ON THE ASYMPTOTIC APPROXIMATION TO INTEGRAL FUNCTIONS OF ZERO ORDER

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(*Communicated by* Rev. E. W. BARNES.)

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1. In spite of the considerable attention which the theory of integral functions of finite order has received for the last fifteen years, the theory of the functions of zero order has been somewhat neglected. Analysis which applies to the theory of finite order usually breaks down in the case of zero order; moreover, the greater part of the few general theorems which exist in the latter theory are particular cases of general theorems applicable to all integral functions.

In his first memoir† on integral functions, M. Hadamard obtained an upper limit for the modulus of the general integral function defined by a Taylor series,‡ and he has given in a subsequent memoir§ a new means of finding such an upper limit. The latter method gives an approximation T for $M(r)$ [the maximum modulus of $|F(z)|$ on the circle $|z| = r$], such that $[M(r)]^{1+\epsilon} > T > [M(r)]^{1-\epsilon}$, when r is sufficiently large, where ϵ is as small as we please. This method applies to the case of functions of zero order.

M. Le Roy|| has given a method by which we can obtain asymptotic expressions for certain functions defined by a Taylor series with real and positive coefficients, the variable also being supposed real and positive. Among these functions are certain functions of zero order, *e.g.*, the function

$$\sum_0^\infty e^{-n^p} z^n \quad (1 < p < 2).$$

The asymptotic expressions of M. Le Roy for these functions afford a

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* The paper has been practically rewritten.

† Memoir crowned by the French Academy, *Jour. de Math.* (*Liouville*), t. IX., 1893.

‡ Hadamard also obtained other general results which hold in the case of zero order. [I] shall, however, only refer to such results as are connected with the subject of this paper.

§ *Bull. de la Soc. Math.*, t. XXIV., p. 186.

|| *Bull. des Sciences Math.*, 2me sér., t. XXIV.

nearer approximation for $M(r)$ than those obtained by the method of M. Hadamard.

M. Lindelöf* has considered a similar class of functions and obtains similar results.

M. Maillet† has extended as follows the definition of *order* in the case of functions of zero order.

Let $$e_{k+1}(m) = a_1^{a_2^{\cdot^{\cdot^{\cdot^{a_{k+1}^{m}}}}}},$$

where $$a_1 = a_2 = \ldots = e.$$

Let $$E(x, k, \rho) = \sum_{m=0}^{\infty} \frac{x^m}{[e_k(m)]^{m/\rho}},$$

and let $M(r)$ be the maximum modulus of a function $f(z)$ on the circle $|z| = r$.

Then, if $M(r) < E(r, k, \rho+\epsilon)$, for all values of r, and if

$$M(r) = E(r, k, \rho-\epsilon_1)$$

for values of r as great as we please, where ϵ and ϵ_1 tend to zero as r tends to infinity, then we say that $f(z)$ is of index k, and of order $(0, k, \rho)$.

He has established theorems concerning the relations between the modulus of the function and the coefficients of the Taylor series, when these coefficients approximate to the form of those of $E(x, k, \rho)$. He also defines "irregular growth," and establishes theorems concerning functions with this property.

The sole particular functions of zero order (the argument being supposed complex) which have been considered in detail, are the function $\prod_{n=0}^{\infty}(1+q^n x)$ and its generalisations

$$\prod_{n=1}^{\infty}(1+q^n x)^{n^{\nu}} \quad \text{and} \quad \prod_{n_1, n_2, \ldots = 0}^{\infty}[1+q_1^{n_1} q_2^{n_2} \cdots q_s^{n_s} x].$$

M. Mellin‡ has given formulæ for the logarithms of these functions. In particular, his formula for $F(x) = \log \Pi(1+q^n x)$ provides an asymptotic expansion for $\log F(x)$ in the case when q is real. We shall give this result in § 12.

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* *Acta Soc. Fenn.*, t. XXXI.

