On Convergence of Series

Author(s): W. N. Roseveare

Source: *The Mathematical Gazette*, Vol. 3, No. 54 (Dec., 1905), pp. 246-250

Published by: The Mathematical Association

Stable URL: <http://www.jstor.org/stable/3603864>

Accessed: 16-06-2016 06:19 UTC

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at
<http://about.jstor.org/terms>

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

The Mathematical Association is collaborating with JSTOR to digitize, preserve and extend access to *The Mathematical Gazette*

III. ON CONVERGENCE OF SERIES.

HAVING established the expansion of functions without any reference to tests of convergence, we are able to enunciate the following simple test for the convergence of any series that may arise.

(We confine our attention to a series of which all the terms that we need to consider have the same sign.)

u_n being the n term of a series,

If $\frac{u_n}{u_{n-1}}$ can be expanded in (integral or fractional) powers of $1/n$, none of the terms after n^{-1} affect the convergence.

If $\frac{u_n}{u_{n-1}} \equiv 1 - \frac{a}{n} \dots$ where a is greater than 1, the series is convergent.

If $\frac{u_n}{u_{n-1}} \equiv 1 - \frac{1}{n} \dots$, the series is divergent.

1. The following theorem is well known: If Σu_i is a series, having $\frac{u_i}{u_{i-1}} \equiv \rho_i$, and Σv_i is another series having $\frac{v_i}{v_{i-1}} \equiv \sigma_i$, then if $\rho_i < \sigma_i$ for all values of i , and Σv is convergent, Σu is convergent; and if $\rho_i > \sigma_i$ and Σv is divergent, Σu is divergent.

For $\sum_{n+1}^N u_i = u_n [\rho_{n+1} + \rho_{n+1}\rho_{n+2} + \dots + \rho_{n+1}\rho_{n+2} \dots \rho_N]$,
and $\sum_{n+1}^N v_i = v_n [\sigma_{n+1} + \sigma_{n+1}\sigma_{n+2} + \dots + \sigma_{n+1}\sigma_{n+2} \dots \sigma_N]$.

If, therefore, every $\rho <$ the corresponding σ , $\Sigma u < \frac{u_n}{v_n} \Sigma v$;

$\therefore$ if Σv is finite when N is infinite, so is Σu .

That is, Σu is convergent if Σv is.

Similarly, if every $\rho >$ the corresponding σ , and Σv is divergent, so is Σu .

Thus, if we note the values of ρ for known series we can determine the convergence of unknown series.

2. If u_i is any 'derived' function $f'(i)$ which increases or diminishes with increasing i ,

$$fi - f(i-1) < \left| \frac{f''i}{f'(i-1)} \right|, \text{ and } f(i+1) - fi < \left| \frac{f''(i+1)}{f'i} \right|;$$

$$\therefore f'i < \left| \frac{f(i+1) - fi}{fi - f(i-1)} \right|; \therefore \sum_{n+1}^N f'i < \left| \frac{f(N+1) - f(n+1)}{fN - fn} \right|.$$

If, therefore, fN is infinite when N is indefinitely increased $\Sigma f'i$ is divergent, but if fN is finite, the series is convergent.

3. *Examples of known series.*

A. If $fx \equiv x^p$, then $f'x \equiv px^{p-1}$;

$$\therefore \sum_{n+1}^N i^{p-1} < \frac{1}{p} \left| \frac{(N+1)^p - (n+1)^p}{N^p - n^p} \right|.$$

If p is negative, the series is convergent, because N^p is zero; and $\rho_n \equiv \left(1 - \frac{1}{n}\right)^{-p+1}$, which $> 1 - \frac{1-p}{n}$;

$\therefore$ any series for which $\rho_n \leq 1 - \frac{1+\alpha}{n}$ where α is positive, is convergent.

B. If $p=0$, $\frac{N^p}{p}$ becomes indeterminate; the form $\frac{1}{0}(N^0-1)$ suggests the logarithm; but the series in this form is better investigated afresh.

Thus if $fx \equiv \log x$, $f'x \equiv \frac{1}{x}$,

$$\text{for } \log x - \log(x-1) = -\log\left(1 - \frac{1}{x}\right) < \left| \frac{\frac{1}{x-1}}{\frac{1}{x}} \right|;$$

$$\therefore \sum_{n+1}^N \frac{1}{i} < \left| \frac{\log(N+1) - \log(n+1)}{\log N - \log n} \right|.$$

$\therefore \sum \frac{1}{i}$ is (logarithmically) infinite.

Also, $\rho_n = 1 - \frac{1}{n}$.

