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LXXVIII. *The Modification of the Parabolic Trajectory on the Theory of Relativity.* By W. B. MORTON, M.A., *Queen's University, Belfast**.

THE simplest case of accelerated motion, that in which a particle moves in a straight line with constant "rest-acceleration," was investigated by Born† who gave it the name of "hyperbolic motion" because the distance-time graph, or "world-line," has that form. The next in order of simplicity is that which corresponds to the parabolic path under gravity—*i. e.*, when the particle has rest-acceleration constant in magnitude and direction, but possesses a velocity-component perpendicular to this direction.

Let the origin be taken at the point where the velocity is perpendicular to the acceleration. It will be convenient to speak of the tangent to the path at this point as "horizontal" and the direction of the acceleration as the downwards vertical. Let the axes of x and y be taken in these directions respectively, u is the velocity at the origin, v the velocity at any point, and c the speed of light. The inclination of the tangent to the horizontal will be denoted by ϕ .

The rest-acceleration is derived from the ordinary acceleration as estimated from a "fixed" standpoint by multiplying the tangential component by γ^3 and the normal component by γ^2 where $\gamma = 1/\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}$.

Therefore in the present case the resultant of $\gamma^3 \frac{v dv}{ds}$ along the tangent and $\gamma^2 \frac{v^2}{\rho}$ along the normal is in the direction of Oy and has the constant magnitude f say. The equations of motion are

$$\gamma^3 \frac{v dv}{ds} \cos \phi - \gamma^2 \frac{v^2}{\rho} \sin \phi = 0 \quad . \quad . \quad . \quad (1)$$

$$\text{and} \quad \gamma^2 \frac{v^2}{\rho} = f \cos \phi \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equation (1) gives $\gamma \frac{dv}{d\phi} = v \tan \phi$.

Put $v = c \sin \theta$ so that $\gamma = \sec \theta$, then $\operatorname{cosec} \theta d\theta = \tan \phi d\phi$, giving the relation

$$\tan \frac{1}{2} \theta \cdot \cos \phi = \text{const.} = \tan \frac{1}{2} \theta_0, \text{ say,} \quad . \quad . \quad (3)$$

where $\sin \theta_0 = u/c$.

* Communicated by the Author.

† Born, *Ann. d. Phys.* xxx. p. 1 (1909).

When θ is small this equation obviously reduces to $v \cos \phi = \text{const.}$ in accordance with the elementary theory.

The subsequent analysis is made much neater by the introduction of an angular parameter α defined by

$$\sin \alpha = \tan \frac{1}{2} \theta_0 = \{c - (c^2 - u^2)^{\frac{1}{2}}\}/u.$$

Using this equation (3) gives the connexion between magnitude and direction of v in the form

$$v/c = \sin \theta = 2 \sin \alpha \cos \phi / (\cos^2 \phi + \sin^2 \alpha). \quad (4)$$

From equation (2) $\rho = c^2 \tan^2 \theta \sec \phi / f$.

The linear scale of the path may be expressed in terms of the parameter of the parabola which would be described on the old theory,

$$a = u^2 / 2f = 2c^2 \sin^2 \alpha / f (1 + \sin^2 \alpha)^2.$$

Using this, and the relation (3), we express the radius of curvature in terms of ϕ and the constants a, α ,

$$\rho = ds/d\phi = 2a(1 + \sin^2 \alpha)^2 \cos \phi / (\cos^2 \phi - \sin^2 \alpha)^2.$$

The integral of this is the intrinsic equation of the path

$$s = \frac{a(1 + \sin^2 \alpha)^2}{2 \cos^3 \alpha} \left\{ \frac{2 \cos \alpha \sin \phi}{\cos(\phi + \alpha) \cos(\phi - \alpha)} + \log \frac{\cos \alpha + \sin \phi}{\cos \alpha - \sin \phi} \right\}. \quad (5)$$

Integrating

$$dx/d\phi = \cos \phi \cdot ds/d\phi \quad \text{and} \quad dy/d\phi = \sin \phi \cdot ds/d\phi,$$

we obtain for the coordinates of a point on the path the expressions

$$x = \frac{a(1 + \sin^2 \alpha)^2}{2 \cos^2 \alpha} \left\{ \frac{\sin 2\phi}{\cos(\phi + \alpha) \cos(\phi - \alpha)} + \frac{2}{\sin 2\alpha} \log \frac{\cos(\phi - \alpha)}{\cos(\phi + \alpha)} \right\} \quad (6)$$

$$y = a(1 + \sin^2 \alpha)^2 \{ \sec(\phi + \alpha) \sec(\phi - \alpha) - \sec^2 \alpha \}. \quad (7)$$

