XXVI.—*On the Theory of Waves. Part I.* By The Rev. P. KELLAND, M.A., F.R.SS.L. & E., F.C.P.S., late Fellow of Queens' College, Cambridge; Professor of Mathematics, &c. in the University of Edinburgh.

(Read April 1. 1839.)

It is my intention to undertake a series of Memoirs on that branch of Hydrodynamics which treats of the transmission of reciprocating motion. It may appear superfluous at first, that any further investigation should be bestowed on the general problem, when it is remembered that two of the most able mathematicians of the present age, M. POISSON and M. CAUCHY, have devoted so much of their talents and attention to the subject. Yet, when advances are rapidly making in our experimental knowledge, we generally find that the complex and apparently general results, deduced by those who have paid the greatest attention to the science as it presented itself to their view, are in reality only particular cases, when they have to be applied to experiments in their actual form. This circumstance detracts greatly from the value of almost all *general* investigations; and in this instance it does so in a remarkable degree. The prosecution of the research has been extended in one direction—the prosecution of experimental knowledge has taken another; and thus, in order to meet the challenge of experimenters to keep pace with them, it behoves the theorist frequently to return on his path, and cross over to the line of their march.

I doubt much, however, whether such men as LAPLACE and LAGRANGE would have been induced, with the expectation of joining experiment on her lower and more trodden fields, to reconsider and remodel their investigations; nor have I any reason to hope, that such men as POISSON and CAUCHY will quit the delectable atmosphere in which they are involved, of abstruse analysis, for the more humble, but not less important task of endeavouring to treat the simpler problems in a manner not made general arbitrarily to lead to the most elegant formulæ, but general to that extent, and in that mode, in which the problem in nature is so. It may seem strange, but I confess it appears to me, that this problem has hitherto fallen into the hands of men too distinguished,—of men who had attained such an elevation, that the labours of ordinary men, the trivial difficulties which impede the progress of less gigantic minds, do not form part either of their actual research, or of that which the world expects at their hands. Hence their inves-

tigations have been such as apply only to the more abstruse and captivating branches of the science, whilst the simple extension of its fundamental conditions has been almost, if not altogether, overlooked. From these impressions I have been induced to apply myself to the question. My desire is to simplify as well as to extend; not that I purpose to remodel the works of those who have gone before me, but, in certain cases, to derive conclusions previously known in support and illustration of my methods of operation. As far, however, as concerns the present memoir, although the results obtained appear to differ but slightly from those of M. POISSON or M. CAUCHY, yet, on examination, they will be found very little alike. A most essential element enters into them, which could not, from the nature of the methods employed, appear in either of theirs, viz. a variation dependent on the sphere of excursion of the molecules. Nor is this all. Both these philosophers suppose the oscillations to be complete,—indeed, one of the very first elements on which M. POISSON's theory is based, amounts only to the expression of this circumstance. Whatever, therefore, may be the value of investigations such as his (and I consider them most valuable), they do not solve the problem which is most wanted, the problem of oscillatory motion, of large or imperfect vibrations. They push to the extreme verge the problem which they actually do contain, but leave the other branches of the subject comparatively unimproved.

The theory of the tides can derive little assistance from such speculations in the present state of our knowledge, if indeed at all. This, and many other most important divisions of philosophy, if accurately subjected to analysis, require other restrictions than those commonly imposed, and at the same time demand the removal of these. With an ultimate view to problems of this nature, I have drawn up the present memoir, which embraces the motion of regular uniformly sized waves proceeding in one direction, and solves the problem in all its generality. It contains, likewise, an approximation to the motion of a solitary wave, such as the tide wave, or, more properly, that examined by Mr RUSSELL, and designated by him the *Wave of Translation*. How far I have succeeded I am not prepared to state. My solution can be regarded only as an approximation, nor does it very accurately agree with observation; yet the agreement is to my mind very satisfactory, as it shews, as far as I have tested the formula, that the error is due to one, and only one, point: what that point may be, I do not take it on me to conjecture.

This is the whole substance of the present memoir. Since it is so restricted, I shall not be expected, at the present time, to offer a sketch of the progress of the general problem. It will suffice, that a very brief statement be made of the labours of those who have added to our theoretical knowledge of this branch of Hydrodynamics.

NEWTON, in the Second Book of his *Principia*, applied his power of abstracting the point adapted to computation from the mass around it, to the solution of the problem of oscillation in a deep fluid, and his results are still valuable, as his process is ingenious.

LAPLACE afterwards applied a regular analysis to the problem,* but his solution is of a very limited nature. LAGRANGE, in his *Mécanique Analytique*,† gives equations which contain the complete statement of the question, and all that remains is the treatment of these equations. LAGRANGE himself‡ correctly solves the problem for fluids of small depth, but his hypothesis is very far from the truth as applied to the general problem.

It was in this state of the question that the French Institute proposed the subject as its prize essay for 1816.

M. POISSON, who, as he himself states, had for a long time been engaged on this problem, sent first a memoir to the Institute in October 1815, and afterwards a second in December. These memoirs contain an approximate solution of the motion of waves in a canal. We commence first with M. POISSON'S.§

This most important memoir starts with an approximation which (M. CHALLIS thinks) all persons who have attempted the problem have used. The assumption which gives rise to his equation (4), p. 81, also appears to me most essentially to destroy the generality of the results. As to the body of the memoir itself, it contains very little which belongs to our present matter. It is principally occupied in discussing the question of the effects which *presently* follow a disturbance in the fluid. The theoretical results of M. POISSON have been analyzed and tested by M. WEBER, with whose work I am altogether unacquainted.

The prize mentioned above was adjudged to M. CAUCHY, who was thought to have solved the question more generally. As to what the problem was I am not altogether certain. In M. CAUCHY'S memoir, printed in 1827,|| fifteen years after it had obtained the prize, I find it stated thus :

“ Une masse fluide pesante, primitivement en repos, et d'un profondeur infinie, a été mise en mouvement par l'effet d'une cause donnée. On demande, au bout d'un temps déterminé, la forme de la surface extérieure du fluide, et la vitesse des molécules situées à cette même surface.”

I have little doubt this is the form in which the question was actually proposed, and we find a sufficient cause for the detention of the two philosophers on this part of the problem, to the exclusion, or nearly so, of others. M. CAUCHY has added notes to his memoir, partly to explain and partly to enlarge on the

* Mémoires de l'Académie des Sciences, 1776.

† Mécanique Analytique, 2d Partie, Sect. xi.

‡ Ibid., Arts. 35-39.

§ Mémoires de l'Académie des Sciences, 1816.

|| Mémoires des Savans Etrangers, tome i.

processes employed in the body of the work. In two of these, the 16th and 20th, we find solutions of the question now before us, but in both of them the approximations are used which I have before alluded to. The results will be found in equation (141) of note 16, and equation (66) of note 20. They agree closely with those obtained by M. POISSON. This memoir of M. CAUCHY is of itself a large and abstruse work, and consequently, the brief notice taken of it must be understood to have reference only to those portions which belong to our present portion of the problem. To bring the history of the subject down to the present time, I have only to mention the names of Mr CHALLIS,* Mr EARNSHAW,† and Mr GREEN;‡ the two former of whom solve problems nearly connected with our own, and the latter takes a very limited case of the actual problem, viz. that solved approximately by M. LAGRANGE.

The present must be regarded rather in the light of an introduction to a series of Memoirs, than as a complete work in itself. We treat only of motion in a canal of uniform breadth,—nor shall we take even a large portion of that problem. The mode of generating motion,—its effect on the final waves, and on the primary ones,—the variable state of the surface, owing to reflexion from the bottom and sides of the canal,—these, and the like questions, will occupy us hereafter. We have, however, a sufficiently wide field, without having recourse to these comparatively abstract points. Not to mention the application of the results to the theory of the Tides, an application becoming daily more and more tangible by the labours of Mr LUBBOCK and Mr WHEWELL, we have a vast variety of questions to resolve, more obviously belonging to the very threshold of our inquiry. What is the correct velocity of a wave in a canal of variable depth? or in one which is not shallow nor very deep, as compared with the length of a wave? Will the effect be modified if the canal be a closed one at the commencement of motion? Will the length of the wave depend on the depth, on the quantity of fluid first put in motion, or on the space over which that motion takes place, or on all these causes? How will friction modify the form of the wave and the velocity of motion? And, lastly, What will be the motion in a channel which shallows away at its sides, when the channel is broad, as is the case when reference is had to the motion of the tidal wave?

These, and like questions, are of the utmost importance, and demand a careful investigation. In my next memoir, I hope to broach at least one, in addition to that at present before us. In the mean time, we proceed to the

* Transactions of the Cambridge Philosophical Society, vols. iii. and v.

† Transactions of the Cambridge Philosophical Society.

‡ Transactions of the Cambridge Philosophical Society, vol. vi.

ANALYTICAL INVESTIGATION.

SECTION I.—UNIFORM WAVE-MOTION.

1. We commence with the determination of Wave-motion in a fluid of finite depth, on the hypothesis of *parallel sections*.

Let PQ be a portion of the surface of the fluid, PM, QN vertical planes at right angles to the direction of transmission: and let AM = x , MP = z , MD = y , MN = δx . We shall also retain the notation in common use according to which u or $\frac{d\phi}{dx}$ represents the velocity parallel to x , v or $\frac{d\phi}{dy}$ that parallel to y . In order to avoid unnecessary length, we must adopt without demonstration the results which have been arrived at for fluid motion in general. The demonstrations may be found in POISSON'S *Traité de Mécanique*, 2d edition, tome ii. liv. 6, chap. 1; in MOSELEY'S *Hydrodynamics*, chap. vii.; in PRATT'S *Mechanical Philosophy*, Hydrodynamics, chap. i.; or in WEBSTER'S *Theory of Fluids*, chap. x.; to all of which we shall give references, for the sake of saving trouble to the reader.

To find the motion of the portion PN.

Let p be the pressure on an unit in PM; p' that on an unit in QN; then the pressure on $PM = \int_0^z dy p$;

the pressure on $QN = \int_0^{z + \frac{dz}{dx} \delta x} dy \left(p + \frac{dp}{dx} \delta x \right)$;

$$\begin{aligned} \therefore \text{the moving force} &= \int_0^z dy p - \int_0^{z + \frac{dz}{dx} \delta x} dy \left(p + \frac{dp}{dx} \delta x \right) \\ &= \int_0^z dy p - \int_0^z dy \left(p + \frac{dp}{dx} \delta x \right) - \int_z^{z + \frac{dz}{dx} \delta x} dy \left(p + \frac{dp}{dx} \delta x \right) \\ &= -\delta x \int_0^z \frac{dp}{dx} dy - \int_z^{z + \frac{dz}{dx} \delta x} p dy. \end{aligned}$$

Let us suppose that all the parts in a given vertical move forwards equally at a given time; then $u = \frac{d\phi}{dx}$ is independent of y .

We have now to solve the problem of finding the motion of fluid in a tube which is continually expanding. The small difference between MP and NQ may, in this investigation, be neglected.

2. To find the descent of fluid in a tube MQ, where MP is a fixed side, but NQ is moveable by means of the pressure.

Let D be any point in the fluid; v = the velocity at D in the direction of y , u = that in the direction of x ; then, if α be the thickness MN at the time t , $\alpha + \delta \alpha$ at the time $t + \delta t$, the quantity DG will have been pressed from δy to $\delta y \cdot \frac{\alpha}{\alpha + \delta \alpha}$.

Therefore the whole DN will be exhibited in $y \cdot \frac{\alpha}{\alpha + \delta \alpha}$;

and D will descend by a space $-y \cdot \frac{\alpha}{\alpha + \delta \alpha} + y$

$$\begin{aligned} &= y \frac{\delta \alpha}{\alpha + \delta \alpha} \\ &= y \frac{\delta \alpha}{\alpha} \text{ nearly :} \end{aligned}$$

hence the velocity of D is $\frac{y}{\alpha} \cdot \frac{d\alpha}{dt}$ (downwards)

$$\therefore v = -\frac{y}{\alpha} \cdot \frac{d\alpha}{dt}.$$

3. This will, however, lead us to no result, except we assume the nature of the motion to be defined. Let us, then, make the hypothesis that the motion is a *wave-motion*. This amounts to the substitution of

$$h + a \sin \frac{2\pi}{\lambda} (c t - x) + a' \sin \frac{4\pi}{\lambda} (c t - x) + \&c.$$

for z .

In this formula, h is the original depth;

λ the length of a wave;

c the velocity of transmission.

Now, if we retain only the first term in the variable part of this expression, we obtain

$$z = h + a \sin \theta,$$

θ being equal to $\frac{2\pi}{\lambda} (c t - x)$.

$$\therefore \frac{dz}{dx} = -\frac{2\pi a}{\lambda} \cos \theta,$$

from the supposition that the velocity in the direction parallel to x is uniform through any vertical section, and that consequently u is a function of x and t only.

From this consideration, it follows that $\frac{du}{dx}$ is independent of y : and conse-

quently $\frac{dv}{dy}$, which is equal to $-\frac{du}{dx}$,* is also independent of y .

v will in consequence assume the form $v=yf(x, t)+\phi(x, t)$; but when $y=0$, v is always $=0$, hence $v=yf(x, t)$; and, by reference to Art. 2, it appears that $v = -\frac{y}{a} \cdot \frac{d\alpha}{dt}$.

But
$$v = \frac{dy}{dt},$$

$$\therefore \frac{dy}{dt} = -\frac{y}{a} \cdot \frac{d\alpha}{dt};$$

hence

$$\frac{dz}{dt} = -\frac{z}{a} \cdot \frac{d\alpha}{dt};$$

$$\therefore v = \frac{y}{z} \cdot \frac{dz}{dt}.$$

Cor.
$$\frac{1}{z} \frac{dz}{dt} = -\frac{1}{a} \frac{d\alpha}{dt}.$$

4. Now z has been assumed to be equal to $h+a \sin \theta$.

$$\therefore \frac{dz}{dt} = \frac{2\pi ca}{\lambda} \cdot \cos \theta - \frac{2\pi a}{\lambda} \cdot \cos \theta \cdot \frac{dx}{dt}$$

$$\therefore v = \frac{y}{z} \frac{2\pi ca}{\lambda} \cos \theta - \frac{2\pi a}{\lambda} \cdot \frac{y}{z} \cdot \cos \theta \cdot \frac{dx}{dt}$$

$$\frac{dv}{dy} = \frac{2\pi ca}{\lambda z} \cdot \cos \theta - \frac{2\pi a}{\lambda} \cdot \frac{\cos \theta}{z} \cdot \frac{dx}{dt}$$

$$\frac{du}{dx} = -\frac{2\pi ca}{\lambda z} \cdot \cos \theta + \frac{2\pi a}{\lambda z} \cdot \cos \theta \cdot \frac{dx}{dt}$$

$$= -\frac{2\pi ca}{\lambda z} \cdot \cos \theta + \frac{2\pi a}{\lambda z} \cdot \cos \theta \cdot u.$$

Now let $u=b \sin \theta$;

then
$$-b \frac{2\pi}{\lambda} \cdot \cos \theta = -\frac{2\pi ca}{\lambda h} \cdot \cos \theta + \frac{2\pi ab}{\lambda h} \cdot \sin \theta \cos \theta + \frac{2\pi ca^2}{\lambda h^2} \cdot \sin \theta \cos \theta;$$

but, from the hypothesis already made, the last two terms must be omitted; hence

$$bh=ca;$$

$$\therefore u = \frac{ca}{h} \cdot \sin \theta$$

$$v = \frac{2\pi ca}{\lambda h} \cdot y \cdot \cos \theta.$$

5. The hypothesis relative to the value of z , by means of which the preceding results have been obtained, is that which belongs to the most simple case of wave-

* POISSON, art. 649; MOSELEY, art. 205; PRATT, art. 564; WEBSTER, art. 108. The equation is

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$$

motion. To solve the problem more generally, we should assume for z a series of sines of multiples of θ . If we restrict ourselves to two sines, we shall have the following equation :

$$z = h + a \sin \frac{2\pi}{\lambda} (ct - x) + e \sin 2\theta ;$$

which gives

$$v = \frac{y}{z} \cdot \frac{dz}{dt} = \frac{y}{z} \cdot \frac{2\pi}{\lambda} \cdot \left(a \cos \theta \cdot c - \frac{dx}{dt} + 2e \cos 2\theta \cdot c - \frac{dx}{dt} \right)$$

$$\frac{du}{dx} = - \frac{2\pi}{\lambda z} \left(a \cos \theta \cdot c - \frac{dx}{dt} + 2e \cos 2\theta \cdot c - \frac{dx}{dt} \right).$$

The same limitation as to the extent of the series gives

$$u = b \sin \theta + f \sin 2\theta.$$

Differentiating this, and equating it to the former,

$$b \cos \theta + 2f \cos 2\theta = \frac{1}{z} (a \cos \theta + 2e \cos 2\theta) \left(c - \frac{dx}{dt} \right) ;$$

$$\therefore (b \cos \theta + 2f \cos 2\theta) (h + a \sin \theta + e \sin 2\theta)$$

$$= (a \cos \theta + 2e \cos 2\theta) (c - b \sin \theta - f \sin 2\theta),$$

$$\text{or } bh \cos \theta + 2fh \cos 2\theta + \frac{ab}{2} \sin 2\theta + af(\sin 3\theta - \sin \theta) + \frac{eb}{2} (\sin 3\theta + \sin \theta) + ef \sin 4\theta$$

$$= ac \cos \theta + 2ec \cos 2\theta - \frac{ba}{2} \sin 2\theta - eb(\sin 3\theta - \sin \theta) - \frac{af}{2} (\sin 3\theta + \sin \theta) - ef \sin 4\theta.$$

By equating the coefficients of $\cos \theta$ and $\cos 2\theta$, we get

$$bh = ac, 2fh = 2ec;$$

hence we learn that, if z be known, u is known.

With respect to the terms involving sines, it is clear that no equations can be made, since the same quantities will occur again.

6. We now proceed to determine the motion parallel to the axis of x by an independent method. We will conceive the portion PQ to become solid for an instant, and calculate the force by which it is urged in the direction of the axis of x . That force will consist of two parts, totally independent of each other; the one the difference of statical pressure on the two planes PM, QN; the other the difference of the dynamical impulses on the same planes.

