{\sin \rho_I (\cos \rho_I + \cos \rho_{II})} \\ &= \cot \lambda \frac{1 - \sin \rho_I \sin \rho_{II} - (1 - \sin^2 \rho_I)}{\sin \rho_I (\cos \rho_I + \cos \rho_{II})} \\ &= \cot \lambda \frac{\sin \rho_I - \sin \rho_{II}}{\cos \rho_I + \cos \rho_{II}} \\ &= \cot \lambda \tan \frac{1}{2} \rho_I - \rho_{II} \dots\dots\dots (14) \end{aligned}$$

And in the same way we find

$$\begin{aligned}\cos \text{PZR} &= \frac{\cos \rho - \cos \lambda \cos \rho_{II}}{\sin \lambda \sin \rho_{II}} = \cot \lambda \tan \frac{1}{2} (\rho_{II} - \rho_I) \\ &= -\cot \lambda \tan \frac{1}{2} (\rho_I - \rho_{II}) \dots (15)\end{aligned}$$

And by a comparison of (14) (15) we find at once that $\text{PZN} + \text{PZR} = \pi$, or the azimuths PZN , PZR are supplementary.

2. Take $\rho_I \infty \rho_{II}$. Then

$$\cos \text{PZN} = \frac{\cos \rho - \cos \lambda \cos \rho_I}{\sin \lambda \sin \rho_I} = \cot \lambda \tan \frac{1}{2} (\rho_I + \rho_{II}) \dots (16)$$

$$\cos \text{PZR} = \frac{\cos \rho - \cos \lambda \cos \rho_{II}}{\sin \lambda \sin \rho_{II}} = \cot \lambda \tan \frac{1}{2} (\rho_I + \rho_{II}) \dots (17)$$

Hence in this case the azimuths are equal; or the points in which the sun crosses the two almucantars are in the same vertical great circle,—a remarkable property.

To the case of the shortest twilight problem, the $\rho_I + \rho_{II}$ applies, and gives the same value as found by Delambre and others: but as it depends upon the respective values of ρ_I and ρ_{II} whether the $+$ or ∞ designates the minimum time between the almucantars, we cannot say that the supplemental or the equal relation of the azimuths belongs either to the maximum or to the minimum only.

In the supplementary case we may readily find the difference of the azimuths as follows:—

Let ξ_I and ξ_{II} denote the azimuths: then by the common properties of angles, we have

$$\cos \xi_I + \cos \xi_{II} = 2 \cos \frac{1}{2} (\xi_I + \xi_{II}) \cos \frac{1}{2} (\xi_I - \xi_{II}) \dots (a)$$

$$-\cos \xi_I + \cos \xi_{II} = 2 \sin \frac{1}{2} (\xi_I + \xi_{II}) \sin \frac{1}{2} (\xi_I - \xi_{II}) \dots (b)$$

But equation (a) is $= 0$, and in equation (b) we have

$$2 \sin \frac{1}{2} (\xi_I + \xi_{II}) = 1, \text{ and hence also}$$

$$-\cos \xi_I + \cos \xi_{II} = 2 \sin \frac{1}{2} (\xi_I - \xi_{II}) \dots (18)$$

But by (14) (15) we have

$$-\cos \xi_I + \cos \xi_{II} = -2 \cot \lambda \tan \frac{1}{2} \overline{\rho_I - \rho_{II}}, \text{ and hence}$$

$$\sin \frac{1}{2} (\xi_I - \xi_{II}) = -\cot \lambda \tan \frac{1}{2} \overline{\rho_I - \rho_{II}} \dots (19)$$

In the other case, or that in which azimuths are equal, we have also

$$\sin \frac{1}{2} (\xi_I - \xi_{II}) = \cot \lambda \tan \frac{1}{2} \overline{\rho_I + \rho_{II}} \dots (20)$$

Delambre, after Cagnoli and some other authors, shows that the angles at the sun are equal in the case of a minimum twilight. They are also equal or supplementary in all cases.