Hence any series for which $\rho_n \leq 1 - \frac{1}{n}$ is divergent.

We have now to see for what values of α *infinitesimally* greater than 0, $1 - \frac{1+\alpha}{n}$ belongs to a convergent series, e.g. $\alpha = n^{-\beta}$ or $\alpha = (\log n)^{-\beta}$, where β is positive.

C. For this purpose it is natural to assume fN infinite in some smaller degree. Let fx be $(\log x)^p$.

Now $(\log x)^p - (\log y)^p < p(\log x - \log y) \left| \frac{(\log x)^{p-1}}{(\log y)^{p-1}} \right|$

$$< p(x-y) \left| \frac{\frac{1}{y}}{\frac{1}{x}} \right| \left| \frac{(\log x)^{p-1}}{(\log y)^{p-1}} \right|.$$

Hence, if $(p-1)$ is negative,

$$(\log x)^p - (\log y)^p < p(x-y) \left| \frac{\frac{1}{y}(\log y)^{p-1}}{\frac{1}{x}(\log x)^{p-1}} \right|;$$

$$\therefore f'x \text{ is in this case } p \cdot \frac{1}{x}(\log x)^{p-1};$$

$$\therefore \sum_{n+1}^N \frac{1}{i(\log i)^{p+1}} < \frac{1}{-p} \left| \frac{(\log \overline{N+1})^{-p} - (\log \overline{n+1})^{-p}}{(\log N)^{-p} - (\log n)^{-p}} \right|,$$

$\therefore$ the series is convergent if $p > 0$, but divergent if p is a negative fraction.

Now, if $p > 0$,

$$\rho_n = \frac{n-1}{n} \cdot \left(\frac{\log \overline{n-1}}{\log n} \right)^{p+1}, \text{ which } > \left(1 - \frac{1}{n} \right) \left(1 - \frac{1}{n-1 \log n} \right)^{p+1},$$

$$\text{which } > \left(1 - \frac{1}{n} \right) \left(1 - \frac{p+1}{n-1 \log n} \right), \text{ that is } 1 - \frac{1 + \frac{p+1}{\log n}}{n};$$

$\therefore$ any series for which $\alpha \geq \frac{p+1}{\log n}$ where p is positive, is convergent.

D. If $p=0$, $\frac{1}{p} \{(\log N)^p - (\log n)^p\}$ becomes indeterminate, but suggests the logarithm, and we find

$$\sum_{n+1}^N \frac{1}{i \log i} < \left| \frac{\log \log(N+1) - \log \log(n+1)}{\log \log N - \log \log n} \right|;$$

$\therefore$ the series is divergent.

$$\text{Also} \quad \rho_n = \frac{n-1}{n} \cdot \frac{\log \overline{n-1}}{\log n},$$

$$\text{which } < \left(1 - \frac{1}{n} \right) \left(1 - \frac{1}{n \log n} \right) < 1 - \frac{1 + \left(1 - \frac{1}{n} \right) \frac{1}{\log n}}{n}.$$

$\therefore$ any series for which $\alpha < \left(1 - \frac{1}{n} \right) \frac{1}{\log n}$ is divergent.

Hence, if α is of the order $n^{-\beta}$ where β is positive, the series is divergent.

Hence the rule at the head of this article.

5. This test covers all ordinary cases; but the above investigation continued on similar lines by taking for fx

$$(\log \log x)^p, (\log \log \log x)^p, \text{ etc.,}$$

leads us to the closer test:

If ρ_n expanded in descending order of magnitude is, or is greater than,

$$1 - \frac{1}{n} - \frac{1}{n \log n} - \frac{1}{n \cdot \log n \cdot \log \log n} \dots,$$

the series is divergent: if ρ_n is less than this it is convergent.

6. *Examples.* (From Chrystal, Part II., Exercise VIII., pp. 158, 159.)