† *Journal de l'Ecole Polytechnique*, 1904 and 1905 ; *Jour. de Math.* (*Liouville*), t. x., 1903–4.

‡ *Acta Soc. Fenn.*, t. XXIX., p. 14, formulæ (23), (25).

This function $F(x)$ has also been considered by Dr. Barnes* and Mr. Hardy.†

Mr. Hardy obtains the formula (in our notation)

$$F(x) = \frac{H}{\sqrt{a}} e^{(\log a)^2/\log(q^{-1})} \sin\left(\frac{\pi \log a}{\log(q^{-1})}\right),$$

where $-a = x = -re^{\iota\phi}$, $-\delta < \phi < \delta$, where δ is a small positive number, $\log a = \log r + \iota\phi$, and $K_1 < H < K_2$, where K_1, K_2 are constants. This formula gives an approximation for $F(x)$ near the line of zeros. The part of the plane for which this formula holds includes the part of the plane from which x is supposed excluded in the asymptotic expansion of M. Mellin (cf. § 12).

Dr. Barnes‡ has shown that in the case of certain functions of finite non-zero order we can obtain a complete asymptotic expansion of the logarithm of the function, *i.e.*, an expansion proceeding in descending powers of z. The analysis, however, breaks down for zero order. Moreover Dr. Barnes concludes§ that the complete asymptotic expansion of $\log F(z)$ in the case of functions $F(z)$, the modulus of whose n-th zero is a function of n of high order in n, is impossible without the introduction of functions at present unknown in analysis.

It is the object of the present paper to show that the functions of zero order have simple and characteristic properties, which do not generally hold for functions of finite order, that they should be studied by different methods than those applicable to other integral functions, and that the arithmetical method is specially applicable to their theory.

I wish to acknowledge my indebtedness to Dr. Barnes for his kind assistance and advice. My thanks are also due to one of the referees, who has given me a number of suggestions with reference to arrangement and choice of expression.

2. The most general integral function of zero order is

$$Fz = C \prod_1^\infty (1 + z/a_s),$$

where $a_1, a_2, \ldots, a_s, \ldots$ is the most general sequence of numbers arranged in

* *Phil. Trans. Roy. Soc.* (A), Vol. 199, pp. 411-500; *Camb. Phil. Trans.*, Vol. XIX., pp. 322-355.

† *Quarterly Journal*, 1905.

‡ *Loc. cit.* and *Proc. London Math. Soc.*, Ser. 2, Vol. 3, Part 4.

§ *Phil. Trans.* and *Camb. Phil. Trans.*, *loc. cit.*

order of increasing moduli, subject to the condition that, if λ be any positive number, then $\sum_1^\infty |a_s|^{-\lambda}$ is convergent, however near to zero λ may be.

Let a_n be written for $|a_n|$, and let $\log a_n/\log n = \phi(n)$.

Then $$a_n = n^{\phi(n)}.\text{*}$$

Now we must have, for a function of zero order,

$$\operatorname*{Lt}_{n=\infty} \phi(n) = \infty.$$

For suppose this were not the case; then we should have a sequence $n_1, n_2, \ldots, n_r, \ldots$ of values of n, for which $\phi(n) < h$, when h is some finite number.

The series $\sum_1^\infty \frac{1}{a_n^{1/2h}}$ must converge. But the n_r-th term $> \frac{1}{n_r^{h/2h}} > \frac{1}{n_r^{\frac{1}{2}}}$, and therefore, since the terms of the series do not increase, the sum of the first n_r terms $> n_r \frac{1}{n_r^{\frac{1}{2}}} > n_r^{\frac{1}{2}}$. Thus, as n_r can be taken as large as we please, the series $\Sigma 1/a_n^{1/2h}$ diverges, which is contrary to our hypothesis.