It only remains to find the relation between ϕ and the time. This can be obtained from (4) by writing

$$v = ds/dt = ds/d\phi \times d\phi/dt,$$

leading to

$$cdt/d\phi = a(1 + \sin^2 \alpha)^2 (\cos^2 \phi + \sin^2 \alpha) / \sin \alpha (\cos^2 \phi - \sin^2 \alpha)^2,$$

the integral of which is

$$ct = \frac{a(1 + \sin^2 \alpha)^2}{2 \sin \alpha \cos^2 \alpha} \left\{ \frac{\sin 2\phi}{\cos(\phi + \alpha) \cos(\phi - \alpha)} + \tan \alpha \log \frac{\cos(\phi - \alpha)}{\cos(\phi + \alpha)} \right\}. \quad (8)$$

The following properties of the motion are seen immediately from the equations. In the first place as ϕ approaches the value $\left(\frac{\pi}{2} - \alpha\right)$, x, y , and t become infinite and v approaches

c . That is to say, the direction of motion, instead of swerving

through a right angle and approaching coincidence with the acceleration, as in the parabolic case, is turned through $\left(\frac{\pi}{2} - \alpha\right)$ and the particle approaches a steady motion at the speed of light in a direction inclined at angle α to that of the constant "rest-acceleration." The following table gives the values of α for different values of u/c :—

u/c .	α .
1	$2^{\circ} 52'$
2	$5^{\circ} 48'$
3	$8^{\circ} 50'$
4	$12^{\circ} 2'$
5	$15^{\circ} 33'$
6	$19^{\circ} 28'$
7	$24^{\circ} 6'$
8	$30^{\circ} 0'$
9	$38^{\circ} 49'$
1.0	90°

$$\text{For } u/c = .8, \quad \sin \alpha = \{c - (c^2 - u^2)^{\frac{1}{2}}\} / u = \frac{1}{2}.$$

To gain a clearer view of the terminal conditions, which have a rather paradoxical appearance at first sight, it is worth while to evaluate the components of the (ordinary) acceleration. We find, reinstating f ,

$$v dv/ds = f \sin \phi (\cos^2 \phi - \sin^2 \alpha)^3 / (\cos^2 \phi + \sin^2 \alpha)^3,$$

$$v^2/\rho = f \cos \phi (\cos^2 \phi - \sin^2 \alpha)^2 / (\cos^2 \phi + \sin^2 \alpha)^2,$$

$$\text{and} \quad \gamma = (\cos^2 \phi + \sin^2 \alpha) / (\cos^2 \phi - \sin^2 \alpha),$$

so that γ^3 times the former component and γ^2 times the latter combine to give f . This verifies the analysis.

Now put $\phi = \pi - \alpha - \chi$ where χ is small,

$$v dv/ds = f \cos \alpha \cdot \cot^3 \alpha \cdot \chi^3,$$

$$\frac{v^2}{\rho} = f \sin \alpha \cdot \cot^2 \alpha \cdot \chi^2,$$

$$\gamma = \tan \alpha / \chi.$$

The tangential acceleration (ordinary) is small compared with the normal, but in passing to the rest-acceleration it is multiplied by a higher power of the large quantity γ , so that the modified components are of the same order of magnitude and have the finite ratio $\tan \alpha$.

I have taken the case $u/c = .8$, $\alpha = 30^{\circ}$, and have calculated x, y for different values of ϕ and plotted the path shown on the diagram. The parabola with the same " a " is shown for comparison. The figures marked above the path give the values of v/c at the corresponding points, and those below give the time, the unit in the latter case being the time required for light to describe the parameter " a ." These points were obtained by calculating the values of v/c

and ct for each ϕ and plotting against x (shown on the other curves). The points on these curves corresponding to even values of v/c and ct could then be projected on the path.

Fig. 1.

It will be noticed that the graph for ct against x is nearly straight, showing that the equable description of horizontal distance is not seriously departed from even at the high velocities taken. The analytical reason for this is seen on inspection of the formulæ (6) and (8) for x and ct . It will be seen that the expressions are built up of the same two functions of ϕ with different constant multipliers. The first term in ct is $\text{cosec } \alpha$ times the term in x , whereas in the second term the ratio is $\sin \alpha$. The two functional expressions are roughly of the same order of magnitude in x for the smaller values of ϕ , but as the limiting value is approached, the second term becomes insignificant compared with the first. Evidently the ratio of ct to x is mainly determined by the first term and approaches the constant value $\text{cosec } \alpha$.