1. Take $\rho_I - \rho_{II}$: then

$$\begin{aligned}\cos \text{PNZ} &= \frac{\cos \lambda - \cos \rho \cos \rho_I}{\sin \rho \sin \rho_I} \\ &= \frac{\cos \lambda \left\{ 1 - \frac{1 + \cos \rho_I \cos \rho_{II} + \sin \rho_I \sin \rho_{II} \cos \rho_I}{\cos \rho_I + \cos \rho_{II}} \right\}}{\sin \rho \sin \rho_I} \\ &= \frac{\cos \lambda \{ \cos \rho_I + \cos \rho_{II} - \cos \rho_I - \cos^2 \rho_I \cos \rho_I - \cos \rho_I \sin \rho_I \sin \rho_{II} \}}{\sin \rho \sin \rho_I (\cos \rho_I + \cos \rho_{II})} \\ &= \frac{\cos \lambda \sin \overline{\rho_I - \rho_{II}}}{\sin \rho (\cos \rho_I + \cos \rho_{II})} = \frac{\cos \lambda}{\sin \rho} \cdot \frac{\sin \frac{1}{2} \overline{\rho_I - \rho_{II}}}{\cos \frac{1}{2} \rho_I + \rho_{II}} \dots\dots\dots (21)\end{aligned}$$

In a similar way, we find

$$\cos \text{PRZ} = \frac{\cos \lambda \sin \frac{1}{2} \overline{\rho_{II} - \rho_I}}{\sin \rho \cos \frac{1}{2} \rho_{II} - \rho_I} = - \frac{\cos \lambda}{\sin \rho} \cdot \frac{\sin \frac{1}{2} \overline{\rho_I - \rho_{II}}}{\cos \frac{1}{2} \rho_I + \rho_{II}} \dots (22)$$

And it follows at once that $\text{PNZ} + \text{PRZ} = \pi$, or the angles at the sun are supplementary of one another.

2. Take $\overline{\rho_I + \rho_{II}}$: then,

$$\begin{aligned}\cos \text{PNZ} &= \frac{\cos \lambda \left\{ 1 - \frac{1 + \cos \rho_I \cos \rho_{II} - \sin \rho_I \sin \rho_{II} \cos \rho_I}{\cos \rho_I + \cos \rho_{II}} \right\}}{\sin \rho \sin \rho_I} \\ &= \frac{\cos \lambda}{\sin \rho} \cdot \frac{\sin \frac{1}{2} \overline{\rho_I + \rho_{II}}}{\cos \frac{1}{2} \rho_I - \rho_{II}} \dots\dots\dots (23)\end{aligned}$$

And in the same way, we find

$$\cos \text{PRZ} = \frac{\cos \lambda}{\sin \rho} \cdot \frac{\sin \frac{1}{2} \overline{\rho_{II} + \rho_I}}{\cos \frac{1}{2} \rho_{II} - \rho_I} \dots\dots\dots (24)$$

Hence, since $\cos \frac{1}{2} \overline{\rho_I - \rho_{II}} = \cos \frac{1}{2} \overline{\rho_{II} - \rho_I}$, these angles are equal.

We may here remark, that Delambre's proof is insufficient to justify his inference of the equality of these angles, in the case he has considered. His argument is,

$$\text{since } \sin \text{ZNP} = \sin \text{PZN} \frac{\sin \lambda}{\sin \rho},$$

$$\sin \text{ZRP} = \sin \text{PZN} \frac{\sin \lambda}{\sin \rho},$$

$$\text{and } \sin \text{PZN} = \sin \text{RZN};$$

hence also $\text{ZNP} = \text{ZRP}$, and *therefore* the angles ZNP, ZRP are themselves equal. But manifestly *this does not show*

that they might not be supplemental instead of equal. Hence his deduction is not authorized. It is, however, true, as may be seen by putting the corresponding values $\frac{\pi}{2}$ and $\frac{\pi}{2} + 2a$ for ρ_I and ρ_{II} in the above equations: *vide* Leybourn's Edition of the Diaries, *ut supra*, or Delambre's *Histoire de l'Astronomie Ancienne*.

It has been objected to the solution of Bernoulli, by Delambre, that it does not give the time of duration of the shortest twilight, but the declination of the sun when the twilight is shortest. By referring, however, to Bernoulli's own words, we shall find that he proposed to determine "*le jour de plus petit crépuscule*," and not to find the *duration* on that day. As an objection, then, Delambre's amounts to this—that Bernoulli, having proposed to himself one problem, did not substitute the solution of another *instead* of the one he had proposed! This is travelling out of the way to find imperfections, most assuredly. It would, indeed, at first sight seem difficult to account for such an insignificant objection being urged at all; but doubtless it originated in the somewhat exaggerated expressions of admiration bestowed upon Bernoulli's solution by Montucla, and the fact of the historian of mathematics having greatly undervalued the original solution of Nunez. It is a fact, however, that cannot have escaped the notice of every one who has had occasion to consult Delambre's Histories, that there is a feeling not altogether friendly displayed by him towards both Montucla and Bailly; and that points of comparative insignificance are often dwelt upon at very incommensurable length, where he has discovered, or believes he has discovered, those authors to be in error. The present is one of them. Bernoulli did not, indeed, solve the other problems that were connected with the one he chose to attempt; but having solved that one, all the others might have been followed out without difficulty,—of which Bernoulli, of course, was well aware. Nor is it more difficult in the general case.

