TYPES OF COVARIANTS OF ANY DEGREE IN THE COEFFICIENTS OF EACH OF ANY NUMBER OF BINARY QUANTICS OF FINITE ORDER

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INTRODUCTION (§§ 1–3).

1.

It is proposed to give a method of determining the type forms for the “complete system mod $(ab)^{\lambda_n}$ ” of any number of binary quantics. The following theorems are known:—

I.^{*} Any perpetuant linear in the coefficients of each of the binary quantics $a_{1,}^{n_1}, a_{2,}^{n_2}, \dots, a_{\delta,}^{n_\delta}$, where $n_1, n_2, \dots, n_\delta$ are all infinite, is linearly expressible in terms of

(i.) symbolical products of the form $(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \dots (a_{\delta-1} a_\delta)^{\lambda_{\delta-1}}$, where $\lambda_1 \geq 2^{\delta-2}$, $\lambda_2 \geq 2^{\delta-3}$, $\dots$, $\lambda_{\delta-1} \geq 1$, and the sequence of the letters $a_1, a_2, \dots, a_\delta$ is fixed beforehand;

(ii.) products of perpetuants of lower total degrees.

II.⁺ Any covariant linear in the coefficients of each of the binary quantics $a_{1,}^{n_1}, a_{2,}^{n_2}, \dots, a_{\delta,}^{n_\delta}$, where $n_1, n_2, \dots, n_\delta$ are all finite, is linearly expressible in terms of

(i.) symbolical products of the form $(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \dots (a_{\delta-1} a_\delta)^{\lambda_{\delta-1}}$, where $\lambda_1 \geq 2^{\delta-2}$, $\lambda_2 \geq 2^{\delta-3}$, $\dots$, $\lambda_{\delta-1} \geq 1$, and the sequence of the letters $a_1, a_2, \dots, a_\delta$ is fixed beforehand;

(ii.) covariants having a factor of the form $(a_i a_k)^\lambda (a_k a_l)^{n_k - \lambda}$;

(iii.) products of covariants of lower total degrees.

These results depend ultimately *only* on the repeated application of fundamental identities of the form $(bc) + (ca) + (ab) \equiv 0$, a, b, c being any three of the letters $a_1, a_2, \dots, a_\delta$, the letters being all taken as referring to different quantics.

Camille Jordan[‡] laid down the dictum that the process of determining type forms must be performed in two distinct steps: first, we must

* Grace, *Proc. London Math. Soc.*, Vol. xxxv.

† A. Young, *Ibid.*, Ser. 2, Vol. 1.

‡ Liouville, 1876.

determine the type forms solely by the use of identities such as the above ; and, secondly, we must find how the forms thus obtained are modified by the interchanges of letters referring to the same quantic. We know* that in Theorems I. and II. we have obtained all the possible results derivable from the application of the fundamental identity : the next step is to determine how the type forms (i.) in these theorems are modified when certain of the letters $a_1, a_2, \dots, a_\delta$ refer to the same quantic, and may therefore be freely interchanged.

2.

Grace† has obtained the result for perpetuants, by interchanging equivalent letters in the result of Theorem (I.), in the following form :—

III. (i.) If the symbols $a_1, a_2, \dots, a_\delta$ all refer to the same quantic. then the type form of an irreducible perpetuant is

$$(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \dots (a_{\delta-1} a_\delta)^{\lambda_{\delta-1}},$$

where

$$\lambda_{\delta-r} = 2^{r-1} + (\xi_{\delta-1} + \xi_{\delta-2} + \dots + \xi_{\delta-r}), \quad r = 1, 2, \dots, \delta-2,$$

$$\lambda_1 = 2^{\delta-2} + 2(\xi_{\delta-1} + \xi_{\delta-2} + \dots + \xi_2 + \xi_1),$$

the ξ 's being positive integers or zeros.