With respect to the latter, it may be remarked, that it must be absolutely independent of v . The real effective part of this pressure is, in fact, nothing more than the *resistance* on either of the planes due to the velocity of the fluid in a direction at right angles to it.

In order to obtain the value of this force, we shall not attempt to treat of it separately as a problem of resistances, but apply directly to it the same argument as is commonly applied to establish the theory of resistances. We have already stated that it must be independent of v : this can only be on the supposition that the

force results from the impulse of the fluid behind that under consideration. We suppose, in fact, that it is the difference of the impulses on the two sides of the solid PN.

Now the expression for the pressure at any point in the mass is this,*

$$p = g \rho (z - y) - \rho \int dx \frac{d^2 x}{dt^2} - \&c.$$

Of this expression the first term is $g \rho (z - y)$, which is the pressure due to the action of gravity.

The second term is $-\rho \int dx \frac{d^2 x}{dt^2}$, a quantity which depends on the motion of the particles in the neighbourhood of the point.

In the theory of resistances, it is assumed that the motion is such as to admit of our writing $\frac{dx}{dt} dt$ for dx ; that is, it is supposed that the motion of a particle is such as that, when it comes to another point in space after any interval, it shall move just in the same manner as those particles do which are at the present moment at that point. In other words, the motion is conceived to be *steady*, that it is always the same at any particular point of space; so that the integration for one particle during its successive stages shall be identical with the integration for different particles at the same instant.

Such an hypothesis as this is not only convenient, but appears to be absolutely correct in the case before us, for this pressure depends only on the variation of velocity of the particles *immediately* about PMN at the instant under consideration, and if one hypothesis as to the future movements of the particles gives the difference of the pressures,

$$= -g \rho dy - \frac{1}{2} \rho d \left(\frac{dx}{dt} \right)^2 - \&c.$$

in going from point to point, any other hypothesis ought to give the same, provided the variation of velocity be the only thing which affects the pressure.

We may then assume as the value of p ;

$$p = g \rho (z - y) - \frac{1}{2} \rho u^2 + P,$$

P being some function depending on v , but not requisite to our calculation.

Similarly,
$$p' = g \rho (z' - y) - \frac{1}{2} \rho u'^2 + P'$$

$$= g \rho \left(z + \frac{dz}{dx} \alpha - y \right) - \frac{1}{2} \rho \left(u^2 + 2u \frac{du}{dx} \alpha \right) + P'$$

* POISSON, art. 647; MOSELEY, art. 203; PRATT, art. 562; WEBSTER, art. 117. The general equation is,

$$\frac{dp}{\rho} = \left(x - \frac{d^2 x}{dt^2} \right) dx + \left(y - \frac{d^2 y}{dt^2} \right) dy + \left(z - \frac{d^2 z}{dt^2} \right) dz.$$

Now, the moving force on the solid PN

$$\begin{aligned}
 &= \int_0^z dy p - \int_0^{z+\frac{dz}{dx}a} dy p' \\
 &= \int_0^z dy g \rho (z-y) - \frac{1}{2} \int_0^z dy \rho u^2 + Q \\
 &\quad - \int_0^{z+\frac{dz}{dx}a} dy g \rho \left(z + \frac{dz}{dx}a - y \right) + \frac{1}{2} \rho \int_0^{z+\frac{dz}{dx}a} dy \left(u^2 + 2u \frac{du}{dx}a \right) + Q' \\
 &= \int_0^z dy g \rho (z-y) \\
 &\quad - \int_z^{z+\frac{dz}{dx}a} dy g \rho \left(z + \frac{dz}{dx}a - y \right) + \rho \int_0^z dy u \frac{du}{dx} a.
 \end{aligned}$$

We have omitted the part $\frac{1}{2} \rho \int_z^{z+\frac{dz}{dx}a} dy u^2 + Q + Q'$ from the circumstance that it does not depend on the difference of velocity or of resistance, and, therefore, is no part of our force.

Perhaps it would be well to call p the horizontal pressure, instead of the whole pressure, as by this means we should have been spared the apparent incorrectness of omitting a part of the results; but I have preferred retaining the above, as the more usual mode of proceeding.

By integrating the above expression between limits, we obtain for the moving force

$$\frac{1}{2} g \rho z^2 - \frac{1}{2} g \rho \left(z + \frac{dz}{dx}a \right)^2 + \rho a z u \frac{du}{dx};$$

and the mass is $\rho a z$; hence the accelerating force is

$$\left(\frac{du}{dt} \right) = -g \frac{dz}{dx} + u \frac{du}{dx};$$

where $\frac{du}{dt}$ is the total differential coefficient of u with respect to t .

By substituting for u its value

$$\frac{ca}{h} \cdot \sin \frac{2\pi}{\lambda} ct - x \quad \text{or} \quad \frac{ca}{h} \cdot \sin \theta$$

we obtain

$$\frac{2\pi ca}{h\lambda} \cdot \cos \theta \left(c - \frac{dx}{dt} \right) = g \cdot \frac{2\pi a}{\lambda} \cdot \cos \theta - u \frac{2\pi ac}{\lambda h} \cos \theta;$$

by equating coefficients, we obtain $\frac{c^2}{h} = g$ as the first part, and the other part is an identity.

7. We may vary the last part of the process in the following manner. Since the moving force is

$$\int_0^z p dy - \int_0^{z+\frac{dz}{dx}a} p' dy,$$

it is

$$\begin{aligned} &= \int_0^z (p-p') dy - \int_z^{z+\frac{dz}{dx}a} p dy \\ &= -g \rho \int_0^z \frac{dz}{dx} a dy - \int_z^{z+\frac{dz}{dx}a} p dy \\ &\quad + \frac{1}{2} \rho \int_0^z \left\{ \left(\frac{d(x+a)}{dt} \right)^2 - \left(\frac{dx}{dt} \right)^2 \right\} a, \\ &= -g \rho a z \frac{dz}{dx} + \frac{1}{2} \rho \frac{2 dx}{dt} \cdot \frac{da}{dt} \cdot z; \end{aligned}$$

since the term $\int_z^{z+\frac{dz}{dx}a} g \rho (z-y) dy$ involves a^2 as a factor.

Hence the accelerating force is

$$\frac{d^2 x}{dt^2} = -g \frac{dz}{dx} + \frac{dx}{dt} \cdot \frac{da}{dt} \cdot \frac{1}{a}.$$

But it has been shewn (Art. 2), that

$$\begin{aligned} &\frac{1}{a} \cdot \frac{da}{dt} = -\frac{1}{z} \cdot \frac{dz}{dt} \\ \therefore \frac{d^2 x}{dt^2} &= -g \frac{dz}{dx} - \frac{dx}{dt} \cdot \frac{dz}{dt} \cdot \frac{1}{z}. \end{aligned}$$

But

$$z = h + a \sin \frac{2\pi}{\lambda} (ct - x)$$

and

$$\frac{dx}{dt} = b \sin \frac{2\pi}{\lambda} (ct - x)$$

where b is supposed undetermined. By substituting these values, we get

$$\frac{2\pi b}{\lambda} \cdot \cos \theta \left(c - \frac{dx}{dt} \right) = g a \cdot \frac{2\pi}{\lambda} \cos \theta - \frac{\frac{dx}{dt}}{h} \cdot \frac{2\pi a c}{\lambda} \cdot \cos \theta$$

or
$$\frac{2\pi b}{\lambda} \cdot \cos \theta (c - b \sin \theta) = \frac{2\pi}{\lambda} g a \cos \theta - \frac{2\pi a c b}{\lambda h} \sin \theta \cos \theta;$$

and, equating coefficients, we get

$$bc = ga; \quad b^2 = \frac{abc}{h};$$

whence

$$\left. \begin{aligned} b &= \frac{ac}{h} \\ c^2 &= gh \end{aligned} \right\}$$

the values which we obtained before.

8. Next, having these approximate formulæ as our guide, let us proceed to the general solution of the problem.

Retaining the same notation,

$$\frac{dp}{dx} = \rho \left(- \left(\frac{du}{dt} \right) \right)^*$$

$$\frac{dp}{dy} = \rho \left\{ -g - \left(\frac{dv}{dt} \right) \right\}$$

$\left(\frac{du}{dt} \right), \left(\frac{dv}{dt} \right)$ being the complete differentials of u and v divided by dt .

These equations, again, give as their result,

$$\frac{d}{dy} \left(\frac{du}{dt} \right) = \frac{d}{dx} \left(\frac{dv}{dt} \right)$$

which being combined with the equation $\frac{du}{dx} + \frac{dv}{dy} = 0$, the motion will be obtained.

Now,

$$\left(\frac{du}{dt} \right) = \frac{du}{dt} + u \frac{du}{dx} + v \frac{du}{dy}$$

$$\left(\frac{dv}{dt} \right) = \frac{dv}{dt} + u \frac{dv}{dx} + v \frac{dv}{dy}.$$

But if the wave be oscillatory, we may assume for u and v a series of terms of the following form :

$$u = f(y) \cdot \sin \frac{2\pi}{\lambda} (ct - x)$$

$$v = Fy \cdot \cos \frac{2\pi}{\lambda} (ct - x).$$

For it is obvious, without any calculation, that, since $\frac{du}{dx} + \frac{dv}{dy} = 0$, if $\frac{du}{dx}$ involve $\cos \frac{2\pi}{\lambda} (ct - x)$, v must do so too, and consequently v will contain only cosines of quantities, of which u contains sines. By substitution in the equation $\frac{du}{dx} + \frac{dv}{dy} = 0$, we get the following result :

$$-\frac{2\pi}{\lambda} \cos \frac{2\pi}{\lambda} (ct - x) f y + \cos \frac{2\pi}{\lambda} (ct - x) F' y = 0$$

or
$$F' y = \frac{2\pi}{\lambda} f y, \quad (1).$$

Let
$$\left. \begin{array}{l} \frac{2\pi}{\lambda} \text{ be denoted by } \alpha \\ \frac{2\pi}{\lambda} (ct - x) \text{ by } \theta \end{array} \right\}$$

then the values of $\left(\frac{du}{dt} \right)$ and $\left(\frac{dv}{dt} \right)$ become (retaining only this term of the value

of u),
$$\left(\frac{du}{dt} \right) = \alpha f y \cos \theta (c - f y \sin \theta) + f' y F y \sin \theta \cos \theta,$$

$$\left(\frac{dv}{dt} \right) = -\alpha F y \sin \theta (c - f y \sin \theta) + F' y F y \cos^2 \theta.$$

* See the references in Art. 6.

Hence the equation $\frac{d}{dy} \cdot \left(\frac{du}{dt} \right) = \frac{d}{dx} \cdot \left(\frac{dv}{dt} \right)$ gives

$$\alpha c \cdot \cos \theta f'y - 2\alpha \sin \theta \cos \theta f y f'y + \sin \theta \cos \theta (F y f'' y + F'y f'y) \\ = \alpha^2 c F y \cos \theta - 2\alpha^2 F y f y \sin \theta \cos \theta + \alpha F'y F y \sin 2\theta.$$

Equating coefficients, we obtain

$$f'y = \alpha F y \quad (2)$$

$$-\alpha f y f'y + \frac{F y f'' y + F'y f'y}{2} + \alpha^2 F y f y - F'y F y \alpha = 0$$

$$\text{or} \quad -\frac{1}{\alpha} F'y F'' y + \frac{1}{2} \left(\frac{1}{\alpha} F y F''' y + \frac{1}{\alpha} F'y F'' y \right) = 0$$

$$\text{or} \quad \frac{1}{\alpha} F y F''' y - \frac{1}{\alpha} F'y F'' y = 0$$

$$\text{or} \quad F y F''' y = F'y F'' y \quad (3)$$

$$\text{but} \quad \therefore f'y = \alpha F y \\ F'y = \frac{1}{\alpha} f'' y$$

$$\text{and} \quad \alpha F'y = \alpha^2 f y \\ \therefore f'' y = \alpha^2 f y$$

$$\text{or} \quad \frac{d^2 f y}{dy^2} = \alpha^2 \cdot f y$$

The complete solution of this equation is

$$f y = b e^{\alpha y} + b' e^{-\alpha y}$$

$$\therefore F'y = \alpha (b e^{\alpha y} + b' e^{-\alpha y}); F y = b e^{\alpha y} - b' e^{-\alpha y} \quad (2)$$

If we substitute this value of $f y$ in equation (3), we obtain

$$(b e^{\alpha y} - b' e^{-\alpha y}) \alpha^2 (b e^{\alpha y} + b' e^{-\alpha y}) = \alpha (b e^{\alpha y} + b' e^{-\alpha y}) \alpha^2 (b e^{\alpha y} - b' e^{-\alpha y})$$

an identity.

Thus all the conditions are satisfied. Our solution, then, of the equations is

$$u = \left(b e^{\frac{2\pi}{\lambda} y} + b' e^{-\frac{2\pi}{\lambda} y} \right) \cdot \sin \frac{2\pi}{\lambda} (c t - x),$$

$$v = \left(b e^{\frac{2\pi}{\lambda} y} - b' e^{-\frac{2\pi}{\lambda} y} \right) \cdot \cos \frac{2\pi}{\lambda} (c t - x).$$

If it should appear more general to affix a constant to u , it will add no difficulty to the investigation.

The value of b' may be determined by supposing the origin of co-ordinates to be placed at the bottom of the fluid, so that $v=0$ when $y=0$: this process gives $b'=b$.

9. Our next step is to find p .

By the equations

$$\frac{dp}{dx} = \rho \left(- \left(\frac{du}{dt} \right) \right)$$

$$\frac{dp}{dy} = \rho \left(-g - \left(\frac{dv}{dt} \right) \right),$$

we get

$$\begin{aligned} dp &= \frac{dp}{dx} dx + \frac{dp}{dy} dy \\ &= \rho \left\{ - \left(\frac{du}{dt} \right) \cdot dx - \left(\frac{dv}{dt} \right) dy - g dy \right\} \\ &= \rho \left\{ -b\alpha \cos \theta (e^{\alpha y} + e^{-\alpha y}) (c - b \cdot \overline{e^{\alpha y} + e^{-\alpha y} \sin \theta}) \right. \\ &\quad \left. - b^2 \alpha (e^{\alpha y} - e^{-\alpha y})^2 \sin \theta \cos \theta \right\} dx \\ &\quad + \rho \left\{ b(e^{\alpha y} - e^{-\alpha y}) \alpha \sin \theta (c - b \cdot \overline{e^{\alpha y} + e^{-\alpha y} \sin \theta}) \right. \\ &\quad \left. - b^2 \alpha (e^{2\alpha y} - e^{-2\alpha y}) \cos^2 \theta \right\} dy \\ &\quad - \rho g dy \\ \therefore p &= -\rho g y + \rho \int b \alpha c (-\overline{e^{\alpha y} + e^{-\alpha y} \cos \theta} dx + \overline{e^{\alpha y} - e^{-\alpha y} \sin \theta} dy) \\ &\quad + \delta b^2 \alpha \int \left\{ 2 \sin 2 \theta dx - (e^{2\alpha y} - e^{-2\alpha y}) dy \right\} \\ &= -\rho g y + \rho b c (e^{\alpha y} + e^{-\alpha y}) \sin \theta \\ &\quad + \rho b^2 \left(\cos 2 \theta - \frac{1}{2} (e^{2\alpha y} + e^{-2\alpha y}) \right) + P \quad (4) \end{aligned}$$

This equation contains the value of p , and the use we purpose to make of it is this. The quantity P is a function of t , and the depth of the fluid for the value of x fixed on. If this depth be called z , we have

$$\begin{aligned} p &= \rho g (z - y) + \rho b c \sin \theta (e^{\alpha y} + e^{-\alpha y}) - \rho b c \sin \theta (e^{\alpha z} + e^{-\alpha z}) \\ &\quad + \rho b^2 \left\{ -\frac{1}{2} (e^{2\alpha y} + e^{-2\alpha y}) + \frac{1}{2} (e^{2\alpha z} + e^{-2\alpha z}) \right\} \quad (5) \end{aligned}$$

Now the value of $\frac{dp}{dx}$, or the expression for the force parallel to the axis of x , has been already formed, but another value of it may be obtained from this final equation. If we equate the two, we get

$$\left\{ \frac{dp}{dx} \right\} = \frac{dp}{dx} + \frac{dp}{dz} \cdot \frac{dz}{dx}$$

where the quantity within brackets is the value of $\frac{dp}{dx}$ from equation (4).

But if equation (4) be written $p = \phi(xy) + P$, and (5) $p = \phi(xy) - \phi(xz)$, we get

$$\frac{d\phi(xy)}{dx} = \frac{d\phi(xy)}{dx} - \frac{d\phi(xz)}{dz} \cdot \frac{dz}{dx} - \frac{d\phi(xz)}{dx} \quad (6)$$

hence
$$\frac{d\phi(xz)}{dz} \cdot \frac{dz}{dx} + \frac{d\phi(xz)}{dx} = 0 \quad (7).$$

10. To prevent confusion, however, we will at first write down all the terms of equation (6), and afterwards strike out those which occur on both sides, so as to obtain the form of equation (7).