$$\begin{aligned} \text{Ex. 5. } u_n &= \frac{1}{\sqrt{n^2-n} \{ \sqrt{n} - \sqrt{n-1} \}}, \\ \rho_n &= \sqrt{\frac{n^2-3n+2}{n^2-n}} \cdot \frac{\sqrt{n-1} - \sqrt{n-2}}{\sqrt{n} - \sqrt{n-1}} \\ &= \left(1 - \frac{3}{n} + \frac{2}{n^2}\right)^{\frac{1}{2}} \left(1 - \frac{1}{n}\right)^{-\frac{1}{2}} \left\{ \left(1 - \frac{1}{n}\right)^{\frac{1}{2}} - \left(1 - \frac{2}{n}\right)^{\frac{1}{2}} \right\} \left\{ 1 - \left(1 - \frac{1}{n}\right)^{\frac{1}{2}} \right\}^{-1} \\ &= \left(1 - \frac{3}{2n} \dots\right) \left(1 + \frac{1}{2n} \dots\right) \\ &\quad \times \left\{ 1 - \frac{1}{2n} - \frac{1}{8n^2} - 1 + \frac{1}{n} + \frac{1}{2n^2} \dots \right\} \left\{ \frac{1}{2n} + \frac{1}{8n^2} \right\}^{-1} \\ &= \left(1 - \frac{1}{n} \dots\right) \left\{ 1 + \frac{3}{4n} \dots \right\} \left\{ 1 - \frac{1}{4n} \dots \right\} \\ &= 1 - \frac{1}{2n} \dots; \therefore \text{divergent.} \end{aligned}$$

$$\text{Ex. 12. } u_n = \frac{1}{n^a} \sum_1^n \frac{1}{i^a};$$

$$\begin{aligned} \therefore \rho_n &= \left(\frac{n-1}{n}\right)^a \left\{ 1 + \frac{1}{n^a S_{n-1}} \right\}, \text{ where } S_{n-1} \text{ is written for } \sum_1^{n-1} \frac{1}{i^a} \\ &= \left(1 - \frac{a}{n} \dots\right) \left(1 + \frac{1}{n^a S_{n-1}}\right). \text{ This } > 1 - \frac{a}{n}; \\ &\therefore \text{ if } a \leq 1, \text{ the series is divergent;} \end{aligned}$$

but if $a > 1$, we have in the article above,

$$S_{n-1} < 1 + \frac{1}{a-1} \left| \frac{2^{-a+1} - n^{-a+1}}{1^{-a+1} - (n-1)^{-a+1}} \right|;$$

$$\therefore n^a S_{n-1} < \text{const. } n^a - \frac{1}{a-1} \left| n \left(1 - \frac{1}{n} \right)^{-a+1} \right|;$$

$\therefore \frac{1}{n^a S_{n-1}} = \frac{\text{const.}}{n^a} + \dots$, which does not affect the convergence.

$\therefore \rho_n = 1 - \frac{a}{n} \dots$, and the series is convergent.

$$7. \text{ If } f(x_i) - f(x_{i-1}) < (x_i - x_{i-1}) \left| \frac{f'x_i}{f'x_{i-1}} \right|,$$

$$\text{Then } \sum_{n+1}^N (x_i - x_{i-1}) f'x_i > f x_N - f x_n,$$

$$\text{and } \sum_{n+1}^N (x_{i+1} - x_i) f'x_i < f x_{N+1} - f x_{n+1}.$$

Cauchy's Condensation Test is the particular case of this when $x_i = k^i$ and $k > 1$. W. N. ROSEVEARE.

REVIEWS.

Vorlesungen über Technische Mechanik. By Dr. A. FÖPPL. Vol. i. **Einführung in der Mechanik.** 3rd ed. pp. 424. 10 m. Vol. iii. **Festigkeitslehre.** 3rd ed. pp. 434. 12 m. (Teubner, 1905.)

In the first volume of this work Professor Föppl's object is to introduce a student who is familiar with the experimental facts of mechanics, to the mathematical treatment of the subject. A working knowledge of the *meaning* of such symbols as $\frac{ds}{dt}$ and $\int Pds$ is assumed, though little analytical skill is demanded of the reader.

The feature which distinguishes Professor Föppl's treatise from most English works on the same subject is the use of vectorial algebra and not merely vector addition. Both in these volumes and in his Introduction to Maxwell's Theory of Electricity, Professor Föppl constantly employs a vectorial notation and vectorial methods. He adopts the convention that, for scalar multiplication $i^2 = j^2 = k^2 = +1$.

Vector multiplication has at present received little attention from teachers of elementary mathematics in this country. This is probably due in part to the unfortunate controversy as to notation. [See Professor Henrici's communication to the British Association. *B.A. Report*, 1903, p. 55.]

But the use of almost any vector notation is of value as an aid to the memory and as directing attention to a physical quantity in itself instead of merely to its components along the axes.

An analogy is furnished by determinants. Who does not find it easier to recollect the result of eliminating x , y , and z from three equations (1), $a_1x + b_1y + c_1z = 0$, (2), and (3), in the form of a determinant?

To illustrate in a similar manner the use of vector multiplication; suppose a system of forces $R_1 \dots R_n$ acting at points in space $r_1 \dots r_n$.