(ii.) If the symbols $a_1, a_2, \dots, a_\delta$ refer to one quantic, and the symbols $b_1, b_2, \dots, b_\epsilon$ refer to a second quantic, then the type form of an irreducible perpetuant of partial degrees δ and ϵ is

$$(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \dots (a_{\delta-1} a_\delta)^{\lambda_{\delta-1}} (a_\delta b_1)^{\lambda_\delta} (b_1 b_2)^{\mu_1} \dots (b_{\epsilon-1} b_\epsilon)^{\mu_{\epsilon-1}},$$

where

$$\mu_{\epsilon-r} = 2^{r-1} + \eta_\epsilon + \eta_{\epsilon-1} + \dots + \eta_{\epsilon-r+1}, \quad r = 1, 2, \dots, \epsilon-1,$$

$$\lambda_\delta = 2^{\epsilon-1} + \eta_\epsilon + \eta_{\epsilon-1} + \dots + \eta_2 + \eta_1,$$

$$\lambda_{\delta-r} = 2^{\epsilon+r-1} + \xi_{\delta-1} + \xi_{\delta-2} + \dots + \xi_{\delta-r}, \quad r = 1, 2, \dots, \delta-2,$$

$$\lambda_1 = 2^{\delta+\epsilon-2} + 2(\xi_{\delta-1} + \xi_{\delta-2} + \dots + \xi_2 + \xi_1),$$

the ξ 's and η 's being positive integers or zeros.

The corresponding result for the perpetuant type form of any degree in each of any number of binary quantics can be written down at once from the preceding result for two quantics : it should be remarked that

* Wood, "On the Irreducibility of Perpetuant Types," *Proc. London Math. Soc.*, Ser. 2, Vol. 1.

† *Proc. London Math. Soc.*, Vol. xxxv. Reference should be made to the early part of this paper for a statement of the general principles underlying the investigation.

the generating functions obtained in this way agree with those found independently by Stroh* and MacMahon.†

3.

It is the purpose of the present paper to show that, if in dealing with quantics of finite order we neglect covariants with a factor $(a_h a_k)^\lambda (a_\kappa a_l)^{n-\lambda}$ (that is, forms of the second class in Theorem II. above), then we obtain exactly the same results as in Theorem III. for types of covariants of quantics of finite order: we shall, in fact, show that

IV. Any covariant of degree δ in the coefficients of

$$a_x^m (\equiv a_{1_x}^m \equiv a_{2_x}^m \equiv \dots \equiv a_{\delta_x}^m)$$

and of degree ϵ in the coefficients of $b_x^n (\equiv b_{1_x}^n \equiv b_{2_x}^n \equiv \dots \equiv b_{\delta_x}^n)$ is linearly expressible in terms of

(i.) symbolical products of the form

$$(a_1 a_2)^{\lambda_1} \dots (a_\delta b_1)^{\lambda_\delta} (b_1 b_2)^{\mu_1} \dots (b_{\epsilon-1} b_\epsilon)^{\mu_{\epsilon-1}},$$

where the indices λ, μ satisfy the conditions given in III., (ii.);

(ii.) covariants having a factor $(a\alpha)^\lambda (a\beta)^{m-\lambda}$ or a factor $(b\alpha)^\lambda (b\beta)^{n-\lambda}$;

(iii.) products of covariants of lower total degrees.

It will be seen that the method of proof admits of extension to the case of covariants of any degree in the coefficients of each of any number of quantics: we shall throughout speak of covariants included in class (ii.) of Theorem II. as "covariants of the second class." The succeeding proof consists only of a suitable modification and combination of the proofs of Theorems II. and III. as given by their respective authors.

INTRODUCTORY LEMMAS (§§ 4-6).

4.

It will be sufficient in the first place to give an outline of such portions of Young's proof of Theorem III. as are essential to the present investigation: the two succeeding Lemmas consist only of a modified form of the corresponding Lemmas, as originally given by Grace in the course of the proof of Theorem III.