By (4)
$$\frac{1}{\rho} \frac{dp}{dx} = -\alpha b c \cos \theta (e^{\alpha y} + e^{-\alpha y}) + 2 \cdot b^2 \alpha \sin 2\theta :$$

and by (5)
$$\begin{aligned} \frac{1}{\rho} \frac{dp}{dx} = & g \frac{dz}{dx} - \alpha b c \cos \theta (e^{\alpha y} + e^{-\alpha y} - e^{\alpha z} - e^{-\alpha z}) \\ & - \alpha b c \sin \theta (e^{\alpha z} - e^{-\alpha z}) \frac{dz}{dx} + \alpha b^2 (e^{2\alpha z} - e^{-2\alpha z}) \frac{dz}{dx} : \end{aligned}$$

hence
$$\begin{aligned} & -\alpha b c \cos \theta (e^{\alpha y} + e^{-\alpha y}) + 2 b^2 \alpha \sin 2\theta \\ = & g \frac{dz}{dx} - \alpha b c \cos \theta (e^{\alpha y} + e^{-\alpha y} - e^{\alpha z} - e^{-\alpha z}) \\ & - \alpha b c \sin \theta (e^{\alpha z} - e^{-\alpha z}) \frac{dz}{dx} + b^2 \alpha (e^{2\alpha z} - e^{-2\alpha z}) \frac{dz}{dx} \end{aligned}$$

that is, as we should have deduced at once from equation (7),

$$\begin{aligned} & + \frac{4\pi b^2}{\lambda} \sin 2\theta = \\ & \frac{dz}{dx} \cdot \left\{ g - \frac{2\pi}{\lambda} b c \sin \theta \left(e^{\frac{2\pi}{\lambda} z} - e^{-\frac{2\pi}{\lambda} z} \right) + \frac{2\pi}{\lambda} b^2 \left(e^{\frac{4\pi}{\lambda} z} - e^{-\frac{4\pi}{\lambda} z} \right) \right\} \\ & + \frac{2\pi}{\lambda} b c \cos \theta \left(e^{\frac{2\pi}{\lambda} z} + e^{-\frac{2\pi}{\lambda} z} \right) ; \end{aligned}$$

or
$$\begin{aligned} \frac{4\pi}{\lambda} b^2 \sin 2\theta = & \frac{2\pi}{\lambda} b c \cos \theta \left(e^{\frac{2\pi}{\lambda} z} + e^{-\frac{2\pi}{\lambda} z} \right) \\ & + \frac{dz}{dx} \left\{ g - \frac{2\pi}{\lambda} b c \sin \theta \left(e^{\frac{2\pi}{\lambda} z} - e^{-\frac{2\pi}{\lambda} z} \right) + \frac{2\pi}{\lambda} b^2 \left(e^{\frac{4\pi}{\lambda} z} - e^{-\frac{4\pi}{\lambda} z} \right) \right\} \quad (8). \end{aligned}$$

Let
$$z = h + \left(a \cdot e^{\frac{2\pi}{\lambda} z} + f e^{-\frac{2\pi}{\lambda} z} \right) \sin \theta$$

$$\begin{aligned} \therefore \frac{dz}{dt} = & \sin \theta \frac{2\pi}{\lambda} \left(a e^{\frac{2\pi}{\lambda} z} - f \cdot e^{-\frac{2\pi}{\lambda} z} \right) \frac{dz}{dt} \\ & + \left(a e^{\frac{2\pi}{\lambda} z} + f e^{-\frac{2\pi}{\lambda} z} \right) \frac{2\pi}{\lambda} \cos \theta \left(c - b \left(e^{\frac{2\pi}{\lambda} z} + e^{-\frac{2\pi}{\lambda} z} \right) \sin \theta \right) : \end{aligned}$$

But from the circumstance that $v = \frac{dz}{dt}$ when $y = z$, this gives,

$$\left(1 - \frac{2\pi}{\lambda} \cdot a \overline{e^{\alpha z} - f e^{-\alpha z}} \sin \theta\right) b \cos \theta (e^{\alpha z} - e^{-\alpha z}) =$$

$$\frac{2\pi}{\lambda} \cos \theta (a e^{\alpha z} + f e^{-\alpha z}) \{c - b \sin \theta (e^{\alpha z} + e^{-\alpha z})\} :$$

or

$$b (e^{\alpha z} - e^{-\alpha z}) - \frac{2\pi}{\lambda} b (e^{\alpha z} - e^{-\alpha z}) (a e^{\alpha z} - f e^{-\alpha z}) \sin \theta$$

$$= \frac{2\pi}{\lambda} c (a e^{\alpha z} + f e^{-\alpha z}) - \frac{2\pi}{\lambda} b \sin \theta (e^{\alpha z} + e^{-\alpha z}) (a e^{\alpha z} + f e^{-\alpha z})$$

Hence

$$f = -a \text{ and } b = \frac{2\pi}{\lambda} a c :$$

$$\therefore z = h + a (e^{\alpha z} - e^{-\alpha z}) \sin \theta$$

$$\frac{dz}{dx} = -\frac{2\pi}{\lambda} a \cos \theta (e^{\alpha z} - e^{-\alpha z})$$

$$+ \frac{2\pi}{\lambda} a \sin \theta \frac{dz}{dx} (e^{\alpha z} + e^{-\alpha z})$$

$$= -\frac{\frac{2\pi}{\lambda} a \cos \theta (e^{\alpha z} - e^{-\alpha z})}{1 - \frac{2\pi}{\lambda} a \sin \theta (e^{\alpha z} + e^{-\alpha z})}$$

which being substituted in equation (8), reduces it to

$$-\frac{2\pi}{\lambda} b \left(4b \sin \theta - (e^{\alpha z} + e^{-\alpha z}) c\right) \cos \theta \left(1 - \frac{b}{c} \sin \theta \overline{(e^{\alpha z} + e^{-\alpha z})}\right)$$

$$= \frac{b}{c} \cos \theta (e^{\alpha z} - e^{-\alpha z}) \left\{g - \frac{2\pi}{\lambda} b c \sin \theta (e^{\alpha z} - e^{-\alpha z}) + \frac{2\pi}{\lambda} b^2 (e^{2\alpha z} - e^{-2\alpha z})\right\}$$

that is

$$-\frac{2\pi b}{\lambda c} \left\{4b c \sin \theta - c^2 (e^{\alpha z} + e^{-\alpha z}) - 4b^2 \sin^2 \theta (e^{\alpha z} + e^{-\alpha z}) + b c \sin \theta (e^{\alpha z} + e^{-\alpha z})^2\right\}$$

$$= \frac{b}{c} \left\{g \overline{e^{\alpha z} - e^{-\alpha z}} - \frac{2\pi}{\lambda} b c \sin \theta (e^{\alpha z} - e^{-\alpha z})^2 + \frac{2\pi}{\lambda} b^2 (e^{\alpha z} - e^{-\alpha z}) (e^{2\alpha z} - e^{-2\alpha z})\right\}$$

or, therefore,

$$-\frac{8\pi}{\lambda} b c \sin \theta + \frac{2\pi}{\lambda} c^2 (e^{\alpha z} + e^{-\alpha z}) + \frac{8\pi}{\lambda} b^2 \sin^2 \theta (e^{\alpha z} + e^{-\alpha z}) - \frac{2\pi}{\lambda} b c \sin \theta (e^{\alpha z} + e^{-\alpha z})^2$$

$$= g (e^{\alpha z} - e^{-\alpha z}) - \frac{2\pi}{\lambda} b c \sin \theta (e^{\alpha z} - e^{-\alpha z})^2 + \frac{2\pi}{\lambda} b^2 (e^{\alpha z} - e^{-\alpha z}) (e^{2\alpha z} - e^{-2\alpha z}) \quad (9)$$

Let $\theta=0$, then $z=h$, and the equation gives

$$\frac{2\pi}{\lambda} c^2 (e^{\alpha h} + e^{-\alpha h}) = g (e^{\alpha h} - e^{-\alpha h}) + \frac{2\pi}{\lambda} b^2 (e^{\alpha h} - e^{-\alpha h}) (e^{2\alpha h} - e^{-2\alpha h})$$

$$= g (e^{\alpha h} - e^{-\alpha h}) + \left(\frac{2\pi}{\lambda}\right)^3 a^2 c^2 (e^{\alpha h} - e^{-\alpha h}) (e^{2\alpha h} - e^{-2\alpha h})$$

or
$$\frac{2\pi}{\lambda} c^2 (e^{\alpha h} + e^{-\alpha h}) \left\{ 1 - \left(\frac{2\pi}{\lambda} a \right)^2 (e^{\alpha h} - e^{-\alpha h})^2 \right\} = g (e^{\alpha h} - e^{-\alpha h})$$

$$\therefore \frac{2\pi}{\lambda} c^2 = g \cdot \frac{e^{\alpha h} - e^{-\alpha h}}{e^{\alpha h} + e^{-\alpha h}} \cdot \frac{1}{1 - \left(\frac{2\pi}{\lambda} a \right)^2 (e^{\alpha h} - e^{-\alpha h})^2}$$

$$= g \cdot \frac{\frac{2\pi}{\lambda} h + \frac{2\pi}{\lambda} h}{2} \quad \text{nearly}$$

hence $c^2 = g h$.

11. Our result has been obtained from equation (9) by putting $\theta = 0$. This mode of proceeding, of course, gives only one result; we shall, therefore, pursue another process in order to obtain a second.

Since
$$z = h + a (e^{\alpha h} - e^{-\alpha h}) \sin \theta$$

$$= h + m \sin \theta \quad \text{suppose,}$$

let us substitute this value of z in the equation (9), and expand the exponential functions.

It will appear readily that

$$e^{\alpha z} + e^{-\alpha z} = e^{\alpha h} + e^{-\alpha h} + \frac{2\pi}{\lambda} m \sin \theta (e^{\alpha h} - e^{-\alpha h})$$

$$e^{\alpha z} - e^{-\alpha z} = e^{\alpha h} - e^{-\alpha h} + \frac{2\pi}{\lambda} m \sin \theta (e^{\alpha h} + e^{-\alpha h})$$

if we omit powers of $m \sin \theta$ greater than the first.

But equation (9), by means of these values, becomes

$$\begin{aligned} & -\frac{8\pi}{\lambda} b c \sin \theta + \frac{2\pi}{\lambda} c^2 (e^{\alpha h} + e^{-\alpha h}) + \frac{4\pi^2}{\lambda^2} c^2 a \sin \theta (e^{\alpha h} - e^{-\alpha h})^2 \\ & + \frac{8\pi}{\lambda} b^2 \sin^2 \theta (e^{\alpha h} + e^{-\alpha h}) - \frac{2\pi}{\lambda} b c \sin \theta (e^{\alpha h} + e^{-\alpha h})^2 \\ & - \frac{8\pi^2}{\lambda^2} b c \sin^2 \theta a (e^{\alpha h} + e^{-\alpha h}) (e^{\alpha h} - e^{-\alpha h})^2 = \\ & g (e^{\alpha h} - e^{-\alpha h}) + g \frac{2\pi}{\lambda} a \sin \theta (e^{2\alpha h} - e^{-2\alpha h}) \\ & - \frac{2\pi}{\lambda} b c \sin \theta (e^{\alpha h} - e^{-\alpha h})^2 - \frac{8\pi^2}{\lambda^2} b c a \sin^2 \theta (e^{\alpha h} + e^{-\alpha h}) (e^{\alpha h} - e^{-\alpha h})^2 \\ & + \frac{2\pi}{\lambda} b^2 \left\{ e^{\alpha h} - e^{-\alpha h} + \frac{2\pi}{\lambda} a \sin \theta (e^{2\alpha h} - e^{-2\alpha h}) \right\} \\ & \times \left\{ e^{2\alpha h} - e^{-2\alpha h} + \frac{4\pi}{\lambda} a \sin \theta (e^{2\alpha h} + e^{-2\alpha h}) (e^{\alpha h} - e^{-\alpha h}) \right\} \end{aligned}$$

This equation will furnish the two results mentioned above. The first, derived from the parts which do not contain θ , is

$$\frac{2\pi}{\lambda} c^2 (e^{\alpha h} + e^{-\alpha h}) = g (e^{\alpha h} - e^{-\alpha h}) + \frac{2\pi}{\lambda} b^2 (e^{\alpha h} - e^{-\alpha h}) (e^{2\alpha h} - e^{-2\alpha h}):$$

and by substituting $a^2 a^2 c^2$ for b^2 , and transposing it, becomes

$$\frac{2\pi}{\lambda} c^2 (e^{\alpha h} + e^{-\alpha h}) \left(1 - \frac{2\pi}{\lambda} a \right)^2 (e^{\alpha h} - e^{-\alpha h})^2 = g (e^{\alpha h} - e^{-\alpha h})$$

$$\therefore \frac{2\pi}{\lambda} c^2 = g \cdot \frac{e^{\alpha h} - e^{-\alpha h}}{e^{\alpha h} + e^{-\alpha h}} \cdot \frac{1}{1 - \left(\frac{2\pi}{\lambda} a \right)^2 (e^{\alpha h} - e^{-\alpha h})^2}$$

the same result as in art. 10.

The second result is

$$-\frac{8\pi}{\lambda} b c + \frac{4\pi^2}{\lambda^2} c^2 a (e^{\alpha h} - e^{-\alpha h})^2 + \frac{8\pi}{\lambda} b^2 (e^{\alpha h} + e^{-\alpha h})$$

$$-\frac{8\pi}{\lambda} b c = g \frac{2\pi a}{\lambda} (e^{2\alpha h} - e^{-2\alpha h})$$

or

$$-\frac{32\pi^2}{\lambda^2} a c^2 + \frac{4\pi^2}{\lambda^2} a c^2 (e^{\alpha h} - e^{-\alpha h})^2$$

$$+ \frac{32\pi^3}{\lambda^3} a^2 c^2 (e^{\alpha h} + e^{-\alpha h}) = \frac{2\pi}{\lambda} a g (e^{2\alpha h} - e^{-2\alpha h})$$

$$\text{whence } a = \frac{1}{\frac{16\pi^2}{\lambda^2} c^2 (e^{\alpha h} + e^{-\alpha h})} \left\{ g (e^{2\alpha h} - e^{-2\alpha h}) - \frac{2\pi}{\lambda} c^2 (e^{\alpha h} - e^{-\alpha h})^2 + \frac{16\pi}{\lambda} c^2 \right\}$$

$$= \frac{1}{\frac{8\pi}{\lambda} g (e^{\alpha h} - e^{-\alpha h})} \cdot \left\{ g (e^{2\alpha h} - e^{-2\alpha h}) - g \frac{(e^{\alpha h} - e^{-\alpha h})^3}{e^{\alpha h} + e^{-\alpha h}} + 8g \frac{e^{\alpha h} - e^{-\alpha h}}{e^{\alpha h} + e^{-\alpha h}} \right\}$$

$$= \frac{\lambda}{8\pi} \left\{ e^{\alpha h} + e^{-\alpha h} - \frac{(e^{\alpha h} - e^{-\alpha h})^2}{e^{\alpha h} + e^{-\alpha h}} + \frac{4}{e^{\alpha h} + e^{-\alpha h}} \right\}$$

$$= \frac{\lambda}{\pi (e^{\alpha h} + e^{-\alpha h})}$$

These two results give us, respectively, the velocity of transmission and the height of the wave.

If h be small compared with λ , the above equations give

$$c^2 = g h$$

$$a = \frac{\lambda}{2\pi}$$

as an approximation.

The first result is too well known to require comment; the second, if it have any truth at all, appears to shew that the tendency of waves in shallow water is to become semicircular, measuring from the mean points to the crests.

12. Before we proceed further, it will be convenient to make a trifling alteration, both in the mode of proceeding and in the notation.

Let θ now represent $x - ct$ instead of $ct - x$, and denote u by the sum of a series of terms of the form

$$b_0 + b_1 (e^{\alpha y} + e^{-\alpha y}) \sin \theta + \&c.$$

which is the most general form of which it is susceptible for a wave motion.

Let $u dx + v dy$ be denoted by $d\phi$, then it is a well known theorem that

$$\frac{dp}{\rho} = -g dy - d \cdot \frac{d\phi}{dt} - \frac{1}{2} d \cdot (u^2 + v^2).$$

By means of this equation a value of p is found, which being treated as in art. 10, will give the values of c and b_0 , &c.

We proceed to the most general case.

$$u = b_0 + b_1(e^{\alpha y} + e^{-\alpha y}) \sin \theta + b_2(e^{2\alpha y} + e^{-2\alpha y}) \sin 2\theta + \&c.$$

$$v = -b_1(e^{\alpha y} - e^{-\alpha y}) \cos \theta - b_2(e^{2\alpha y} - e^{-2\alpha y}) \cos 2\theta + \&c.$$

$$\phi = b_0 - \frac{1}{\alpha} \left\{ b_1 \overline{e^{\alpha y} + e^{-\alpha y}} \cdot \cos \theta + \frac{b_2}{2} (e^{2\alpha y} + e^{-2\alpha y}) \cos 2\theta + \dots \right\}$$

$$\frac{d\phi}{dt} = -c \left\{ b_1 \overline{e^{\alpha y} + e^{-\alpha y}} \sin \theta + b_2 (e^{2\alpha y} + e^{-2\alpha y}) \sin 2\theta + \dots \right\}$$

$$\therefore \frac{p}{\rho} = -gy + c \left\{ b_1 (e^{\alpha y} + e^{-\alpha y}) \sin \theta + b_2 (e^{2\alpha y} + e^{-2\alpha y}) \sin 2\theta + \dots \right\}$$

$$- \frac{1}{2} \left[b_0^2 + 2b_0 \left\{ b_1 \overline{e^{\alpha y} + e^{-\alpha y}} \sin \theta + b_2 (e^{2\alpha y} + e^{-2\alpha y}) \sin 2\theta + \dots \right\} \right.$$

$$+ b_1^2 \overline{e^{2\alpha y} + e^{-2\alpha y}} + b_2^2 (e^{4\alpha y} + e^{-4\alpha y}) + \dots$$

$$- 2 \{ b_1^2 \cos 2\theta + b_2^2 \cos 4\theta + \dots \}$$

$$+ 2 \sum b_r b_s (e^{\overline{r+s\alpha y}} + e^{-\overline{r+s\alpha y}}) \cos \overline{r-s\theta}$$

$$- 2 \sum b_r b_s (e^{\overline{r-s\alpha y}} + e^{-\overline{r-s\alpha y}}) \cos \overline{r+s\theta} \Big] + P$$

$$= -gy + c \sum b_r (e^{r\alpha y} + e^{-r\alpha y}) \sin r\theta$$

$$- \frac{1}{2} b_0^2 - b_0 \sum b_r (e^{r\alpha y} + e^{-r\alpha y}) \sin r\theta$$

$$- \frac{1}{2} \sum b_r b_s (e^{\overline{r+s\alpha y}} + e^{-\overline{r+s\alpha y}}) \cos \overline{r-s\theta}$$

$$+ \frac{1}{2} \sum b_r b_s (e^{\overline{r-s\alpha y}} + e^{-\overline{r-s\alpha y}}) \cos \overline{r+s\theta} + P$$

the symbol $\sum$ denoting that all the values of b_r, b_s are to be taken, so that r may = 1, 2 ... and s = 1, 2 ...

$$13. \text{ Now, since } \frac{p}{\rho} = f(x, y) - f(x, z);$$

$$\text{the first differentiation gives us } \frac{1}{\rho} \frac{dp}{dx} = \frac{df(x, y)}{dx};$$

$$\text{the second } \frac{1}{\rho} \frac{dp}{dx} = \frac{df(x, y)}{dx} - \frac{df(x, z)}{dx}.$$

Hence, since the two are equal, we must have $\frac{d \cdot f(x, z)}{dx} = 0$, z being considered a function of x .