$$\begin{aligned}\text{For } \cos(\xi_I - \xi_{II}) &= 1 - 2 \sin^2 \frac{1}{2} (\xi_I - \xi_{II}) \\ &= 1 - 2 \cot^2 \lambda \tan^2 \frac{1}{2} \overline{\rho_I - \rho_{II}}\end{aligned}$$

But taking RN as the arc of a great circle joining the points R, N, we have $\cos RN = \cos \rho_I \cos \rho_{II} + \sin \rho_I \sin \rho_{II} \cos(\xi_I - \xi_{II})$

$$\begin{aligned}\sin^2 \frac{1}{2} (\theta_{II} - \theta_I) &= \frac{1 - \cos RN}{2 \sin^2 \rho} \\ &= \frac{1 - \cos \rho_I \cos \rho_{II} - \sin \rho_I \sin \rho_{II} + 2 \sin \rho_I \sin \rho_{II} \cot^2 \lambda \tan^2 \frac{1}{2} \overline{\rho_I - \rho_{II}}}{2 \sin^2 \rho}\end{aligned}$$

$$\begin{aligned}
&= \frac{1 - \cos \rho_I \infty \rho_{II} + 2 \sin \rho_I \sin \rho_{II} \cot^2 \lambda \tan^2 \frac{1}{2} \rho_I - \rho_{II}}{2 \sin^2 \rho} \\
&= \frac{\sin^2 \frac{1}{2} \rho_I \infty \rho_{II} + \sin \rho_I \sin \rho_{II} \cot^2 \lambda \tan^2 \frac{1}{2} \rho_I - \rho_{II}}{\sin^2 \rho} \\
&= \frac{\tan^2 \frac{1}{2} \rho_I \infty \rho_{II}}{\sin^2 \lambda \sin^2 \rho} \left\{ \cos^2 \frac{1}{2} \rho_I - \rho_{II} \sin^2 \lambda + \cos^2 \lambda \right\} \\
&= \frac{\tan^2 \frac{1}{2} \rho_I \infty \rho_{II}}{\sin^2 \lambda \sin^2 \rho} \left\{ 1 - \sin^2 \lambda \sin^2 \frac{1}{2} \rho_I - \rho_{II} \right\} \dots\dots\dots (25)
\end{aligned}$$

Substituting in (25) the values of ρ adapted to the two cases of $\rho_I \infty \rho_{II}$ and $\rho_I + \rho_{II}$, we shall have expressions for the corresponding times. It is altogether unnecessary to consider them here, or to enter upon a discussion of any other analogies between the properties, which Delambre has collected together, of the diagram employed, and those which belong to the more general problem discussed in this paper; since they do not all hold true, and where they do hold true they are mere matters of mathematical curiosity,—at the same time that they offer no real difficulties to any one who may be tempted to enter upon their investigation.

Bath, August 8, 1833.

* * Mr. Skene, in the prefatory remarks to his solution, mentions an analytical one by Fontana; but he does not say whether it was a separate publication or inserted in any of the foreign scientific Transactions. I have sought it without success. A solution by means of the Stereographic Projection of the figure was given by M. Dandelin of Liege, in vol. ii. of Quetelet's *Correspondance Mathématique et Physique*; but it offers nothing novel in principle to call for special remark. Several neat solutions may also be seen in the Gentleman's Diary for 1817; but they also differ more in form than in principle, from others which had been previously published.

XLIX. *On a Difficulty in the Theory of the Attraction of Spheroids.* By J. MACCULLAGH, F.T.C.D.*

AN approximate theorem discovered by Laplace, and relating to the attraction of a solid slightly differing from a sphere on a point placed at its surface, has given rise to many disputes. It has engaged the attention of Lagrange, Ivory,

* Communicated by the Author.