Summary of Results contained in the Proof of Theorem III.

(i.) Consider the covariant $C_\delta \equiv (a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \dots (a_{\delta-1} a_\delta)^{\lambda_{\delta-1}}$; if we write

* *Math. Ann.*, Bd. xxxvi.

† *Proc. London. Math. Soc.*, Vol. xxvi,

$$a_{4_x}^{n_4-\lambda_4} q_y^\rho \equiv (a_4 a_5)^{\lambda_4} (a_5 a_6)^{\lambda_5} \dots (a_{\delta-1} a_\delta)^{\lambda_{\delta-1}} a_{4_x}^{n_1-\lambda_4} a_{5_y}^{n_5-\lambda_4-\lambda_5} \dots a_{\delta_y}^{n_\delta-\lambda_{\delta-1}},$$

then C_δ is a term of the transvectant

$$\{(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2}, a_{4_x}^{n_4-\lambda_4} q_y^\rho\}_{y=x}^{\lambda_3}$$

and so differs from it or from any other one of its terms by covariants involving more factors in a_1, a_2, a_3 alone: if either λ_1 or λ_2 is less than $2^{\delta-3}$, then it may be shown that this transvectant and each of its terms is linearly expressible in terms of covariants of the second class, of products of covariants, and of covariants involving more factors in a_1, a_2, a_3 alone.

(ii.) If we put $(a_1 a_2)^\mu \equiv a_x^{n_1+n_2-2\mu}$, and consider forms of the second class arising from the $(\delta-1)$ quantics $a_x^{n_1+n_2-2\mu}, a_{3_x}^{n_3}, \dots, a_{\delta_x}^{n_\delta}$, we include among such forms those covariants having a factor

$$(a_h a)^\nu (a a_x)^{n_1+n_2-2\mu-\nu};$$

it may be shown that any such covariant is a form of the second class arising from the original δ quantics $a_{1_x}^{n_1}, a_{2_x}^{n_2}, \dots, a_{\delta_x}^{n_\delta}$.

(iii.) Finally, if in the transvectant $\{(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2}, a_{4_x}^{n_4-\lambda_4} q_y^\rho\}_{y=x}^{\lambda_3}$ any one of n_1, n_2, n_3 is less than $(\lambda_1 + \lambda_2)$, then that transvectant gives rise only to covariants of the second class; so that, if we neglect covariants of the second class, the Jordan Lemma or Stroh's generalized form of the Lemma may be applied to the first member

$$(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2}$$

of the transvectant.

5.

As we are concerned only with the type forms (class i.), we shall henceforth neglect covariants of the second class and products of covariants.

LEMMA I.—The covariant

$$(a_1 a_2)^\lambda (a_2 a_3)^\mu (a_3 a_4)^\nu \dots \equiv (a_1 a_2)^\lambda (a_2 a_3)^\mu P,$$

wherein a_1, a_2, a_3 , refer to the same quantic, is expressible in the form

$$\Sigma (a_1 a_2)^{\lambda'} (a_2 a_3)^{\mu'} P,$$

where λ' is even and $\geq 2\mu'$.

The covariant $(a_1 a_2)^\lambda (a_2 a_3)^\mu P$ is a term of the transvectant

$$T \equiv \{ (a_1 a_2)^\lambda (a_2 a_3)^\mu, \quad a_{4_x}^{n_4 - \lambda_4} q_y^\rho \}_{y=x}^\nu,$$

and therefore differs from T only by covariants involving more factors in a_1, a_2, a_3 alone. Also, by § 4, (iii.), $(a_1 a_2)^\lambda (a_2 a_3)^\mu$ is, by means of the Jordan Lemma, linearly expressible in terms of the covariants of the three sets

$$\left. \begin{aligned} & (a_1 a_2)^\alpha (a_2 a_3)^\beta \\ & (a_2 a_3)^\alpha (a_3 a_1)^\beta \\ & (a_3 a_1)^\alpha (a_1 a_2)^\beta \end{aligned} \right\}, \quad \alpha \geq 2\beta,$$