Now

$$\begin{aligned} f(x, z) = & -g z + c \sum b_r (e^{r\alpha z} + e^{-r\alpha z}) \sin r\theta \\ & - \frac{1}{2} b_o^2 - b_o \sum b_r (e^{r\alpha z} + e^{-r\alpha z}) \sin r\theta \\ & - \frac{1}{2} \sum b_r b_s (e^{\overline{r+s}\alpha z} + e^{-\overline{r+s}\alpha z}) \cos (r-s)\theta \\ & + \frac{1}{2} \sum b_r b_s (e^{\overline{r-s}\alpha z} + e^{-\overline{r-s}\alpha z}) \cos (r+s)\theta \end{aligned}$$

$$\begin{aligned} \therefore 0 = & -g \frac{dz}{dx} + (c - b_o) \alpha \{ b_r (e^{\alpha z} + e^{-\alpha z}) \cos \theta + 2 b_s (e^{2\alpha z} + e^{-2\alpha z}) \cos 2\theta + \dots \} \\ & - 2 \alpha \{ b_r^2 \sin 2\theta + 2 b_s^2 \sin 4\theta + \dots \} \\ & + \alpha \sum \overline{r-s} b_r b_s (e^{\overline{r+s}\alpha z} + e^{-\overline{r+s}\alpha z}) \sin \overline{r-s}\theta \\ & - \alpha \sum \overline{r+s} b_r b_s (e^{\overline{r-s}\alpha z} + e^{-\overline{r-s}\alpha z}) \sin \overline{r+s}\theta \\ & + (c - b_o) \alpha \{ b_r (e^{\alpha z} - e^{-\alpha z}) \sin \theta + 2 b_s (e^{2\alpha z} - e^{-2\alpha z}) \sin 2\theta + \dots \} \frac{dz}{dx} \\ & - \{ b_r^2 (e^{2\alpha z} - e^{-2\alpha z}) + 2 b_s^2 (e^{4\alpha z} - e^{-4\alpha z}) + \dots \} \alpha \frac{dz}{dx} \\ & - \alpha \sum \overline{r+s} b_r b_s (e^{\overline{r+s}\alpha z} - e^{-\overline{r+s}\alpha z}) \cos \overline{r-s}\theta \frac{dz}{dx} \\ & + \alpha \sum \overline{r-s} b_r b_s (e^{\overline{r-s}\alpha z} - e^{-\overline{r-s}\alpha z}) \cos \overline{r+s}\theta \frac{dz}{dx} \end{aligned}$$

or, if we adopt the latter notation, where Σ includes values for r and s , which may be identical, which $\sum$ does not; we get

$$\begin{aligned} 0 = & -g \frac{dz}{dx} + (c - b_o) \alpha \Sigma r b_r (e^{r\alpha z} + e^{-r\alpha z}) \cos r\theta \\ & + \frac{1}{2} \alpha \Sigma \overline{r-s} b_r b_s (e^{\overline{r+s}\alpha z} + e^{-\overline{r+s}\alpha z}) \sin \overline{r-s}\theta \\ & - \frac{1}{2} \alpha \Sigma \overline{r+s} b_r b_s (e^{\overline{r-s}\alpha z} + e^{-\overline{r-s}\alpha z}) \sin \overline{r+s}\theta \\ & + (c - b_o) \alpha \Sigma b_r r (e^{r\alpha z} - e^{-r\alpha z}) \sin r\theta \cdot \frac{dz}{dx} \\ & - \frac{\alpha}{2} \Sigma \overline{r+s} b_r b_s (e^{\overline{r+s}\alpha z} - e^{-\overline{r+s}\alpha z}) \cos \overline{r-s}\theta \cdot \frac{dz}{dx} \\ & + \frac{\alpha}{2} \Sigma \overline{r-s} b_r b_s (e^{\overline{r-s}\alpha z} - e^{-\overline{r-s}\alpha z}) \cos \overline{r+s}\theta \cdot \frac{dz}{dx} \end{aligned}$$

14. From this equation we shall be able to obtain a complete solution of the problem, as far as the assumed form of the velocity expresses the actual state of motion. It will readily appear that z can contain only *sines* of multiples of θ ; for, if it could contain cosines, v would also contain sines, which it is supposed

not to do. Let us then assume

$$z = h + a_1 (e^{\alpha z} \sin \theta) + a_2 e^{2\alpha z} \sin 2\theta + \&c. \\ + f_1 e^{-\alpha z} \sin \theta + f_2 e^{-2\alpha z} \sin 2\theta + \&c.$$

$$\therefore \frac{dz}{dt} = \alpha \overline{u-c} \{ (a_1 e^{\alpha z} + f_1 e^{-\alpha z}) \cos \theta + 2(a_2 e^{2\alpha z} + f_2 e^{-2\alpha z}) \cos 2\theta + \&c. \} \\ + \alpha \frac{dz}{dt} \{ a_1 (e^{\alpha z} - f_1 e^{-\alpha z}) \sin \theta + 2(a_2 e^{2\alpha z} - f_2 e^{-2\alpha z}) \sin 2\theta + \&c. \}$$

or substituting for $\frac{dz}{dt}$ its value

$$(-b_1 \overline{e^{\alpha z} - e^{-\alpha z}} \cos \theta - b_2 \overline{e^{2\alpha z} - e^{-2\alpha z}} \cos 2\theta - \&c.) \times \\ \{ 1 - \alpha \cdot \overline{(a_1 e^{\alpha z} - f_1 e^{-\alpha z}) \sin \theta + 2 \cdot a_2 e^{2\alpha z} - f_2 e^{-2\alpha z} \sin 2\theta + \dots} \} \\ = \alpha \{ (a_1 e^{\alpha z} + f_1 e^{-\alpha z}) \cos \theta + 2(a_2 e^{2\alpha z} + f_2 e^{-2\alpha z}) \cos 2\theta + \dots \} \\ \cdot \{ b_0 - c + b_1 (e^{\alpha z} + e^{-\alpha z}) \sin \theta + b_2 (e^{2\alpha z} + e^{-2\alpha z}) \sin 2\theta + \dots \}.$$

By equating coefficients, we obtain

$$f_1 = -a_1, f_2 = -a_2 \dots \\ \alpha a_1 (b_0 - c) = -b_1, \dots \\ 2\alpha a_2 (b_0 - c) = -b_2, \dots \\ \&c. = \&c. \\ \therefore \alpha a_1 (c - b_0) = b_1, \\ 2\alpha a_2 (c - b_0) = b_2, \\ 3\alpha a_3 (c - b_0) = b_3, \\ \&c. = \&c.$$

So that $z = h + a_1 (e^{\alpha z} - e^{-\alpha z}) \sin \theta + a_2 (e^{2\alpha z} - e^{-2\alpha z}) \sin 2\theta + \dots$

and $\frac{dz}{dx} \{ 1 - \alpha \overline{(a_1 e^{\alpha z} + e^{-\alpha z}) \sin \theta + 2a_2 \cdot e^{2\alpha z} + e^{-2\alpha z} \sin 2\theta + \dots} \} \\ = \alpha \{ a_1 \overline{e^{\alpha z} - e^{-\alpha z}} \cos \theta + 2a_2 \overline{e^{2\alpha z} - e^{-2\alpha z}} \cos 2\theta + \dots \}.$

15. Substituting this value in the equation, we get

$$\left[(c - b_0) \alpha \{ b_1 \overline{e^{\alpha z} + e^{-\alpha z}} \cos \theta + 2b_2 \overline{(e^{2\alpha z} + e^{-2\alpha z}) \cos 2\theta + \dots} \} \right. \\ + \frac{\alpha}{2} \sum \overline{r-s} b_r b_s (e^{\overline{r+s\alpha z}} + e^{-\overline{r+s\alpha z}}) \sin \overline{r-s} \theta \\ \left. - \frac{\alpha}{2} \sum \overline{r+s} b_r b_s (e^{\overline{r-s\alpha z}} + e^{-\overline{r-s\alpha z}}) \sin \overline{r+s} \theta \right] \times \\ \{ 1 - \alpha \overline{(a_1 e^{\alpha z} + e^{-\alpha z}) \sin \theta + 2a_2 \overline{e^{2\alpha z} + e^{-2\alpha z}} \sin 2\theta + \dots} \}$$

$$\begin{aligned}
&= - \left[-g + \overline{c-b_0} \alpha \{ b, \overline{e^{\alpha z} - e^{-\alpha z}} \sin \theta + 2 b_2 \overline{e^{2\alpha z} - e^{-2\alpha z}} \sin 2\theta + \dots \} \right. \\
&\quad - \frac{\alpha}{2} \Sigma \overline{r+s} b_r b_s (\overline{e^{r+s\alpha z} - e^{-r+s\alpha z}}) \cos \overline{r-s} \theta \\
&\quad \left. + \frac{\alpha}{2} \Sigma \overline{r-s} b_r b_s (\overline{e^{r-s\alpha z} - e^{-r-s\alpha z}}) \cos \overline{r+s} \theta \right] \\
&\quad \times \{ \alpha (\overline{a, e^{\alpha z} - e^{-\alpha z}} \cos \theta + 2 a_2 \overline{e^{2\alpha z} - e^{-2\alpha z}} \cos 2\theta + \dots) \}
\end{aligned}$$

the symbol Σ applying to all values of r and s from 1 to ∞ .

$$16. \text{ But since } \frac{b_2}{b_1} = \frac{2a_2}{a_1}, \quad \therefore 2a_2 = \frac{b_2}{b_1} a_1.$$

$$\text{Similarly } 3a_3 = \frac{a_1}{b_1} b_3, \quad 4a_4 = \frac{a_1}{b_1} b_4 \quad \&c.$$

Hence, if we substitute these values in the above equation, it gives us

$$\begin{aligned}
&\left[(c-b_0) \alpha \{ b, \overline{e^{\alpha z} + e^{-\alpha z}} \cos \theta + 2 b_2 \overline{e^{2\alpha z} + e^{-2\alpha z}} \cos 2\theta + \dots \} \right. \\
&\quad + \frac{\alpha}{2} \Sigma (r-s) b_r b_s (\overline{e^{r+s\alpha z} + e^{-r+s\alpha z}}) \sin \overline{r-s} \theta \\
&\quad \left. - \frac{\alpha}{2} \Sigma (r+s) b_r b_s (\overline{e^{r-s\alpha z} + e^{-r-s\alpha z}}) \sin \overline{r+s} \theta \right] \\
&\quad \times \left\{ 1 - \alpha \frac{a_1}{b_1} (b, \overline{e^{\alpha z} + e^{-\alpha z}} \sin \theta + b_2 \overline{e^{2\alpha z} + e^{-2\alpha z}} \sin 2\theta + \dots) \right\} \\
&= - \left[-g + \overline{c-b_0} \alpha \{ b, \overline{e^{\alpha z} - e^{-\alpha z}} \sin \theta + 2 b_2 \overline{e^{2\alpha z} - e^{-2\alpha z}} \sin 2\theta + \dots \} \right. \\
&\quad - \frac{\alpha}{2} \Sigma (r+s) b_r b_s (\overline{e^{r+s\alpha z} - e^{-r+s\alpha z}}) \cos (r-s) \theta \\
&\quad \left. + \frac{\alpha}{2} \Sigma \overline{r-s} b_r b_s (\overline{e^{r-s\alpha z} - e^{-r-s\alpha z}}) \cos \overline{r+s} \theta \right] \\
&\quad \times \frac{\alpha a_1}{b_1} \{ b, \overline{e^{\alpha z} - e^{-\alpha z}} \cos \theta + b_2 (\overline{e^{2\alpha z} - e^{-2\alpha z}}) \cos 2\theta + \dots \}
\end{aligned}$$

17. By multiplying the upper line of the first side of this equation by its factor, we obtain

$$\begin{aligned}
&(c-b_0) \alpha \Sigma r b_r (\overline{e^{r\alpha z} + e^{-r\alpha z}}) \cos r \theta \\
&- \alpha^2 \frac{a_1}{b_1} (c-b_0) \Sigma r b_r b_s (\overline{e^{r\alpha z} + e^{-r\alpha z}}) (\overline{e^{s\alpha z} + e^{-s\alpha z}}) \cos r \theta \sin s \theta :
\end{aligned}$$

the analogous term on the second side of the equation is

$$- \alpha^2 (c-b_0) \frac{a_1}{b_1} \Sigma (\overline{e^{r\alpha z} - e^{-r\alpha z}}) (\overline{e^{s\alpha z} - e^{-s\alpha z}}) b_r b_s r \cos s \theta \sin r \theta ;$$

if this result be brought to the first side of the equation, and combined with the former, we obtain

$$\begin{aligned} & (c-b_0) \alpha \Sigma r b_r (e^{r\alpha z} + e^{-r\alpha z}) \cos r \theta \\ & + \alpha \Sigma (e^{\overline{r+s}\alpha z} + e^{-\overline{r+s}\alpha z}) r b_r b_s \sin \overline{r-s} \theta \\ & - \alpha \Sigma (e^{\overline{r-s}\alpha z} + e^{-\overline{r-s}\alpha z}) r b_r b_s \sin \overline{r+s} \theta \quad (1) \end{aligned}$$

it being observed that $\alpha \frac{a_i}{b_i} \overline{c-b_0} = 1$ by (14).

18. The result of the factor unity is (omitting the part which has been already considered)

$$\begin{aligned} & \frac{\alpha}{2} \Sigma (r-s) b_r b_s (e^{\overline{r+s}\alpha z} + e^{-\overline{r+s}\alpha z}) \sin \overline{r-s} \theta \\ & - \frac{\alpha}{2} \Sigma (r+s) b_r b_s (e^{\overline{r-s}\alpha z} + e^{-\overline{r-s}\alpha z}) \sin \overline{r+s} \theta \quad (2). \end{aligned}$$

The remainder of the upper side is

$$\begin{aligned} & - \frac{\alpha^2 a_i}{2 b_i} \Sigma (r-s) b_r b_s b_t (e^{\overline{r+s}\alpha z} + e^{-\overline{r+s}\alpha z}) (e^{t\alpha z} + e^{-t\alpha z}) \sin \overline{r-s} \theta \cdot \sin t \theta \\ & + \frac{\alpha^2 a_i}{2 b_i} \Sigma (r+s) b_r b_s b_t (e^{\overline{r-s}\alpha z} + e^{-\overline{r-s}\alpha z}) (e^{t\alpha z} + e^{-t\alpha z}) \sin \overline{r+s} \theta \sin t \theta : \end{aligned}$$

and the corresponding terms of the lower side are

$$\begin{aligned} & \frac{\alpha^2 a_i}{2 b_i} \Sigma (r+s) b_r b_s b_t (e^{\overline{r+s}\alpha z} - e^{-\overline{r+s}\alpha z}) (e^{t\alpha z} - e^{-t\alpha z}) \cos \overline{r-s} \theta \cos t \theta \\ & - \frac{\alpha^2 a_i}{2 b_i} \Sigma (r-s) b_r b_s b_t (e^{\overline{r-s}\alpha z} - e^{-\overline{r-s}\alpha z}) (e^{t\alpha z} - e^{-t\alpha z}) \cos \overline{r+s} \theta \cos t \theta ; \end{aligned}$$

which being brought over to the other side, the sum of the respective terms is

$$\begin{aligned} \frac{\alpha^2 a_i}{2 b_i} \Sigma b_r b_s b_t \left[& -(r-s) (e^{\overline{r+s+t}\alpha z} + e^{-\overline{r+s+t}\alpha z} + e^{\overline{r-s-t}\alpha z} + e^{-\overline{r-s-t}\alpha z}) \sin (r-s) \theta \sin t \theta \right. \\ & + (r+s) (e^{\overline{r-s+t}\alpha z} + e^{\overline{r-s-t}\alpha z} + e^{-\overline{r-s+t}\alpha z} + e^{-\overline{r-s-t}\alpha z}) \sin (r+s) \theta \sin t \theta \\ & - (r+s) (e^{\overline{r+s+t}\alpha z} - e^{\overline{r+s-t}\alpha z} + e^{-\overline{r+s+t}\alpha z} - e^{-\overline{r+s-t}\alpha z}) \cos (r-s) \theta \cos t \theta \\ & \left. + (r-s) (e^{\overline{r-s+t}\alpha z} - e^{\overline{r-s-t}\alpha z} + e^{-\overline{r-s+t}\alpha z} - e^{-\overline{r-s-t}\alpha z}) \cos (r+s) \theta \cos t \theta \right] \end{aligned}$$

which becomes, by multiplying out, and adding together, such quantities as will make up cosines of sums or differences of the arcs which appear in the present form,

$$\begin{aligned} & \frac{\alpha^2 a_i}{2 b_i} \Sigma b_r b_s b_t \left[-r (e^{\overline{r+s+t}\alpha z} + e^{-\overline{r+s+t}\alpha z}) \cos \overline{r-s-t} \theta \right. \\ & \quad -s (e^{\overline{r+s+t}\alpha z} + e^{-\overline{r+s+t}\alpha z}) \cos \overline{r-s+t} \theta \\ & \quad +r (e^{\overline{r+s-t}\alpha z} + e^{-\overline{r+s-t}\alpha z}) \cos \overline{r-s+t} \theta \\ & \quad \left. +s (e^{\overline{r+s-t}\alpha z} + e^{-\overline{r+s-t}\alpha z}) \cos \overline{r-s-t} \theta \right] \end{aligned}$$

$$\begin{aligned}
& + r (e^{\overline{r-s+t}\alpha z} + e^{\overline{-r-s+t}\alpha z}) \cos \overline{r+s-t}\theta \\
& - s (e^{\overline{r-s+t}\alpha z} + e^{\overline{-r-s+t}\alpha z}) \cos \overline{r+s+t}\theta \\
& - r (e^{\overline{r-s-t}\alpha z} + e^{\overline{-r-s-t}\alpha z}) \cos \overline{r+s+t}\theta \\
& + s (e^{\overline{r-s-t}\alpha z} + e^{\overline{-r-s-t}\alpha z}) \cos \overline{r+s-t}\theta \quad]
\end{aligned}$$

Or since

$$\begin{aligned}
& \Sigma r \cos \overline{r-s-t}\theta e^{\overline{r+s+t}\alpha z} \\
& = \Sigma s \cos \overline{r-s+t}\theta e^{\overline{r+s+t}\alpha z} \\
& \quad \&c. \quad \&c.
\end{aligned}$$

this term may be written

$$\begin{aligned}
& \frac{\alpha^2 a_r}{2 b_r} \Sigma b_r b_s b_t \left\{ -2r (e^{\overline{r+s+t}\alpha z} + e^{\overline{-r+s+t}\alpha z}) \cos \overline{s+t-r}\theta \right. \\
& \quad + 2r (e^{\overline{r+s-t}\alpha z} + e^{\overline{-r+s-t}\alpha z}) \cos \overline{r+t-s}\theta \\
& \quad + 2r (e^{\overline{r+t-s}\alpha z} + e^{\overline{-r+t-s}\alpha z}) \cos \overline{r+s-t}\theta \\
& \quad \left. - 2r (e^{\overline{s+t-r}\alpha z} + e^{\overline{-s+t-r}\alpha z}) \cos \overline{r+s+t}\theta \right\} :
\end{aligned}$$

which may be written

$$\begin{aligned}
& = \frac{\alpha^2 a_r}{b_r} \Sigma r b_r b_s b_t \left\{ - (e^{f\alpha z} + e^{-f\alpha z}) \cos \overline{f-2r}\theta - (e^{\overline{f-2r/\alpha}z} + e^{-\overline{f-2r/\alpha}z}) \cos f\theta \right. \\
& \quad \left. + (e^{\overline{f-2t/\alpha}z} + e^{-\overline{f-2t/\alpha}z}) \cos (f-2s) \cdot \theta + (e^{\overline{f-2s/\alpha}z} + e^{-\overline{f-2s/\alpha}z}) \cos (f-2t) \theta \right\} \quad (3)
\end{aligned}$$

if $r+s+t$ be denoted by f .

