Nicomachus and his immediate successors took most delight were such as these: "All the powers of five end in five," "Square numbers are the sums of a series of odd numbers." The symbols of Euclid could not be employed in demonstrating such propositions, so that inductive reasoning gradually led the way to algebra.

(To be continued)

GRAPH WORK IN ELEMENTARY ALGEBRA.*

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Much is now being said concerning the use of the graph in elementary algebra. The movement, I believe, is a good one and deserves at least the careful thought of each algebra teacher. In this paper I do not wish to advance a new theory but I do wish to select from the abundant resources a few typical problems which will serve to illustrate some of the advantages to be gained through the use of the graph. By request, I shall attempt to put the thoughts in a form to be used by a teacher in algebra, who may not be familiar with the higher mathematics.

As soon as work in simultaneous equations in two unknowns is reached, I believe it wise to teach the fact that an equation of the first degree in two unknowns always represents a straight line. After an actual trial of the graph in the algebra class room, and that through a number of years, I am led to believe that there is no more satisfactory manner to illustrate and solve some of the simple equations of the first degree than by the use of the graph. To find at one time values for two unknowns is a conclusion at which the pupil is delighted to arrive. A similar process will indicate that two equations of the first degree in two unknowns will give graphs,

Fig. 1.

*This article is a portion of the paper read by Mr. Touton before the Missouri Society of Teachers of Mathematics at their first annual meeting.

which if they intersect at all in finite space, can intersect at but one point. The position of this point, common to both lines, will give values which will satisfy both equations. This may easily be seen by plotting the graphs of the equations $5x-20y=0$ and $3x+12y=-18$, as in Fig. 1. The point of intersection, $x=-2$, $y=-1$, will be common to both lines, hence its coördinates will satisfy the equation; another pair such as $x+2y=10$ and $3x+6y=30$ will invariably give the same straight line, hence the values which satisfy one equation must always satisfy the other.

Fig. 2.

Again in Fig. 3, it will be possible to show that independent equations in two unknowns and first degree, as $2x+5y=10$ and $4x+10y=50$, can have no common values for x and y . This conclusion when reached by the pupil is not questioned, for nothing can be more clear to him than that two parallel straight lines can never meet in finite space. Since the place of meeting determines the values of the

Fig. 3.

unknowns—where there is no common point, there will be no common values for the unknown quantities. Pairs of equations which give lines intersecting in the second, third and fourth quadrants will give an added illustration of value concerning the somewhat intangible negative quantity.

When work in quadratics is reached, the pupil will have no

difficulty in understanding that a quadratic equation always represents a curve which can be cut in two places by a straight line. He will be interested to know the nature of the curve which a certain quadratic equation represents. When the equations $x^2 + y^2 = 101$ and $x + y = 9$ are considered, he will be pleased to note that when plotted, these equations will give graphs (Fig. 4.) of a circle cut in two places by a straight line, at the points (1, -10) and (-10, 1) respectively. He will note that the coördinates of these points will give roots which will satisfy both equations.

In a similar manner it will be found, as in Fig 5, that the equations $xy = 12$ and $x^2 + y^2 = 40$ will give graphs of a circle cut by a hyperbola at four points. Here the coördinates of the four points of intersection $(\pm 2, \pm 6)$ and $(\pm 6, \pm 2)$ will be the roots of the equations, because the set equation have two pairs of double roots.

He will also note that an equation of the form $4x^2 + 25y^2 = 100$ will have an ellipse (Fig. 6) for its graph, while $y = x^2$ (Fig. 7) has a parabola for its graph. It will be apparent at once that the graphs of both the ellipse and the parabola can each be cut

Fig. 4.

Fig. 5.

Fig. 6.

in two places by a straight line.

The quadratic equations will always give rise to a double set of values for x and y . Often in the solution of quadratics the roots will be imaginary. Just here comes up one of the strong points in favor of the graphic representation of points and lines. The pupil is interested to know that it is even possible to plot imaginary roots. To plot the complex number $(-5-8\sqrt{-1})$

(Fig. 8) when (-5) denotes that a distance to be laid off on the x axis five units to the left of the origin while $(-8\sqrt{-1})$ denotes that a distance of eight units should be laid downward on the y axis, passing through (-5) on the x axis, or that the point thus located represents the complex number $-5-8\sqrt{-1}$.

Then, too, the graphic representation of surds is a process of interest to the pupil. The pupil will enjoy the representation of quantities, the exact values of which are usually conceded as impossible. The $\sqrt{5}$ is an incommensurable root, yet in Fig. 9, a distance from the origin to a point located by $x=2$, and $y=1$, will give an exact represented value.

In general, graph work does not need emphasis on the part of the teacher, for interest on the part of the pupil will supply the

Fig. 7.

Fig. 8.

need. The graph work required that the teacher may successfully introduce the work into the algebra class need not be extensive and I believe can be easily mastered by the teacher who has had no previous training in analytics. If the graph work is not included within the range of the algebra text used, there are several sources from which the information can be secured in pamphlet form. In the pamphlets I men-

Fig. 9.

tion, the matter is in such a form as can easily be mastered by the pupil in the study hours. Among such works are the following:

- Graphs, by Aley, D. C. Heath and Company.
- Graphs, by Short, D. C. Heath and Company.
- Graphs, by Nipher, Henry Holt and Company.
- Graphs, by Newson, Ginn and Company.

As to the time to be spent on graph work, I believe that it need not take the time necessary for the mastery of the other methods of handling equations. The time will be well spent and I believe that the solution of equations by graphic methods should at least be given place along with the other methods of solution for stated equations.

Of course, we must be careful that the pendulum shall not swing too far past the center, yet I know that the thoughts presented in this paper when presented in any algebra class, do incite interest on the part of the pupil and at the same time give subject matter of worth. A test upon graph work in the class room is always welcome, for the pupils, without exception, like the work. They will also enjoy the solution of graphic methods of certain written problems where rates of motion are considered. While I advocate the use of the graph, I do not intend to take from the work of any other department nor do I intend to trespass upon the field of analytics with the first year algebra pupil, but I do contend that the work of the high school algebra class shall be such as

will best prepare our pupils to work with a degree of accuracy and understanding the simple problems he is to meet in his science work.

LIMITS IN GEOMETRIC FORMS.

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The current doctrine of the text-book world regards the straight line and the circle as two essentially different things. The straight line is regarded as the limit toward which the circle tends, but which it never reaches. So also the circle is regarded as the limit toward which the regular polygon tends, but which it *never* reaches. The theorems regarding the circumference and the area of a circle are derived on the supposition that the circle is the limit which the regular polygon almost but never quite reaches, and that the error is negligible. But we always have the reservation that the circle is *not* a polygon, say what you will, and that there *is* an error, however small it may be; less than any assignable quantity, but yet an error after all. The difference between the circle and polygon is so small that for all practical purposes we may consider them as one; but, of course, they are not one, and never can be, etc., etc.

And through all this array of verbiage, we feel that there is a fallacy somewhere; it is and it is not, all in the same breath; the error is inexpressible and yet the forms do not coincide. We can push the polygon *almost* to the circle; what is that invisible barrier which keeps it back?

There is no barrier except our own narrow definitions and methods. The straight line is a circle of special form, not the limit of a circle; the circle is a polygon, not the limit of a polygon. There is no residual error. The circle straightens out into a straight line and sweeps over it into a circle on the other side. The inscribed polygon merges into the circle and sweeps over it into a polygon again on the outside.

Let us see what is meant by a limit, and why forms have limits. A limit is that constant value (or form) which a variable value (or form) approaches indefinitely near but never quite