where $\alpha + \beta = \lambda + \mu$; and therefore T is expressible as a sum of transvectants of the three sets

$$\left. \begin{aligned} & \{ (a_1 a_2)^\alpha (a_2 a_3)^\beta, \quad a_{4_x}^{n_4 - \lambda_4} q_y^\rho \}_{y=x}^\nu \\ & \{ (a_2 a_3)^\alpha (a_3 a_1)^\beta, \quad a_{4_x}^{n_4 - \lambda_4} q_y^\rho \}_{y=x}^\nu \\ & \{ (a_3 a_1)^\alpha (a_1 a_2)^\beta, \quad a_{4_x}^{n_4 - \lambda_4} q_y^\rho \}_{y=x}^\nu \end{aligned} \right\}, \quad \alpha \geq 2\beta.$$

Since a_1, a_2, a_3 refer to the same quantic, they are interchangeable and the three sets of transvectants are equivalent to the first set alone. Now the transvectant $\{ (a_1 a_2)^\alpha (a_2 a_3)^\beta, \quad a_{4_x}^{n_4 - \lambda_4} q_y^\rho \}_{y=x}^\nu$ contains either the term $(a_1 a_2)^\alpha (a_2 a_3)^\beta (a_3 a_4)^\nu \dots$, if $\nu \leq n_3 - \beta$, or the term $(a_2 a_3)^\beta (a_3 a_4)^{n_3 - \beta} Q$, if $\nu \geq n_3 - \beta$, Q being any symbolical product of all the letters except a_3 ; the latter of these terms is a covariant of the second class. The whole transvectant differs from one or other of these terms by covariants involving more factors in a_1, a_2, a_3 alone. Hence the covariant

$$(a_1 a_2)^{\lambda_2} (a_2 a_3)^\mu P$$

is linearly expressible in terms of

- (i.) covariants $(a_1 a_2)^\alpha (a_2 a_3)^\beta P$, $\alpha \geq 2\beta$;
- (ii.) covariants with more factors in a_1, a_2, a_3 alone.

The covariants (ii.) may be treated in the same way, and, since the number of factors in a_1, a_2, a_3 alone cannot be increased indefinitely without giving rise to product forms, it follows that $(a_1 a_2)^\lambda (a_2 a_3)^\mu P$ is, neglecting covariants of the second class and product forms, expressible in terms of covariants

$$(a_1 a_2)^{\lambda'} (a_2 a_3)^{\mu'} P, \quad \text{where } \lambda' \geq 2\mu'.$$

Finally, if λ' is odd, it may be increased by unity by interchanging the equivalent letters a_1 and a_2 : hence we obtain the result stated.

6.

LEMMA II.—The covariant of degree δ

$$(a_1 a_2)^\lambda (a_2 a_3)^\mu (a_3 a_4)^\nu \dots \equiv (a_1 a_2)^\lambda (a_2 a_3)^\mu P,$$

wherein a_2, a_3 refer to the same quantic, while a_1 refers to a different quantic, is expressible in the form

$$\Sigma (a_1 a_2)^{\lambda'} (a_2 a_3)^{\mu'} P,$$

where $\lambda' \geq \mu' + 2^{\delta-3}$.