19. The remainder of the lower side of our equation is

$$\frac{g \alpha a_r}{b_r} \Sigma b_r (e^{r\alpha z} - e^{-r\alpha z}) \cos r\theta.$$

By collecting these several terms our equation is reduced to the following very simple form :

$$\begin{aligned}
& (c-b_0) \alpha \Sigma r b_r (e^{r\alpha z} + e^{-r\alpha z}) \cos r\theta \\
& + \alpha \Sigma (e^{\overline{r+s}\alpha z} + e^{\overline{-r+s}\alpha z}) r b_r b_s \sin \overline{r-s}\theta \\
& - \alpha \Sigma (e^{\overline{r-s}\alpha z} + e^{\overline{-r-s}\alpha z}) r b_r b_s \sin \overline{r+s}\theta \\
& + \frac{\alpha}{2} \Sigma (r-s) b_r b_s (e^{\overline{r+s}\alpha z} + e^{\overline{-r+s}\alpha z}) \sin \overline{r-s}\theta \\
& - \frac{\alpha}{2} \Sigma \overline{r+s} b_r b_s (e^{\overline{r-s}\alpha z} + e^{\overline{-r-s}\alpha z}) \sin \overline{r+s}\theta
\end{aligned}$$

$$\begin{aligned}
& + \frac{\alpha^2 a_i}{b_i} \Sigma_r b_r b_s b_t \left\{ -(e^{f\alpha z} + e^{-f\alpha z}) \cos \overline{f-2r} \cdot \theta - (e^{\overline{f-2r} \cdot \alpha z} + e^{-\overline{f-2r} \cdot \alpha z}) \cos f \theta \right. \\
& + (e^{\overline{f-2t} \cdot \alpha z} + e^{-\overline{f-2t} \cdot \alpha z}) \cos \overline{f-2s} \theta + (e^{\overline{f-2s} \cdot \alpha z} + e^{-\overline{f-2s} \cdot \alpha z}) \cos \overline{f-2t} \theta \left. \right\} \\
& = \frac{g \alpha a_i}{b_i} \Sigma b_r (e^{r\alpha z} - e^{-r\alpha z}) \cos r \theta :
\end{aligned}$$

OR

$$\begin{aligned}
& (c-b_0)^2 \alpha \Sigma r b_r (e^{r\alpha z} + e^{-r\alpha z}) \cos r \theta \\
& + 2 \alpha \Sigma (e^{\overline{r+s} \alpha z} + e^{-\overline{r+s} \alpha z}) r b_r b_s \sin \overline{r-s} \theta (c-b_0) \\
& - 2 \alpha \Sigma (e^{\overline{r-s} \alpha z} + e^{-\overline{r-s} \alpha z}) r b_r b_s \sin \overline{r+s} \theta (c-b_0) \\
& + \alpha \Sigma r b_r b_s b_t \left\{ -(e^{f\alpha z} + e^{-f\alpha z}) \cos \overline{f-2r} \cdot \theta - (e^{\overline{f-2r} \alpha z} + e^{-\overline{f-2r} \alpha z}) \cos f \theta \right. \\
& + (e^{\overline{f-2t} \alpha z} + e^{-\overline{f-2t} \alpha z}) \cos \overline{f-2s} \theta + (e^{\overline{f-2s} \alpha z} + e^{-\overline{f-2s} \alpha z}) \cos \overline{f-2t} \theta \left. \right\} \\
& = g \Sigma b_r (e^{r\alpha z} - e^{-r\alpha z}) \cos r \theta :
\end{aligned}$$

or finally

$$\begin{aligned}
& (c-b_0)^2 \Sigma r b_r (e^{r\alpha z} + e^{-r\alpha z}) \cos r \theta \\
& + 2 (c-b_0) \Sigma r b_r b_s \left\{ (e^{\overline{r+s} \alpha z} + e^{-\overline{r+s} \alpha z}) \sin \overline{r-s} \theta - (e^{\overline{r-s} \alpha z} + e^{-\overline{r-s} \alpha z}) \sin \overline{r+s} \theta \right\} \\
& + \Sigma r b_r b_s b_t \left\{ -(e^{f\alpha z} + e^{-f\alpha z}) \cos \overline{f-2r} \theta - (e^{\overline{f-2r} \alpha z} + e^{-\overline{f-2r} \alpha z}) \cos f \theta \right. \\
& + (e^{\overline{f-2t} \alpha z} + e^{-\overline{f-2t} \alpha z}) \cos \overline{f-2s} \theta + (e^{\overline{f-2s} \alpha z} + e^{-\overline{f-2s} \alpha z}) \cos \overline{f-2t} \theta \left. \right\} \\
& = \frac{g}{\alpha} \Sigma b_r (e^{r\alpha z} - e^{-r\alpha z}) \cos r \theta.
\end{aligned}$$

20. This expression is exceedingly simple and symmetrical, and might be very easily applied to any hypothesis respecting the coefficients $b_1, b_2 \dots$. It may be satisfactory, in the first place, to deduce from it the particular form already obtained, art. 10.

Let b , be the only value of b , then $r=s=t=1$, and we get

$$\begin{aligned}
& c^2 b (e^{\alpha z} + e^{-\alpha z}) \cos \theta - 2 c b^2 \cdot 2 \sin 2 \theta \\
& + b^3 \left(-(e^{3\alpha z} + e^{-3\alpha z}) \cos \theta - (e^{\alpha z} + e^{-\alpha z}) \cos 3 \theta + 2 (e^{\alpha z} + e^{-\alpha z}) \cos \theta \right) \\
& = \frac{g}{\alpha} b (e^{\alpha z} - e^{-\alpha z}) \cos \theta
\end{aligned}$$

OR

$$\begin{aligned}
& c^2 b (e^{\alpha z} + e^{-\alpha z}) \cos \theta - 8 c b^2 \sin \theta \cos \theta \\
& - b^3 \left\{ (e^{3\alpha z} + e^{-3\alpha z}) \cos \theta + e^{\overline{\alpha z}} + e^{-\overline{\alpha z}} (\cos 3 \theta + \cos \theta) - 3 e^{\overline{\alpha z}} + e^{-\overline{\alpha z}} \cos \theta \right\} \\
& = \frac{g}{\alpha} b (e^{\alpha z} - e^{-\alpha z}) \cos \theta :
\end{aligned}$$

that is, if we divide by $b \cos \theta$

$$\begin{aligned} & c^2 (e^{\alpha z} + e^{-\alpha z}) - 8 c b \sin \theta - b^2 (e^{3\alpha z} + e^{-3\alpha z}) \\ & - 2 b^2 (e^{\alpha z} + e^{-\alpha z}) \cos 2\theta + 3 b^2 (e^{\alpha z} + e^{-\alpha z}) \\ & = \frac{g}{\alpha} (e^{\alpha z} - e^{-\alpha z}). \end{aligned}$$

Now the expression in art. 10. is this,

$$\begin{aligned} & -8 b c \sin \theta + c^2 (e^{\alpha z} + e^{-\alpha z}) + 2 b^2 \overline{1 - \cos 2\theta} (e^{\alpha z} + e^{-\alpha z}) \\ & - b^2 (\overline{e^{3\alpha z} + e^{-3\alpha z}} - \overline{e^{\alpha z} + e^{-\alpha z}}) = \frac{g}{\alpha} (e^{\alpha z} - e^{-\alpha z}); \end{aligned}$$

which is equivalent to

$$\begin{aligned} & c^2 (e^{\alpha z} + e^{-\alpha z}) - 8 b c \sin \theta - b^2 (e^{3\alpha z} + e^{-3\alpha z}) + 3 b^2 (e^{\alpha z} + e^{-\alpha z}) \\ & - 2 b^2 (e^{\alpha z} + e^{-\alpha z}) \cos 2\theta = \frac{g}{\alpha} (e^{\alpha z} - e^{-\alpha z}) \end{aligned}$$

an equation identical with the one above.

21. Let us now derive from the equation the value of c in terms of $b, b_2 \dots$.

1. In the first place, since b_0 is in every place subtracted from c , never occurring in any other way, we derive the following important conclusion :

That a progressive motion of the fluid does not affect the velocity of transmission of the undulation relative to the position of corresponding particles; in other words, the velocity of transfer of the undulation is exactly equal to the sum of the velocity of progression, and that of undulation in a fluid at rest.

2. If we make $\theta=0$, we obtain

$$\begin{aligned} & (c - b_0)^2 \Sigma r b_r (e^{r\alpha h} + e^{-r\alpha h}) \\ & + \Sigma r b_r b_s b_t \left\{ -(e^{f\alpha h} + e^{-f\alpha h}) - (e^{(f-2r)\alpha h} + e^{-\overline{f-2r}\alpha h}) \right. \\ & \left. + (e^{\overline{f-2t}\alpha h} + e^{-\overline{f-2t}\alpha h}) + (e^{\overline{f-2s}\alpha h} + e^{-\overline{f-2s}\alpha h}) \right\} \\ & = \frac{g}{\alpha} \Sigma b_r (e^{r\alpha h} - e^{-r\alpha h}) \end{aligned}$$

which equation gives $c - b_0$.

The two values so obtained will be equal, but will have opposite signs, since the term which involves the first power of $c - b_0$ does not contain any cosines.

This circumstance that the sines are combined with the odd powers, and cosines with even powers of $c - b_0$ is very remarkable, and it is probably connected with the relation existing between the quantities $b, b_2 \dots$, but we shall not at present enter into a discussion of the subject.

If it be thought more simple to obtain $(c - b_0)^2$ in terms of $a, a_2 \dots$, than in terms of $b, b_2 \dots$, this can be at once effected by means of the equations in art. 14; the result being :

$$\begin{aligned}
& (c-b_0)^2 \Sigma r^2 a_r (e^{r\alpha h} + e^{-r\alpha h}) \\
& + \alpha^2 \Sigma r^2 s t (c-b_0)^2 a_r a_s a_t \left\{ -(e^{f\alpha h} + e^{-f\alpha h}) \right. \\
& \left. - (e^{(f-2r)\alpha h} + e^{-(f-2r)\alpha h}) + e^{(f-2t)\alpha h} + e^{-(f-2t)\alpha h} + e^{(f-2s)\alpha h} + e^{-(f-2s)\alpha h} \right\} \\
& = \frac{g}{\alpha} \Sigma r a_r (e^{r\alpha h} - e^{-r\alpha h}).
\end{aligned}$$

From this equation, we obtain

$$\begin{aligned}
(c-b_0)^2 &= \frac{g}{\alpha} \Sigma r a_r (e^{r\alpha h} - e^{-r\alpha h}) \div \\
& \left[\Sigma r^2 a_r (e^{r\alpha h} + e^{-r\alpha h}) + \alpha^2 \Sigma r^2 s t a_r a_s a_t \times \right. \\
& \left. \left\{ -(e^{f\alpha h} + e^{-f\alpha h}) - (e^{(f-2r)\alpha h} + e^{-(f-2r)\alpha h}) + e^{(f-2s)\alpha h} + e^{-(f-2s)\alpha h} \right. \right. \\
& \left. \left. + e^{(f-2t)\alpha h} + e^{-(f-2t)\alpha h} \right\} \right]
\end{aligned}$$

22. Since v must of necessity be zero at the points where z is a maximum and a minimum, we get for the values of θ at such points, first from the value of $\frac{dz}{dx}$ being 0 at such points,

$$0 = a_1 (e^{\alpha z} - e^{-\alpha z}) \cos \theta + 2 a_2 (e^{2\alpha z} - e^{-2\alpha z}) \cos 2\theta + \dots;$$

and from the circumstance that $v=0$,

$$0 = b_1 (e^{\alpha z} - e^{-\alpha z}) \cos \theta + b_2 (e^{2\alpha z} - e^{-2\alpha z}) \cos 2\theta + \dots$$

But by art. 14,

$$\begin{aligned}
b_1 &= \alpha (c-b_0) a_1 \\
b_2 &= \alpha (c-b_0) 2 a_2 \\
&\dots = \dots
\end{aligned}$$

therefore, the second equation becomes

$$0 = a_1 (e^{\alpha z} - e^{-\alpha z}) \cos \theta + 2 a_2 (e^{2\alpha z} - e^{-2\alpha z}) \cos 2\theta + \dots$$

which is identical with the first. This is strongly confirmatory of the correctness of our operations.

If (as is probably always the case) the wave be symmetrical on both sides, we must have this zero occurring at points distant from each other by $\frac{\lambda}{2}$; hence

$$\theta = \frac{\pi}{2}, \quad \frac{3\pi}{2}, \text{ and } a_2, a_1, \&c. = 0; \quad b_2, b_1, \&c. = 0.$$

The value of u at such points is

$$\begin{aligned}
u &= b_0 + b_1 e^{-\frac{2\pi}{\lambda}y} \sin \theta + b_3 e^{-\frac{6\pi}{\lambda}y} \sin 3\theta + \dots \\
\theta = \frac{\pi}{2} &\left\{ \begin{aligned} &= b_0 + b_1 e^{-\frac{2\pi}{\lambda}y} - b_3 e^{-\frac{6\pi}{\lambda}y} + b_5 e^{-\frac{10\pi}{\lambda}y} - \&c. \end{aligned} \right. \\
\theta = \frac{3\pi}{2} &\left\{ \begin{aligned} &= b_0 - b_1 e^{-\frac{2\pi}{\lambda}y} + b_3 e^{-\frac{6\pi}{\lambda}y} - \&c. \end{aligned} \right.
\end{aligned}$$

2. In the case of common oscillatory waves, there ought one to be positive and the other negative, equal in magnitude,

$$\therefore b_o = 0 :$$

from which it follows, that b_o is a quantity depending *entirely* on the progressive motion.

SECTION II.—VARIABLE WAVE MOTION.

23. Hitherto we have confined our attention to motion in two dimensions, limiting the channel to one of uniform breadth and depth.

We proceed next to the more general case, where the breadth is variable but the depth constant.

To take the same order as before, we will commence with the hypothesis that all vertical sections have the same horizontal velocity at any time, the section being perpendicular to the direction of transmission.

By reference to arts. 2, 6, and 7, the process which follows will be perfectly intelligible, without any lengthened explanation.

We commence with that case wherein the section perpendicular to the direction of transmission is a triangle, one side of which we may suppose vertical, but it will not affect the calculation.

Let the axes be called those of x , y , and z ; the introduction of a third being requisite in this case. In the former calculations, we adopted z as the representation of the depth: in the present case, we shall use the letter s for the same purpose.

The annexed figure is supposed to be a section of the fluid perpendicular to the direction of motion, and it is supposed that all sections so made are similar. Our object is to determine the motion of a portion of the fluid enclosed between two planes perpendicular to the direction of translation.

Let α be the distance between these planes at the time t ,

$\alpha + \delta \alpha$ at the time $t + \delta t$;

s , the depth of the fluid at t ,

$s + \delta s$ at $t + \delta t$;

r the breadth at the top at the time t ,

$r + \delta r$ at $t + \delta t$;

then, from the property of the triangle, r and s bear a constant ratio: let $r = m s$.

Also, since the quantity of fluid between the planes is supposed to remain unchanged, we obtain

$$r s \alpha = (r + \delta r) (s + \delta s) (\alpha + \delta \alpha)$$

$$\begin{aligned} \text{or} \quad s^2 \alpha &= (s + \delta s)^2 (\alpha + \delta \alpha) \\ &= s^2 \alpha + s^2 \delta \alpha + 2 s \alpha \delta s \quad \text{nearly} \end{aligned}$$

$$\begin{aligned} \therefore \quad \delta s &= -\frac{s^2}{2 s \alpha} \delta \alpha \\ &= -\frac{s}{2 \alpha} \delta \alpha \end{aligned}$$

$$\text{hence} \quad \frac{ds}{dt} = -\frac{s}{2 \alpha} \frac{d\alpha}{dt}.$$

24. Now, the moving force on this mass is

$$\begin{aligned} & \iint dy \, dz \, p - \iint dy \, dz \left(p + \frac{dp}{dx} \delta x \right) \\ &= \int_0^s dy \int_0^{my} dz \, p - \int_0^{s + \frac{ds}{dx} \delta x} dy \int_0^{my} dz \left(p + \frac{dp}{dx} \delta x \right); \end{aligned}$$

and

$$p = g \rho \cdot \overline{s - y} - \frac{1}{2} \rho u^2,$$

$$\therefore \quad \frac{dp}{dx} = g \rho \frac{ds}{dx} - \rho u \frac{du}{dx}.$$

Hence moving force $\cdot \frac{1}{\rho}$

$$\begin{aligned} &= \int_0^s dy \int_0^{my} dz \left\{ g \overline{s - y} - \frac{1}{2} u^2 \right\} \\ &- \int_0^{s + \frac{ds}{dx} \alpha} dy \int_0^{my} dz \left\{ g \left(\overline{s + \frac{ds}{dx} \alpha - y} \right) - \frac{1}{2} \left(u^2 + 2u \frac{du}{dx} \alpha \right) \right\} \\ &= \int_0^s dy \left\{ g \overline{s - y} my - \frac{1}{2} u^2 my \right\} \\ &- \int_0^{s + \frac{ds}{dx} \alpha} dy \left\{ g \left(\overline{s + \frac{ds}{dx} \alpha - y} \right) my - \frac{1}{2} \overline{u^2 + 2u \frac{du}{dx} \alpha} my \right\} \\ &= g m \frac{s^3}{2} - g m \frac{s^3}{3} - \frac{1}{4} u^2 m s^2 \\ &- g \overline{s + \frac{ds}{dx} \alpha}^3 \cdot \frac{m}{2} + g m \frac{\overline{s + \frac{ds}{dx} \alpha}^3}{3} + \frac{1}{4} \left(u^2 + 2u \frac{du}{dx} \alpha \right) m \overline{s + \frac{ds}{dx} \alpha}^2 \\ &= -\frac{3}{2} g m s^2 \cdot \frac{ds}{dx} \alpha + g m s^2 \frac{ds}{dx} \alpha + \frac{1}{2} m \left(u^2 s \frac{ds}{dx} \alpha + s^2 u \frac{du}{dx} \alpha \right) \\ &= -\frac{1}{2} m g \alpha s^2 \frac{ds}{dx} + \frac{1}{2} m \alpha \left(u^2 s \frac{ds}{dx} + s^2 u \frac{du}{dx} \right) \end{aligned}$$

and mass moved $= \frac{m}{2} s^2 \alpha \rho$

therefore, accelerating force $= -g \frac{ds}{dx} + \frac{u^2 \frac{ds}{dx} + s u \frac{du}{dx}}{s}$

25. Also,

$$u = b \sin \frac{2\pi}{\lambda} (ct - x)$$

$$s = h + a \sin \frac{2\pi}{\lambda} (ct - x):$$

$$\therefore \frac{du}{dx} = -b \frac{2\pi}{\lambda} \cos \frac{2\pi}{\lambda} (ct - x),$$

$$\frac{ds}{dx} = -a \frac{2\pi}{\lambda} \cos \frac{2\pi}{\lambda} (ct - x),$$

$$\frac{du}{dt} = \frac{2\pi}{\lambda} bc \cos \theta - \frac{2\pi}{\lambda} b^2 \sin \theta \cos \theta.$$

Consequently,
$$\frac{2\pi}{\lambda} bc \cos \theta - \frac{2\pi}{\lambda} b^2 \sin \theta \cos \theta = \frac{2\pi}{\lambda} ag \cos \theta$$

$$- \frac{2\pi}{\lambda} b^2 \cdot \frac{a \cos \theta \sin^2 \theta + \sin \theta \cos \theta (h + a \sin \theta)}{h + a \sin \theta}$$

Multiplying out, this gives

$$(ag - bc + b^2 \sin \theta) (h + a \sin \theta) = b^2 a \sin^2 \theta + b^2 \sin \theta (h + a \sin \theta),$$

or
$$(ag - bc) (h + a \sin \theta) = b^2 a \sin^2 \theta.$$

Whence we get

$$hag - hbc = 0;$$

$$\therefore ag = bc.$$

But

$$\left(\frac{dx}{dt}\right)^2 = u^2,$$

$$\therefore \left(\frac{dx + d\alpha}{dt}\right)^2 = \left(u + \frac{du}{dx} \alpha\right)^2$$

and

$$\frac{dx}{dt} \cdot \frac{d\alpha}{dt} = u \frac{du}{dx} \alpha;$$

hence

$$\frac{d\alpha}{dt} = \alpha \frac{du}{dx},$$

$$\therefore \frac{du}{dx} = \frac{1}{\alpha} \frac{d\alpha}{dt} = -\frac{2}{s} \cdot \frac{ds}{dt};$$

whence, by substitution,

$$(h + a \sin \theta) b \cos \theta = 2a \cos \theta (c - b \sin \theta) + \dots$$

$$\therefore hb = 2ac$$

or if we put for b its value found above,

$$\frac{hag}{c} = 2ac$$

$$hg = 2c^2$$

$$c = \sqrt{\frac{gh}{2}}.$$

That is, the square of the velocity of transmission in a triangular channel, is half the square of the velocity in a rectangular channel of the same maximum depth.

It may be thought necessary to verify this result, although, from its simplicity, I suspect it must have been known to Poisson and others. As, however, I have never seen it, or met with any reference to such an investigation, I have added a few examples from Mr RUSSELL's *Report on Waves*, in the last volume of the Reports of the British Association. The wave which Mr RUSSELL has examined, is different from that which has been assumed as the basis of calculation; but the difference makes no alteration in this result, as will be seen hereafter.

From the analysis which Mr RUSSELL has given of his experiments, I select the following results; being those for which the height of the fluid in the triangular tube most nearly coincides with that in the corresponding experiment in the quadrilateral one.

Depth of Fluid in inches.	Velocity observed in rectangular channel.	Velocity observed in triangular channel.	Do. computed from those in the second column.	Difference.
4.15	3.20	2.19	2.2	
4.23	3.35	2.42	2.4	
4.32	3.40	2.43	2.4	
4.38	3.40	2.46	2.4	
5.59	4.05	2.66	2.8	.1
6.26	4.08	2.88	2.9	
6.38	4.04	2.85	2.8	
6.62	4.05	3.02	2.8	.2
7.21	4.32	3.02	3.0	
7.36	4.39	3.02	3.1	

The only discrepancies are those which I have placed in the table; but it may be remarked that, in all cases, the circumstances are very different in the two experiments, depending partly on the height and partly on the length of the wave. That the two cases fixed on should be at variance, is not to be wondered at, when it is remarked that, in the rectangular channel, the velocity is the same for both cases, whilst the depths differ by about an inch; and it is by means of the depth alone, that we have compared the results in a triangular channel with those in a rectangular.

On the whole, I conceive the coincidence between the different results as a striking confirmation of the process which has been employed, approximative as that process confessedly is.

26. Let us now take the general case of any shape whatever to the vertical section.

Let K represent the area of the section of the fluid at the point x, y, z .

Then
$$K \alpha = \left(K + \frac{dK}{dt} \delta t \right) \left(\alpha + \frac{d\alpha}{dt} \delta t \right),$$

$$\therefore 0 = K \frac{d\alpha}{dt} + \alpha \frac{dK}{dt}.$$

Now, K is a function of the height s ; therefore, s is a function of K .

Let

$$s = f K$$

$$\begin{aligned} \therefore \frac{ds}{dt} &= f' K \cdot \frac{dK}{dt} \\ &= -\frac{K}{\alpha} f' K \cdot \frac{d\alpha}{dt}. \end{aligned}$$

Now the moving force, on the mass of which the thickness is α

$$= \int_0^s dy \int_0^{\phi y} dz p - \int_0^{s + \frac{ds}{dx} \alpha} dy \int_0^{\phi y} dz \left(p + \frac{dp}{dx} \delta x \right)$$

where $\phi y = z$ is the equation to the generating curve of the boundary of the section.

And

$$p = g \rho (s - y) - \frac{1}{2} \rho u^2,$$

$$\frac{dp}{dx} = g \rho \frac{ds}{dx} - \rho u \frac{du}{dx}.$$

$$\begin{aligned} \therefore \frac{\text{moving force}}{\rho} &= \int_0^s dy \int_0^{\phi y} dz \left(g \overline{s - y} - \frac{1}{2} u^2 \right) \\ &\quad - \int_0^{s + \frac{ds}{dx} \alpha} dy \int_0^{\phi y} dz \left\{ g \left(\overline{s + \frac{ds}{dx} \alpha - y} \right) - \frac{1}{2} \left(u^2 + 2u \frac{du}{dx} \alpha \right) \right\} \\ &= \int_0^s dy \left(g \overline{s - y} - \frac{1}{2} u^2 \right) \phi y \\ &\quad - \int_0^{s + \frac{ds}{dx} \alpha} dy \left(g \left(\overline{s + \frac{ds}{dx} \alpha - y} \right) - \frac{1}{2} \overline{u^2 + 2u \frac{du}{dx} \alpha} \right) \phi y \\ &= \int_0^s dy \left\{ -g \frac{ds}{dx} \alpha + u \frac{du}{dx} \alpha \right\} \phi y \\ &\quad - \int_s^{s + \frac{ds}{dx} \alpha} dy \left\{ g \overline{s - y} - \frac{1}{2} u^2 \right\} \phi y. \end{aligned}$$

Let

$$\int \phi y dy = F y$$

$$\int y \phi y dy = f y$$

$$\begin{aligned} \therefore \frac{\text{moving force}}{\rho} &= - \left(g \frac{ds}{dx} - u \frac{du}{dx} \right) \alpha F(s) \\ &\quad - \left(g s - \frac{1}{2} u^2 \right) \alpha \frac{ds}{dx} \cdot F' s + \alpha \frac{ds}{dx} g f' s \end{aligned}$$

$$\begin{aligned}
 &= - \left(g \frac{ds}{dx} - u \frac{du}{dx} \right) \alpha F s - \left(g s - \frac{1}{2} u^2 \right) \alpha \frac{ds}{dx} \phi s + \alpha s \frac{ds}{dx} g \phi s \\
 &= - \left(g \frac{ds}{dx} - u \frac{du}{dx} \right) \alpha F s + \frac{1}{2} \alpha u^2 \frac{ds}{dx} \phi s
 \end{aligned}$$

and mass moved

$$\begin{aligned}
 &= K \alpha \rho \\
 &= \alpha \rho F y = \alpha \rho F s
 \end{aligned}$$

therefore moving force $= -g \frac{ds}{dx} + u \frac{du}{dx} + \frac{1}{2} u^2 \cdot \frac{ds}{dx} \cdot \frac{\phi s}{F s} \quad (1)$

This is an expression which can be applied to different shaped channels with great facility.

Also, as in other cases,

$$\left(\frac{dx + d\alpha}{dt} \right)^2 = \left(u + \frac{du}{dx} \alpha \right)^2$$

$$\therefore \frac{d\alpha}{dt} = \frac{du}{dx} \alpha$$

and

$$\frac{du}{dx} = \frac{1}{\alpha} \frac{d\alpha}{dt} = -\frac{1}{K} \frac{dK}{dt}$$

Now

$$\begin{aligned}
 K &= F s \\
 \therefore \frac{dK}{dt} &= F s \cdot \frac{ds}{dt} \\
 &= \phi s \cdot \frac{ds}{dt} \\
 \therefore \frac{du}{dx} &= -\frac{\phi s}{F s} \cdot \frac{ds}{dt} \quad (2)
 \end{aligned}$$

By means of these equations the velocity and motion are discovered.

27. Ex. Let us take one example for the sake of illustration.

Suppose the canal to have the parabolic form, then

$$z = \sqrt{m y} = \phi y$$

and

$$F y = \frac{2}{3} \sqrt{m} \cdot y^{\frac{3}{2}} \quad \text{and} \quad \frac{\phi s}{F s} = \frac{3}{2} \frac{1}{s}$$

By the substitution of this value in equation (1), we obtain

$$\frac{du}{dt} = -g \frac{ds}{dx} + u \frac{du}{dx} + \frac{1}{2} u^2 \frac{ds}{dx} \frac{3}{2} \cdot \frac{1}{s}$$

or

$$\frac{2\pi}{\lambda} b c \cdot \cos \theta - \frac{2\pi}{\lambda} b^2 \sin \theta \cos \theta =$$

$$\frac{2\pi}{\lambda} a g \cos \theta - \frac{2\pi}{\lambda} b^2 \sin \theta \cos \theta - \frac{3}{4} \frac{b^2 \sin^2 \theta \frac{2\pi}{\lambda} a \cos \theta}{h + a \sin \theta}$$

therefore $ag - bc = 0$ by equating the large terms as in other cases.

And from equation (2)

$$\frac{du}{dx} = -\frac{3}{2s} \frac{ds}{dt}$$

$$\therefore (h + a \sin \theta) b \cos \theta = \frac{3}{2} a \cos \theta (c - b \sin \theta + \dots)$$

$$hb = \frac{3}{2} ac.$$

If we put for b this value in the former equation, we get

$$ag = \frac{3}{2} \frac{ac^2}{h}$$

$$c^2 = \frac{2}{3} gh$$

$$c = \sqrt{\frac{2}{3} gh}$$

or the square of the velocity in a parabolic vessel is $\frac{2}{3}$ of its value in a rectangular one.

28. We may readily deduce the more general results, viz.

$$ag - bc = 0 \quad (1)$$

$$b \cos \theta = a \cos \theta (c - b \sin \theta) \frac{\phi s}{F s}$$

$$b = ac \frac{\phi h}{F h} \quad (2)$$

$$\therefore ag = ac^2 \frac{\phi h}{F h}$$

$$c^2 = g \frac{F h}{\phi h}$$

$$= g \cdot \frac{\text{area of vertical section}}{\text{breadth at surface}}$$

29. The investigation supposes the curve a continuous curve, but, except in extreme cases, it will apply equally well to others. For instance, it will apply to the cases examined by Mr RUSSELL, viz. when the areas are trapeziums. We will apply the formula to one or two of these experiments of Mr RUSSELL, and then quit the subject.

In the channel M, the breadth is 12 inches, and the depth of the triangular part 4 inches: therefore area of triangular section equals 24 inches. And if we take the height to the centre of the wave as the height corresponding to h , we get for experiments xc, xciii, mentioned p. 444;

$$h = 6.21,$$

$$\text{area of parallelogram} = 26.52,$$

$$\text{therefore area of section} = 50.52$$

and

$$\begin{aligned} c^2 &= g \cdot \frac{50.52}{144} \text{ in feet} \\ &= \frac{32.2 \cdot (50.52)}{144} = \frac{(32.2) 50.5}{144} \text{ suppose} \\ c &= \frac{40.3}{12} = 3.3 \dots \end{aligned}$$

Experiment gives it 3.08.

The discrepancy is due entirely to our not knowing in *any* case the exact height (h) which gives $c^2 = g h$ for a square formed canal.

This discrepancy is not much greater than that between two waves of the same height in a rectangular channel.

By computing the velocity of the next waves given by Mr RUSSELL, we obtain

$$c = 3.45.$$

By observation c is equal to 3.50.

For the next and last set we obtain 3.56, which by observation is 3.86.

30. I propose, in the last place, to deduce a first approximation to motion in a channel of variable breadth, taking only that case for which the channel diminishes very slowly and uniformly.

Let z be the breadth at the point whose other variables are x and t .

$$\therefore z = m(l-x)$$

l being the whole length of the channel from the origin.

Adopting all the previous notation, we obtain

$$\begin{aligned} z s \alpha &= z + \delta z \quad s + \delta s \quad \alpha + \delta \alpha \\ \text{or} \quad \alpha s \frac{dz}{dt} + \alpha z \frac{ds}{dt} + z s \frac{d\alpha}{dt} &= 0 \\ \text{or} \quad \frac{1}{\alpha} \frac{d\alpha}{dt} &= -\frac{1}{z} \frac{dz}{dt} - \frac{1}{s} \frac{ds}{dt} \\ \therefore \frac{du}{dx} &= \frac{1}{z} m \frac{dx}{dt} - \frac{1}{s} \frac{ds}{dt} \quad (1) \end{aligned}$$

31. Now, we may find the variation in the height of the wave, by supposing its length to remain constant, an hypothesis which must be considered as merely approximative.

The volume of the wave will vary as

$$\int_0^z \int_0^\pi a \sin \frac{2\pi}{\lambda} (ct-x) dx dz$$

$$\text{or as} \quad a \lambda z + C$$

hence, if a' be the value of a at the origin,

$$a' \lambda \cdot l = a \lambda (l-x)$$

$$a = a' \cdot \frac{l}{l-x}$$

$$\therefore s = h + \frac{a' l}{l-x} \sin \theta$$

$$u = b \sin \theta.$$

By substituting these values in equation (1), we obtain

$$-\frac{2\pi}{\lambda} b \cos \theta + \sin \theta \frac{db}{dx} = \frac{mb \sin \theta}{m(l-x)}$$

$$-\frac{1}{h + \frac{a'l}{l-x} \sin \theta} \left\{ \frac{a'l}{(l-x)^2} \sin \theta - \frac{2\pi}{\lambda} \frac{a'l}{l-x} \cos \theta \right\} \frac{dx}{dt} + \frac{2\pi}{\lambda} \frac{a'cl}{l-x} \cos \theta \}$$

Equating coefficients of like functions of x , we obtain approximately

$$bh = \frac{a'cl}{l-x}$$

and

$$\frac{db}{dx} = \frac{b}{l-x}$$

$$\therefore \log b = \log \frac{C}{l-x}$$

$$\frac{b}{b'} = \frac{l}{l-x}$$

$$b = \frac{b'l}{l-x}$$

b' being the value of b at the origin.