The covariant $(a_1 a_2)^\lambda (a_2 a_3)^\mu P$ is a term of the transvectant

$$T \equiv \{ (a_1 a_2)^\lambda (a_2 a_3)^\mu, \quad a_{4_x}^{n_4-\lambda_4} q_{y\}^{\rho\} \}_{y=z}^\nu,$$

and therefore differs from it by covariants involving more factors in a_1, a_2, a_3 alone; also, by § 4, (iii.), $(a_1 a_2)^\lambda (a_2 a_3)^\mu$ may, by means of Stroh's generalized form of the Jordan Lemma, be expressed linearly in terms of covariants

$$(a_1 a_2)^\omega, \quad (a_1 a_2)^{\omega-1} (a_2 a_3), \quad \dots, \quad (a_1 a_2)^{\omega-r} (a_2 a_3)^r,$$

$$(a_1 a_3)^\omega, \quad (a_1 a_3)^{\omega-1} (a_3 a_2), \quad \dots, \quad (a_1 a_3)^{\omega-r} (a_3 a_2)^r,$$

$$(a_2 a_3)^\omega, \quad (a_2 a_3)^{\omega-1} (a_3 a_1), \quad \dots, \quad (a_2 a_3)^{\omega-r'} (a_3 a_1)^{r'},$$

where $\omega = \lambda + \mu$, and, if p is chosen arbitrarily, r is a positive integer given by

$$2(r+1) + p + 1 = \omega + 1 \text{ or } \omega + 2.$$

[If r is given by $2(r+1) + p + 1 = \omega + 2$, there is a linear relation connecting the $(\omega+2)$ covariants above.]

Hence T is expressible linearly in terms of transvectants of the sets

$$\left\{ (a_1 a_2)^{\omega-r'} (a_2 a_3)^{r'}, \quad a_{4_x}^{n_4-\lambda_4} q_{y\}^{\rho\} \right\}_{y=z}^\nu, \quad r' = 0, 1, 2, \dots, r,$$

$$\left\{ (a_1 a_3)^{\omega-r'} (a_3 a_2)^{r'}, \quad a_{4_x}^{n_4-\lambda_4} q_{y\}^{\rho\} \right\}_{y=z}^\nu,$$

$$\left\{ (a_2 a_3)^{\omega-r'} (a_3 a_1)^{r'}, \quad a_{4_x}^{n_4-\lambda_4} q_{y\}^{\rho\} \right\}_{y=z}^\nu, \quad p' = 0, 1, 2, \dots, p:$$

and, since a_2 and a_3 refer to the same quantic, the first two sets are equivalent to the first set alone.

Take $p = 2^{\delta-3} - 1$; then, by § 4, (ii.), each transvectant of the third set

is expressible as a sum of covariants of the second class, of products of covariants, and of covariants with more factors in a_1, a_2, a_3 alone; also

$$\omega - 2r = p + 1 \quad \text{or} \quad p + 2,$$

and therefore

$$\omega - 2r \geq 2^{\delta-3}.$$

Now the transvectant $\{(a_1 a_2)^{\omega-r'} (a_2 a_3)^{r'}, a_{4_x}^{n_4-\lambda_4} q_y^{\rho}\}_{y=x}^{\nu}$ contains either the term $(a_1 a_2)^{\omega-r'} (a_2 a_3)^{r'} (a_3 a_4)^{\nu} \dots$, if $\nu \leq n_3 - r'$, or the term $(a_2 a_3)^{r'} (a_3 a_4)^{n_3-r'} Q$, if $\nu \geq n_3 - r'$, Q being any symbolical product of all the letters except a_3 ; and the latter term is a covariant of the second class. Therefore this transvectant differs from one or other of these terms by covariants involving more factors in a_1, a_2, a_3 alone.

Hence the covariant $(a_1 a_2)^{\lambda} (a_2 a_3)^{\mu} P$ is expressible linearly in terms of covariants $(a_1 a_2)^{\lambda'} (a_2 a_3)^{\mu'} P$, where $\lambda' \geq \mu' + 2^{\delta-3}$, and of covariants with more factors in a_1, a_2, a_3 alone: the latter may be treated in the same way, and so finally we obtain the result stated.

COVARIANTS OF ANY DEGREE IN THE COEFFICIENTS OF A SINGLE QUANTIC (§§ 7-8).

7.

We shall first establish inductively the results for covariants of a single quantic: the treatment of the general case follows immediately.

Covariants of Degree 3.