Again,
$$\frac{du}{dt} = -g \frac{ds}{dx} + u \frac{du}{dx} + \&c.$$

$$\therefore -\frac{2\pi}{\lambda} bc \cos \theta - \frac{2\pi}{\lambda} b^2 \sin \theta \cos \theta + \sin^2 \theta b \frac{db}{dx}$$

$$= \frac{2\pi}{\lambda} \frac{a'gl}{l-x} \cos \theta - \frac{ga'l}{(l-x)^2} \sin \theta + \&c.:$$

or equating only those coefficients which belong to the large terms,

$$bc = \frac{a'gl}{l-x}.$$

But we have shewn that

$$bh = \frac{a'cl}{l-x}$$

$$\therefore \frac{c}{h} = \frac{g}{c}$$

or

$$c^2 = gh,$$

Thus it appears, that the velocity is not altered, whilst the height of the wave increases in harmonic progression. This result does not agree even roughly with Mr RUSSELL's experiments; the reason for which is, that his waves were of considerable length, so that the variation of the channel through the length of a single wave cannot be neglected.

To attempt the solution of the more general problem, would lead us into very complex analysis. It must consequently be reserved for another memoir.

SECTION III.—SOLITARY WAVE MOTION.

32. The subject for investigation in the ensuing section, is the transmission of a solitary wave. Waves of this kind are so generated, that, throughout the whole length of the wave, the velocity parallel to x is positive.

As to the vertical motion, it does not appear probable that any difference would be caused in it by the horizontal motion; we may then conceive all the circumstances to remain the same as before, with the exception that the whole wave has a transmission parallel to the axis of x . By art. 8, it appears that the only functions which will satisfy the conditions are,

$$u = b(e^{\alpha y} + e^{-\alpha y}) \sin \theta + \phi y,$$

$$v = -b(e^{\alpha y} - e^{-\alpha y}) \cos \theta,$$

θ being $= x - ct$.

We shall hereafter discuss the variation which these formulæ admit of, but there does not appear to be any other *form* capable of satisfying the necessary conditions.

We may remark that p is not now, as in the former case, a complete differential, except approximately; it becomes, then, a question in what manner to vary or increase the formulæ, &c., to render it so accurately.

This discussion will form a distinct branch of inquiry, into which I forbear to enter at present.

Let us return to our equations.

The condition to be satisfied is, that, when $t=0$ and $x = -\frac{\lambda}{4}$, u shall $=0$, and $v=0$: this gives

$$\phi y = e^{\alpha y} + e^{-\alpha y}$$

and we get

$$u = b(e^{\alpha y} + e^{-\alpha y})(1 + \sin \theta)$$

$$v = -b(e^{\alpha y} - e^{-\alpha y}) \cos \theta.$$

33. Lest it should be thought that, in the case before us, the assumption which we made in art. 8, that the form of the circular function is $\frac{\sin 2\pi}{\cos \lambda} \frac{x - ct}{\lambda}$, is inapplicable here, I offer the following demonstration of the point.

Take the most general form involving only one circular function:

$$u = (e^{\alpha y} + e^{-\alpha y})(\sin \alpha x f t - \cos \alpha x \phi t + C + c)$$

$$v = -(e^{\alpha y} - e^{-\alpha y})(\cos \alpha x f t + \sin \alpha x \phi t + H)$$

where the circumstance that $v=0$ when $y=0$ for all values of x , gives the form of the exponentials; and the relation $\frac{du}{dx} + \frac{dv}{dy} = 0$, gives that of the circular functions. C and H are supposed to be functions of t .

Now, when $t=0$ and $\alpha x = -\frac{\pi}{2}$, $v=0$, and $u=0$,

$$\therefore 0 = -fo + C_o + c; \quad \phi(o) = H_o$$

H_o being the value of H when $t=0$.

$$\text{Let} \quad z = h + (e^{\alpha z} - e^{-\alpha z}) (D + \cos \alpha x F t + \sin \alpha x \psi t)$$

$$\begin{aligned} \therefore \frac{dz}{dt} &= \alpha (e^{\alpha z} + e^{-\alpha z}) \frac{dz}{dt} (D + \cos \alpha x F t + \sin \alpha x \psi t) + (e^{\alpha z} - e^{-\alpha z}) \left(\frac{dD}{dt} + \dots \right) \\ &\quad - (e^{\alpha z} - e^{-\alpha z}) (\alpha \sin \alpha x F t - \alpha \cos \alpha x \psi t) \\ &\quad \times (e^{\alpha z} + e^{-\alpha z}) (\sin \alpha x f t - \cos \alpha x \phi t + C) \end{aligned}$$

hence we obtain, by putting for $\frac{dz}{dt}$ its value $(v)_{y=z}$

$$\begin{aligned} &(e^{\alpha z} - e^{-\alpha z}) (\cos \alpha x f t + \sin \alpha x \phi t + H) \\ &\times \{ \alpha (e^{\alpha z} + e^{-\alpha z}) (D + \cos \alpha x F t + \sin \alpha x \psi t) - 1 \} \\ &= (e^{\alpha z} - e^{-\alpha z}) \left(\frac{dD}{dt} + \cos \alpha x F t + \sin \alpha x \psi t \right) \\ &\quad - \alpha (e^{2\alpha z} - e^{-2\alpha z}) (\sin \alpha x F t - \cos \alpha x \psi t) \\ &\quad \times (\sin \alpha x f t - \cos \alpha x \phi t + C). \end{aligned}$$

Equate separately to zero the coefficients of $e^{\alpha z} - e^{-\alpha z}$, and of $e^{2\alpha z} - e^{-2\alpha z}$, and there results

$$\begin{aligned} &(\cos \alpha x f t + \sin \alpha x \phi t + H) (D + \cos \alpha x F t + \sin \alpha x \psi t) \\ &= -(\sin \alpha x F t - \cos \alpha x \psi t) (\sin \alpha x f t - \cos \alpha x \phi t + C) \end{aligned}$$

$$\text{and} \quad \cos \alpha x f t + \sin \alpha x \phi t + H = -\frac{dD}{dt} - \cos \alpha x F t - \sin \alpha x \psi t.$$

From the former equation, we obtain

$$\begin{aligned} &D (\cos \alpha x f t + \sin \alpha x \phi t) + \cos^2 \alpha x f t F t + \sin^2 \alpha x \phi t \psi t \\ &+ \sin \alpha x \cos \alpha x (f t \psi t + F t \phi t) + DH + H \cos \alpha x F t \\ &+ H \sin \alpha x \psi t = -C (\sin \alpha x F t - \cos \alpha x \psi t) \\ &- \sin^2 \alpha x F t f t - \cos^2 \alpha x \phi t \psi t + \sin \alpha x \cos \alpha x (F t \phi t + f t \psi t); \end{aligned}$$

in which, if we equate the coefficients of sines and cosines of αx , $2\alpha x$, &c., we get

$$D f t + H F t = C \psi t \dots \dots \dots (1),$$

$$D \phi t + H \psi t = -C F t \dots \dots \dots (2),$$

$$f t \psi t + F t \phi t = F t \phi t + f t \psi t \quad \text{an identity};$$

$$2DH + f t F t + \phi t \psi t = -F t f t - \phi t \psi t$$

$$\text{or} \quad DH + f t F t + \phi t \psi t = 0 \dots \dots \dots (3)$$

$$f t F t - \phi t \psi t = F t f t - \phi t \psi t \quad \text{an identity.}$$

From the second equation, we obtain, in like manner,

$$\frac{dD}{dt} = -H \dots \dots \dots (4)$$

$$ft = -F't \dots\dots\dots(5)$$

$$\phi t = -\psi' t \dots\dots\dots(6).$$

34. Let us next solve these six equations. To find C, we must combine (1), (2), and (3).

From (1) and (2),

$$C(\phi t \psi t + ft F t) = H(\phi t F t - \psi t ft) \quad (a)$$

and from (3),

$$-CDH = C(ft F t + \phi t \psi t)$$

$$\therefore -CD = (\phi t F t - \psi t ft) \quad (b)$$

To find H, we eliminate C by (1) and (2), and obtain

$$H\{(F t)^2 + (\psi t)^2\} + D(ft F t + \phi t \psi t) = 0 \quad (c)$$

Putting for H its value from (3), we get

$$(F t)^2 + (\psi t)^2 = D^2 \quad (d)$$

Also, by (5) and (6),

$$ft F t + \phi t \psi t = -(F t F' t + \psi t \psi' t) \quad (e)$$

$$= -\frac{1}{2} \frac{d}{dt} (\overline{F t}^2 + \overline{\psi t}^2)$$

$$= -\frac{1}{2} \frac{d}{dt} D^2 \text{ by } (d)$$

$$= -D \frac{dD}{dt} = DH$$

But by (3),

$$ft F t + \phi t \psi t = -DH$$

$$\therefore DH = 0, ft F t + \phi t \psi t = 0.$$

35. We may satisfy the equation $DH=0$ in two ways: 1. by making $D=0$, in which case, by virtue of equation (d), both Ft and ψt equal zero, or z is itself constant. This case corresponds to equilibrium, and we have nothing to do with it. 2. By making $H=0$; which hypothesis requires that D should be a constant, independent of the time, by equation (4). To prevent error, in consequence of a quantity equal to zero appearing in our equations, it will be desirable to write them down again, omitting H . We shall also omit t for the sake of brevity, and write f instead of ft , ϕ instead of ϕt , &c.

The equations are

$$Df = C\psi \dots\dots\dots(1)$$

$$D\phi = -Cf \dots\dots\dots(2)$$

$$fF + \phi\psi = 0 \dots\dots\dots(3)$$

$$f = -F' \dots\dots\dots(5)$$

$$\phi = -\psi' \dots\dots\dots(6)$$

The equation (3), combined with (4) and (5), gives

$$\begin{aligned} F^2 + \psi^2 &= a \text{ const}, \\ &= E^2 \text{ suppose:} \end{aligned}$$

$$\text{but} \quad F^2 + \psi^2 = \frac{D^2}{C^2} (f^2 + \phi^2) \quad \text{by (1) and (2);}$$

$$\therefore f^2 + \phi^2 = \frac{C^2 E^2}{D^2}$$

$$\text{or} \quad F^2 + \psi^2 = \frac{C^2 E^2}{D^2} \quad \text{by (5) and (6).}$$

If $F = E \sin \gamma$, ψ will $= E \cos \gamma$;

$$\therefore \left(\frac{d\gamma}{dt} \right)^2 = \frac{C^2}{D^2}.$$

36. Again, since

$$\begin{aligned} \psi &= \sqrt{E^2 - F^2} \\ \psi' &= \frac{-F F'}{\sqrt{E^2 - F^2}} \end{aligned}$$

by substituting which in the equation $F'^2 + \psi'^2 = \frac{C^2 E^2}{D^2}$ we get

$$F'^2 + \frac{F^2 F'^2}{E^2 - F^2} = \frac{C^2 E^2}{D^2}$$

or

$$\frac{E^2 F'^2}{E^2 - F^2} = \frac{C^2 E^2}{D^2}$$

$$\frac{F'^2}{E^2 - F^2} = \frac{C^2}{D^2}$$

$$\frac{F'}{\sqrt{E^2 - F^2}} = \frac{C}{D};$$

$$\therefore \sin^{-1} \frac{F}{E} = \frac{1}{D} \int C dt,$$

and

$$F = E \sin \left(\frac{1}{D} \int C dt \right).$$

Similar values may be obtained for the other quantities f , ϕ ..., and thus the number of arbitrary functions will be reduced to one, viz. C . This is the general solution of the problem.

For the present, however, we prefer the examination of the following particular case:

37. Let

$$F t = e^{-mt} (f \cos \alpha c t + g \sin \alpha c t + k)$$

$$\psi t = e^{-mt} (p \cos \alpha c t + q \sin \alpha c t + r)$$

$$\begin{aligned} \therefore f t &= m e^{-mt} (f \cos . + g \sin . + k) + \alpha c e^{-mt} (f \sin . - g \cos .) \\ &= e^{-mt} (\overline{m f - \alpha c g} . \cos . + \overline{m g + \alpha c f} . \sin . + m k) \end{aligned}$$

and

$$\begin{aligned} \phi t &= m e^{-mt} (p \cos . + q \sin . + r) + \alpha c e^{-mt} (p \sin . - q \cos .) \\ &= e^{-mt} \{ \overline{m p - \alpha c q} \cos . + \overline{m q + \alpha c p} \sin . + m r \} : \end{aligned}$$

by writing $\cos .$ for $\cos \alpha c t$ and so on for shortness.

And the equation $fF + \phi\psi = 0$ gives us

$$(f \cos. + g \sin. + k) (\overline{mf - acg} \cos. + \overline{mg + acf} \sin. + mk) \\ + (p \cos. + q \sin. + r) (\overline{mp - acq} \cos. + \overline{mq + acp} \sin. + mr) = 0.$$

Equating coefficients we get

$$f(mf - acg) + g(mg + acf) + p(mp - acq) \\ + q(mq + acp) + 2mk^2 + 2mr^2 = 0 \dots\dots\dots (1)$$

$$f(mf - acg) - g(mg + acf) + p(mp - acq) \\ - q(mq + acp) = 0 \dots\dots\dots (2)$$

$$f(mg + acf) + g(mf - acg) \\ + p(mq + acp) + q(mp - acq) = 0 \dots\dots\dots (3)$$

$$mkf + k(mf - acg) + mrp + r(mp - acq) = 0 \dots\dots\dots (4)$$

$$mkg + k(mg + acf) + mrq + r(mq + acp) = 0 \dots\dots\dots (5)$$

38. From combining (1) and (2), there arises the following equation :

$$f(mf - acg) + p(mp - acq) + m(k^2 + r^2) = 0 \dots\dots\dots (I.)$$

From equation (2) alone we get

$$m(f^2 + p^2 - g^2 - q^2) = 2ac(fg + pq) \dots\dots\dots (II.)$$

From (3) $2m(fg + pq) + ac(f^2 + p^2 - g^2 - q^2) = 0 \dots\dots\dots (III.)$

From (4) $2m(kf + rp) = ac(kg + rq) \dots\dots\dots (IV.)$

From (5) $2m(kg + rq) + ac(kf + pr) = 0 \dots\dots\dots (V.)$

If it be allowable to eliminate $f^2 + p^2 - g^2 - q^2$ between II. and III., we obtain

$$\frac{m}{2ac} = -\frac{ac}{2m}$$

or

$$m^2 + a^2c^2 = 0$$

$$\therefore m = 0, ac = 0;$$

and all the equations are satisfied without giving any other conditions. In fact, it is obvious this case is that in which $Ft, \psi t$ are constant.

Now, the only reason which can operate to prevent this elimination being effected, is the circumstance that one of the quantities $f^2 + p^2 - g^2 - q^2$ or $fg + pq$ is equal to zero.

And, by means of the same equations, it appears that *both* the quantities must equal zero ; or

$$f^2 + p^2 - g^2 - q^2 = 0 \quad (1')$$

$$fg + pq = 0 \quad (2')$$

But the first equation gives $m(f^2 + p^2 + k^2 + r^2) = 0$ by means of (2') ; which again, combined with (1'), gives

$$m(g^2 + q^2 + k^2 + r^2) = 0 \quad (3')$$

Either therefore $m = 0$, or g, q, k, r, f, p , are all separately equal to 0 ; but the latter condition cannot be true,

$$\therefore m = 0;$$

also equation (IV.) gives

$$k g + r q = 0 \quad (4');$$

and equation (V)

$$k f + p r = 0 \quad (5');$$

If k and r are not each equal to zero, we obtain by equations (4') and (5')

$$\frac{f}{g} = \frac{p}{q}$$

and since by (2')

$$f g = -p q,$$

$$\therefore \frac{f}{g} = -\frac{f g}{q^2} \text{ or } q^2 = -g^2$$

$$\therefore q = 0, g = 0.$$

But, in this case, by (1'),

$$f = 0, p = 0.$$

Hence our functions are reduced to

$$F = k, \psi = r.$$

This solution is one with which we have no concern; it belongs to a state of rest, or of uniform motion. If, however, one of the quantities, as k , is equal to zero, then $r p = 0$ and $r q = 0$; either, therefore, $r = 0$ or $p = 0, q = 0$. But if $p = 0, q = 0$ we have by (1') $f^2 = g^2$, and by (2') $f g = 0$:

$$\therefore f = 0, g = 0;$$

and we are reduced to the same state as before.

But if $r = 0$, the only equations to be satisfied are (1') and (2'), which are

$$\begin{aligned} f^2 + p^2 - g^2 - q^2 &= 0 \\ f g + p q &= 0; \end{aligned}$$

and thus we reduce our equations to the form

$$\begin{aligned} F t &= f \cos \alpha c t + g \sin \alpha c t \\ \psi t &= p \cos \alpha c t + q \sin \alpha c t \\ f t &= -\alpha c (g \cos \alpha c t - f \sin \alpha c t) \\ \phi t &= -\alpha c (q \cos \alpha c t - p \sin \alpha c t) \end{aligned}$$

39. Now, when $\alpha x = -\frac{\pi}{2}$ and $t = 0$, our assumed conditions are, that $u = 0, v = 0$; and since it has been proved that $H = 0$, it follows that ϕt must be equal to 0 when $t = 0$; hence we must have $q = 0$: wherefore we require to have either $f = 0$ or $g = 0$, in order to satisfy equation (2').