The forms to be considered are $(a_1 a_2)^{\lambda} (a_2 a_3)^{\mu}$, and two cases of importance occur:—

(i.) $a_1 \neq a_2 = a_3$.—Here we have $\mu \geq 1$, and, by Lemma II. (§ 6), $\lambda \geq \mu + 1$; so

$$\mu = 1 + \eta, \quad \lambda = 2 + \xi + \eta,$$

ξ, η being positive integers or zero.

(ii.) $a_1 = a_2 = a_3$.—Here we have $\mu \geq 1$, and, by Lemma I. (§ 5), λ is even and $\geq 2\mu$; so

$$\mu = 1 + \eta, \quad \lambda = 2 + 2(\xi + \eta),$$

ξ, η being positive integers or zero.

8. *Covariants of Degree δ .*

Assume that the type forms $(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \dots (a_{\delta-2} a_{\delta-1})^{\lambda_{\delta-2}}$ of degree $(\delta-1)$ are already determined as follows:—

(i.) $a_1 \neq a_2 = \dots = a_{\delta-1}$.—

$$\lambda_{\delta-r} = 2^{r-2} + \xi_{\delta-2} + \xi_{\delta-3} + \dots + \xi_{\delta-r}, \quad r = 2, 3, \dots, \delta-1.$$

(ii.) $a_1 = a_2 = a_3 = \dots = a_{\delta-1}$.—

$$\lambda_{\delta-r} = 2^{r-2} + \xi_{\delta-2} + \xi_{\delta-3} + \dots + \xi_{\delta-r}, \quad r = 2, 3, \dots, \delta-2;$$

$$\lambda_1 = 2^{\delta-3} + 2(\xi_{\delta-2} + \xi_{\delta-3} + \dots + \xi_2 + \xi_1).$$

These are, in the first instance, the two most important cases of equivalence among the letters: it will be sufficient, since we have verified these results for degree 3, to show that the corresponding type forms of degree δ are determined in the same way. The forms to be considered are $C_\delta \equiv (a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \dots (a_{\delta-1} a_\delta)^{\lambda_{\delta-1}}$; putting

$$a_x^{n_1+n_2-2\lambda_1} \equiv (a_1 a_2)^{\lambda_1},$$

we see that C_δ differs from

$$(a a_3)^{\lambda_2} (a_3 a_4)^{\lambda_3} \dots (a_{\delta-1} a_\delta)^{\lambda_{\delta-1}}$$

only by covariants with more factors $(a_1 a_2)$. Here $a_3, a_4, \dots, a_\delta$ refer to one quantic and a refers to another, and therefore by hypothesis, from (ii.),

$$\lambda_{\delta-r} = 2^{r-1} + \xi_{\delta-1} + \xi_{\delta-2} + \dots + \xi_{\delta-r}, \quad r = 1, 2, \dots, \delta-2.$$

[In applying this result for degree $(\delta-1)$, we are neglecting covariants of the second class among which will be included forms with a factor

$$(a_h a)^\nu (a a_x)^{n_1+n_2-2\lambda_1-\nu};$$

but, by § 4, (ii.), such forms are covariants of the second class as originally defined.]

The covariants with more factors in $(a_1 a_2)$ may be treated in the same way. The first exponent λ_1 is determined thus:—

(i.) $a_1 \neq a_2 = a_3 = \dots = a_\delta$.—By Lemma II. (§ 6),

$$\lambda_1 \geq \lambda_2 + 2^{\delta-3};$$

and therefore $\lambda_1 = 2^{\delta-2} + \xi_{\delta-1} + \xi_{\delta-2} + \dots + \xi_2 + \xi_1.$

(ii.) $a_1 = a_2 = a_3 = \dots = a_\delta$.—By Lemma I. (§ 5),

$$\lambda_1 \text{ is even and } \geq 2\lambda_2;$$

$$\text{and therefore } \lambda_1 = 2^{\delta-2} + 2(\xi_{\delta-1} + \xi_{\delta-2} + \dots + \xi_2 + \xi_1).$$

Hence the results are true universally.

If the order of the binary quantic is n , then every covariant of the second class is of grade $\geq \frac{1}{2}n$; so that the type forms enumerated above constitute the “complete system mod $(ab)^{\frac{1}{2}n}$ ” for the binary quantic in question.

COVARIANTS OF ANY DEGREE IN THE COEFFICIENTS OF EACH OF TWO QUANTICS (§ 9).

9.

The treatment of covariants of two quantics proceeds inductively on similar lines: assuming that the restrictions on the indices λ and μ as given in Theorem IV. (§ 3) are true for the partial degrees $(\delta-1)$ and ϵ , we show that they are true for the partial degrees δ and ϵ . We have shown (§ 8) that, for the type form of an irreducible perpetuant

$$(a_1 b_1)^{\lambda_1} (b_1 b_2)^{\mu_1} (b_2 b_3)^{\mu_2} \dots (b_{\epsilon-1} b_\epsilon)^{\mu_{\epsilon-1}}$$

of partial degrees unity and ϵ ,

$$\mu_{\epsilon-r} = 2^{r-1} + \eta_\epsilon + \eta_{\epsilon-1} + \dots + \eta_{\epsilon-r+1}, \quad r = 1, 2, \dots, \epsilon-1;$$

$$\lambda_1 = 2^{\epsilon-1} + \eta_\epsilon + \eta_{\epsilon-1} + \dots + \eta_2 + \eta_1.$$

Next consider the form $(a_1 a_2)^{\lambda_1} (a_2 b_1)^{\lambda_2} (b_1 b_2)^{\mu_1} \dots (b_{\epsilon-1} b_\epsilon)^{\mu_{\epsilon-1}}$; putting $a_x^{\eta_1 + \eta_2 - 2\lambda_1} \equiv (a_1 a_2)^{\lambda_1}$, we can show, as in § 8, that, by the result just given,

$$\mu_{\epsilon-r} = 2^{r-1} + \eta_\epsilon + \eta_{\epsilon-1} + \dots + \eta_{\epsilon-r+1}, \quad r = 1, 2, \dots, \epsilon-1,$$

$$\lambda_2 = 2^{\epsilon-1} + \eta_\epsilon + \eta_{\epsilon-1} + \dots + \eta_2 + \eta_1;$$

and then, by Theorem II. (§ 1), we have $\lambda_1 \geq 2^\epsilon$.

Finally, we deduce the type form for partial degrees δ and ϵ , as given in Theorem IV., from that for partial degrees $(\delta-1)$ and ϵ , by putting $(a_1 a_2)^{\lambda_1} \equiv a_x^{\eta_1 + \eta_2 - 2\lambda_1}$ as in § 8, and then determining the value of λ_1 by Lemma I. (§ 8).

The type forms enumerated in Theorem IV. constitute the complete system mod $(a' b')^{\frac{1}{2}m}$ or mod $(a' b')^{\frac{1}{2}n}$ of the two quantics a_x^m and b_x^n .

10.

It is obvious that this inductive method admits of extension to the treatment of covariants of any degree in the coefficients of each of any number of binary quantics; the value of the first exponent is always established either by Theorem II. (§ 1), or by one of the two Lemmas (§§ 5-6), and the remaining exponents are determined by induction by putting $a_x^{n_1+n_2-2\lambda_1} \equiv (a_1 a_2)^{\lambda_1}$ and making use of the result for degree one less in the first quantic. It is easily seen that the exponents thus determined are exactly the same as in the case of perpetuants, since Theorem II. (§ 1), and the two Lemmas (§§ 5-6), by means of which the results are established, differ from Theorem I. (§ 1) and the Lemmas given by Grace for the proof of Theorem III. (§ 2) only by the inclusion of the covariants of the second class.