But

$$F^2 + \psi^2 = \text{a constant}$$

hence

$$(f \cos \alpha c t + g \sin \alpha c t)^2 + p^2 \cos^2 \alpha c t = \text{const}$$

$$\therefore \frac{f^2 - g^2}{2} + \frac{p^2}{2} = 0$$

$$f g = 0$$

consequently

$$p^2 = g^2 - f^2$$

and p^2 is a positive quantity, therefore g^2 cannot be equal to 0, hence $f = 0$;

and

$$p = \pm g = -g \text{ suppose.}$$

We are therefore reduced to precisely the old form, viz. :

$$\begin{aligned} u &= (e^{\alpha y} + e^{-\alpha y}) (-g \alpha c \sin \alpha x \cos \alpha c t + g \alpha c \cos \alpha x \sin \alpha c t + C + c) \\ &= (e^{\alpha y} + e^{-\alpha y}) (b \sin \alpha \cdot \overline{x - c t} + C + c) \end{aligned}$$

if $b = -g \alpha c$

$$\begin{aligned} v &= -(e^{\alpha y} - e^{-\alpha y}) (b \cos \alpha x \cos \alpha c t + b \sin \alpha x \sin \alpha c t) \\ &= -b (e^{\alpha y} - e^{-\alpha y}) \cos \alpha \cdot \overline{x - c t}. \end{aligned}$$

40. The corresponding value of z is

$$\begin{aligned} z &= h + (e^{\alpha z} - e^{-\alpha z}) (D + g \sin \alpha c t \cos \alpha x - g \cos \alpha c t \sin \alpha x) \\ &= h + (e^{\alpha z} - e^{-\alpha z}) \left(D + \frac{b}{\alpha c} \sin \alpha \cdot \overline{x - c t} \right) \end{aligned}$$

Now D is independent of t (35); if, therefore, we put $\alpha x = -\frac{\pi}{2}$, $t=0$, we get by hypothesis $z=h$.

But we obtain
$$z = h + (e^{\alpha h} - e^{-\alpha h}) \left(D - \frac{b}{\alpha c} \right)$$

hence
$$\frac{b}{\alpha c} = D$$

and
$$\begin{aligned} z &= h + (e^{\alpha z} - e^{-\alpha z}) \left(\frac{b}{\alpha c} + \frac{b}{\alpha c} \sin \theta \right) \\ &= h + \frac{b}{\alpha c} (e^{\alpha z} - e^{-\alpha z}) (1 + \sin \theta). \end{aligned}$$

41. Lastly, $C^2 \propto f^2 + \phi^2 \dots (35)$

$$\begin{aligned} &\propto p^2 \alpha^2 c^2 \sin^2 \alpha c t + g^2 \alpha^2 c^2 \cos^2 \alpha c t \\ &\propto p^2 \alpha^2 c^2 \quad \because \quad p^2 = g^2 \end{aligned}$$

$\therefore C$ is independent of t .

And by making $\alpha x = -\frac{\pi}{2}$ and $t=0$, we obtain $u=0$,

$$\therefore C + c = b$$

and
$$u = (e^{\alpha y} + e^{-\alpha y}) b (1 + \sin \theta).$$

42. Having thus obtained values of u , v , and z , it remains that we substitute them in the equations which determine the pressure. Now, in doing this, it must be borne in mind that the values of u , v , and z , are not correctly expressed by the above formulæ, and that consequently we must not expect accurately to satisfy the conditions of integrability of the function which expresses the differential of the pressure. Still whatever variation may be requisite in the above functions, to enable them accurately to satisfy all the conditions, it cannot be doubted that they hold true in the early part of the motion, as far as the large terms are concerned. If, then, in the process of finding the pressure, we take no notice of terms of the second order, our results will be a close approximation to the truth.

We proceed to the determination of the motion. All that remains for us to do, is to substitute the values of u , v , and z , in the equations of art. 8. We obtain

$$\begin{aligned}
 \frac{1}{\rho} \frac{dp}{dx} &= -b (e^{\alpha y} + e^{-\alpha y}) (\alpha \cos \theta) \left(\frac{dx}{dt} - c \right) + \alpha b^2 (e^{\alpha y} - e^{-\alpha y})^2 \cos \theta (1 + \sin \theta) \\
 &= -b \alpha \cos \theta (e^{\alpha y} + e^{-\alpha y}) \{ (b \overline{e^{\alpha y} + e^{-\alpha y}}) (1 + \sin \theta) - c \} \\
 &\quad + \alpha b^2 (e^{\alpha y} - e^{-\alpha y})^2 (1 + \sin \theta) \cos \theta \\
 &= -\alpha b^2 (1 + \sin \theta) \cos \theta \cdot 4 + c \alpha b \cos \theta (e^{\alpha y} + e^{-\alpha y}) \\
 \frac{1}{\rho} \frac{dp}{dy} &= -g + \left\{ -b \overline{e^{\alpha y} - e^{-\alpha y}} \sin \theta \alpha \left(\frac{dx}{dt} - c \right) + \alpha b \cos \theta (e^{\alpha y} + e^{-\alpha y}) \left(\frac{dy}{dt} \right) \right\} \\
 &= -g - \alpha b \sin \theta \{ b \overline{1 + \sin \theta} (e^{2\alpha y} - e^{-2\alpha y}) - c (e^{\alpha y} - e^{-\alpha y}) \} \\
 &\quad - \alpha b^2 \cos^2 \theta (e^{2\alpha y} - e^{-2\alpha y}) \\
 &= -g - \alpha b^2 \sin \theta (e^{2\alpha y} - e^{-2\alpha y}) - \alpha b^2 (e^{2\alpha y} - e^{-2\alpha y}) \\
 &\quad + \alpha b c \sin \theta (e^{\alpha y} - e^{-\alpha y}) \\
 \therefore \frac{dp}{\rho} &= -\alpha b^2 (4 \cos \theta + 2 \sin 2\theta) dx + c \alpha b \cos \theta (e^{\alpha y} + e^{-\alpha y}) dx \\
 &\quad - g dy - \alpha b^2 (1 + \sin \theta) (e^{2\alpha y} - e^{-2\alpha y}) dy + \alpha b c (e^{\alpha y} - e^{-\alpha y}) \sin \theta dy \\
 &\quad = c b d \cdot (e^{\alpha y} + e^{-\alpha y}) \sin \theta - g dy \\
 \frac{p}{\rho} &= b c (e^{\alpha y} + e^{-\alpha y}) \sin \theta - g y + P
 \end{aligned}$$

where P is a function of z .

Applying the formula $\frac{d\phi x z}{dx} + \frac{d\phi x z}{dz} \cdot \frac{dz}{dx} = 0$, which we proved, art. 9, we obtain

$$0 = b c \alpha (e^{\alpha z} + e^{-\alpha z}) \cos \theta + (-g + b c \alpha \sin \theta (e^{\alpha z} - e^{-\alpha z})) \frac{dz}{dx}$$

and from the value

$$z = h + \frac{b}{\alpha c} (e^{\alpha z} - e^{-\alpha z}) (1 + \sin \theta),$$

we obtain

$$\begin{aligned}
 \frac{dz}{dx} &= \frac{b}{c} (e^{\alpha z} - e^{-\alpha z}) \cos \theta + \frac{b}{c} (e^{\alpha z} + e^{-\alpha z}) (1 + \sin \theta) \\
 &= \frac{\frac{b}{c} (e^{\alpha z} - e^{-\alpha z}) \cos \theta}{1 - \frac{b}{c} (e^{\alpha z} + e^{-\alpha z}) (1 + \sin \theta)}
 \end{aligned}$$

By substituting this value in the above equation, it becomes

$$b c \alpha (e^{\alpha z} + e^{-\alpha z}) \cos \theta + \frac{\frac{b}{c} (e^{\alpha z} - e^{-\alpha z}) \cos \theta}{1 - \frac{b}{c} (e^{\alpha z} + e^{-\alpha z}) (1 + \sin \theta)} (-g + b c \alpha \sin \theta \overline{e^{\alpha z} - e^{-\alpha z}}) = 0$$

$$\begin{aligned}
& b c \alpha \cos \theta \left\{ 1 - \frac{b}{c} \overline{1 + \sin \theta} (e^{\alpha z} + e^{-\alpha z}) \right\} (e^{\alpha z} + e^{-\alpha z}) \\
&= \frac{b}{c} (e^{\alpha z} - e^{-\alpha z}) \cos \theta \left\{ g - b c \alpha \sin \theta (e^{\alpha z} - e^{-\alpha z}) \right\} \\
\therefore \quad & c^2 \alpha (e^{\alpha z} + e^{-\alpha z}) - c b \alpha \overline{1 + \sin \theta} (e^{\alpha z} + e^{-\alpha z})^2 \\
&= g (e^{\alpha z} - e^{-\alpha z}) - b c \alpha \sin \theta (e^{\alpha z} - e^{-\alpha z})^2 \\
c^2 \alpha (e^{\alpha h} + e^{-\alpha h}) &= g (e^{\alpha h} - e^{-\alpha h}) + b c \alpha (e^{\alpha h} - e^{-\alpha h})^2 \\
c^2 \alpha &= g \frac{e^{\alpha h} - e^{-\alpha h}}{e^{\alpha h} + e^{-\alpha h}} + b c \alpha \frac{(e^{\alpha h} - e^{-\alpha h})^2}{e^{\alpha h} + e^{-\alpha h}}.
\end{aligned}$$

If e be the semi-elevation, $e = \frac{b}{c \alpha} (e^{\alpha h} - e^{-\alpha h})$

$$\begin{aligned}
\therefore \quad c^2 \alpha \left(1 - e \alpha \frac{e^{\alpha h} - e^{-\alpha h}}{e^{\alpha h} + e^{-\alpha h}} \right) &= g \frac{e^{\alpha h} - e^{-\alpha h}}{e^{\alpha h} + e^{-\alpha h}} \\
c^2 &= \frac{g}{\alpha} \cdot \frac{e^{\alpha h} - e^{-\alpha h}}{e^{\alpha h} + e^{-\alpha h}} \div \left(1 - e \alpha \frac{e^{\alpha h} - e^{-\alpha h}}{e^{\alpha h} + e^{-\alpha h}} \right)
\end{aligned}$$

43. Thus we have obtained the velocity of transmission in a very simple form. As we have before pointed out, the function which properly represents the velocity of vibration is discontinuous; the *value* of this function at points of the fluid which are in actual motion cannot be correct, unless it lead to the conclusion that the motion ceases as soon as the wave has traversed a space equal to its length. This conclusion amounts to the fact which it is incumbent on us to prove, that all the superadded fluid, and no more, will pass onwards in the time occupied by the vibration.

Let Q be the volume of superadded fluid; R the volume of the portion which is carried forwards during the time of vibration.

$$\begin{aligned}
z &= h + \frac{b}{c \alpha} (e^{\alpha z} - e^{-\alpha z}) (1 + \sin \theta) \\
Q &= \int_{-\frac{\lambda}{4}}^{\frac{3\lambda}{4}} dx \frac{b}{c \alpha} (e^{\alpha z} - e^{-\alpha z}) (1 + \sin \theta) \\
e^{\alpha z} &= e^{\alpha h + \alpha m + \alpha m \sin \theta} \\
&= e^{\alpha h + m} \overline{1 + \alpha m \sin \theta} \\
\therefore \quad Q &= \int dx \frac{b}{c \alpha} \left\{ e^{\alpha h} - e^{-\alpha h} + \alpha m \sin \theta (e^{\alpha h} + e^{-\alpha h}) \right\} \overline{1 + \sin \theta} \\
&= \frac{b}{c \alpha} \int dx \left\{ e^{\alpha h} - e^{-\alpha h} + \frac{\alpha m}{2} (e^{\alpha h} + e^{-\alpha h}) + \dots \right\} \\
&= \frac{b \lambda}{c \alpha} \left\{ e^{\alpha h} - e^{-\alpha h} + \frac{\alpha m}{2} (e^{\alpha h} + e^{-\alpha h}) + \dots \right\}
\end{aligned}$$

$$\begin{aligned} R &= b f f d y d t (e^{\alpha y} + e^{-\alpha y}) (1 + \sin \theta) \\ &= \frac{b}{\alpha} f d t (e^{\alpha z} - e^{-\alpha z}) (1 + \sin \theta). \end{aligned}$$

Now x and t enter always together, and the limits $x=p$ $x=q$ correspond to $t = \frac{q}{c} + r$, $\frac{p}{c} + r$; therefore the two are equal, *i. e.* $Q=R$ accurately.

Consequently we deduce the following important conclusion :

“ That all the fluid which was elevated above the statical level has passed on with the wave, and no other fluid with it.”

Combining this with the fact that the velocity of the particles in all vertical sections through points at the nearer extremity of the wave is zero, and that no particle is moving backwards, it follows *that a wave will not be followed by another resulting from the displacement which it has caused amongst the particles.* Hence the functions u , v , z are discontinuous ones, having the values assigned to them when the function $x - ct$ lies between $-\frac{\lambda}{4}$ and $+\frac{3\lambda}{4}$, but being zero in all other cases.

It is not necessary to determine what function this is : for, although the derived functions of certain orders might differ widely from those deduced from the equivalent formulæ, yet the first derived functions of which alone we make use, representing either a velocity or a force, cannot be very different in the wave as expressed by its *value* from that expressed by the form ; they cannot, in fact, differ from those obtained in ordinary undulations, provided we conceive all the fluid in motion in the direction of the axis of x . We may conceive, however, a difference of pressure, and hence probably arises the fact that p is not in the present case a complete differential.

44. The formulæ we have deduced above will give us the velocity of transmission, provided we know the length of the wave.

Mr RUSSELL has not yet published his results as to the length of the wave, nor have I been able at present to deduce it to my satisfaction in terms of the depth of the fluid and the height of the wave. It would appear at first sight probable that the higher the wave for a given depth of fluid, the longer it will be. Yet, from Mr RUSSELL's conclusions, it appears the contrary to this is the case. I have appended a few results deduced from the value of λ , which Mr RUSSELL has favoured me with as the result of his present researches. I must, however, take the liberty of observing, that, although the first part of it is just what I should have expected, the latter part is by no means so. Had I chosen a formula empirically, as that which the nature of the case seems to require, it would have differed from RUSSELL's ; thus, for $2\pi h - 2x$, I should have conceived $2\pi h + \pi x$, or $2\pi \left(h + \frac{x}{2} \right)$, as the probable form. But I forbear any further remarks until I shall

have considered the subject with an especial view to the determination of the length of the wave.

45. The following results are all that I have *attempted* to obtain ; consequently they may be considered as a fair specimen of the whole.

The account of the experiments will be found in the Report of the British Association, vol. vi.

Wave 1.

$$\left. \begin{array}{l} h+2e=4.50 \\ 2e=.53 \end{array} \right\} \therefore h=3.97:$$

$$\text{and} \quad \frac{e^{\alpha h} - e^{-\alpha h}}{e^{\alpha h} + e^{-\alpha h}} = .78.$$

$$\begin{aligned} \text{Hence} \quad c^2 &= \frac{32.2 (.78)}{2\pi} \frac{(2\pi h - 4e)}{12} \div \{1 - e(2\pi - 4e)(.78)\} \\ &= \frac{32.2}{12} h (.9575) \div \left\{1 - \frac{e}{h} \frac{.78}{.9575}\right\} \\ &= \frac{(32.2)(3.97)(.9575)}{12} \div \left\{1 - \frac{(.53)(.39)}{(3.97)(.9575)}\right\} \\ &= \frac{7.785}{1 - \frac{(.53)(.39)}{3.97(.9575)}} \\ &= \frac{7.785}{.94563} \\ c &= 2.8693 = \frac{1}{.34853} \end{aligned}$$

therefore, giving in twelve seconds 35 feet instead of 40.

Wave 27.

$$\begin{aligned} h &= 1, \quad 2e = .30 \\ \alpha &= \frac{2\pi}{2\pi h - 2e} = \frac{1}{h - \frac{.15}{3.1416}} \\ &= \frac{3.1416}{3.1416 - (.15)} = \frac{3.1416}{2.9916} \\ &= 1.05 \\ \frac{e^{2\alpha h} - 1}{e^{2\alpha h} + 1} &= \frac{7.1666}{9.1666} = .7818 \\ e\alpha \cdot \frac{e^{2\alpha h} - 1}{e^{2\alpha h} + 1} &= .15(1.05) \frac{7.1666}{9.1666} = .12314 \end{aligned}$$

$$\begin{aligned}\therefore c^2 &= \frac{32.2}{12} \cdot 7818 \\ &= \frac{32.2 (.6515)}{8.7686} = 2.3924 \\ \therefore c &= 1.547\end{aligned}$$

hence, in eleven seconds, 17 feet ought to have been described instead of 20.

Wave 53.

$$\begin{aligned}h &= 7.04 \\ 2e &= .89 \\ \alpha &= \frac{1}{7.04} \\ \frac{c^2}{7.04} \left\{ 1 - \frac{.89}{14.08} \cdot 7616 \right\} &= \frac{32.2}{12} \cdot 7616 \\ c^2 \cdot \frac{.952}{7.04} &= \frac{32.2}{12} \cdot 7616 \\ c^2 &= \frac{(32.2) (.7616) (7.04)}{12 (.952)} \\ &= \frac{16.1 (.1904) (2.35)}{.476} \\ &= \frac{(16.1) (.0476) (2.35)}{.119} \\ &= 16 \text{ nearly} \\ c &= 4\end{aligned}$$

hence, in nine seconds, 36 feet ought to be described, whilst the result of observation is 40.

Wave in Mr RUSSELL's memoir, entitled "Researches in Hydrodynamics," Trans. Royal Soc. Edin. vol. xiv.

$$\begin{aligned}h &= 13, \quad 2e = 2.2 \\ \lambda &= 2\pi h - 4.4 \\ c^2 &= \frac{(32.2) \cdot 12}{2\pi} (2\pi h - 4.4) P \div \left(1 - \frac{2\pi \cdot (1.1)}{2\pi h - 4.4} P \right) \\ 2\alpha h &= \frac{4\pi h}{2\pi h - 4.4} = \frac{2}{1 - \frac{2.2}{\pi h}} \\ &= \frac{2}{1 - \frac{2.2}{(3.1416)(13)}} = 2.1061 \\ P &= \frac{e^{2\alpha} - 1}{e^{2\alpha} + 1} = \frac{8.2155 - 1}{8.2155 + 1} = \frac{7.2155}{9.2155} = .78297\end{aligned}$$

$$c^2 = \frac{(32.2) 12 (12.3032) (.78297)}{.928703}$$

$$\text{in feet} \quad c^2 = \frac{(32.2) (12.3032) (.78297)}{12 (.928703)}$$

$$c = 5.275$$

remarkably near the truth.

These results, however, are of considerable importance, from the circumstance that they all err in defect; and it appears that the error is nearly of the same magnitude in all, as compared with the whole space, viz. about one-eighth. Whether this can be accounted for by any slight error in the formulæ, or in the value of λ , I leave it to subsequent investigations to determine.

EDINBURGH, *March* 12. 1839.