Hence $$\operatorname*{Lt}_{n=\infty} \phi(n) = \infty.$$

In the case of functions of zero order, defined by a product form, which arise naturally, the moduli $a_1, a_2, \ldots, a_n, \ldots$ of the zeros will increase in a regular manner (the word "regular" being used in a rough sense), and, since $a_n = n^{\phi(n)}$, and $\phi(n)$ ultimately becomes infinite, in the natural cases $\phi(n)$ will increase uniformly after a certain value of n, *e.g.*,

$$a_n = n^{\log n}, \qquad a_n = e^n = n^{n/\log n}.$$

This, of course, only applies to functions which arise from the consideration of the product form. If a function of zero order be defined by a Taylor series, then, however regularly the coefficients of this series may increase, there seems no legitimate presumption that the moduli of the zeros of the function increase at all regularly.

In the present paper we shall chiefly be concerned with functions which are such that $\phi(n)$ increases uniformly with n, after some finite value of n, the word "increasing" always being understood to include the case "not decreasing."

Functions of this class we shall call "functions of standard type."

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* The standard function of finite order ρ ($\rho < 1$) is $\prod_1^\infty (1 + zs^{-1/\rho})$. When $\rho = 0$ this form does not exist, and we introduce the function $\phi(n)$ to replace, to some extent, the constant $1/\rho$.

3. We shall use the following notation:—

$$F(z) = \prod_{s=1}^{\infty}\left(1+\frac{z}{a_s}\right), \quad |a_s| = a_s = s^{\phi(s)} \quad (s > 1), \quad |z| = r.$$

Let n be the integer such that $a_n \leqslant |z| < a_{n+1}$. Then n depends uniquely on r. We shall always use the symbol n with this meaning.

We have

$$\log F(z) = \log\left[\prod_{s=1}^{n}\left(\frac{z}{a_s}\right)\right]+\log\left[\prod_{s=n+2}^{\infty}\left(1+\frac{z}{a_s}\right)\right]$$

$$+\log\left[\prod_{s=1}^{n-1}\left(1+\frac{a_s}{z}\right)\right]+\log\left(1+\frac{z}{a_{n+1}}\right)+\log\left(1+\frac{a_n}{z}\right), \quad (1)$$

where n has its special meaning.

We call the first three terms of the right hand of (1), respectively P, R, S. We also denote the real parts of P, R, S, by $\overline{P}$, $\overline{R}$, $\overline{S}$.

Then $\overline{P} = n\log r - \sum_{s=1}^{n}\log a_s, \qquad \overline{R} = \sum_{s=2}^{\infty}\log\left|1+\frac{z}{a_{n+s}}\right|,$

$$\overline{S} = \sum_{s=1}^{n-1}\log\left|1+\frac{a_s}{z}\right|.$$

Throughout this paper we shall be primarily concerned with $F(z)$, and not with $\log F(z)$, although most of our results are stated in terms of the latter function. We shall, then, not concern ourselves with the proper multiple of $2\pi\iota$ which occurs in equations like (1).

When we deduce an asymptotic expression for $F(z)$ from a corresponding expression for $\log F(z)$ the imaginary part of the latter expression yields a factor $e^{\psi\iota}$ in $F(z)$, where we may suppose $|\psi| \leqslant \pi$.

We shall then regard all possible values of $\log F(z)$ (for the same z) as equivalent, and we shall regard the imaginary part of $\log F(z)$, however it may be expressed, as a finite term in the expression for $\log F(z)$.

We may then re-write (1), after separating the right-hand side into its real and imaginary parts,

$$\log F(z) = \overline{P}+\overline{R}+\overline{S}+\log\left|1+\frac{z}{a_{n+1}}\right|+\log\left|1+\frac{a_n}{z}\right|+\iota\gamma_z, \quad (2)$$

where we suppose that $|\gamma_z| \leqslant \pi$.

4. We shall establish the following general theorem:—

Let $F(z) = C\,.\prod_{s=1}^{\infty}(1+z/a_s)$ be any integral function of zero order, where $a_1, a_2, \ldots$ are arranged in order of non-decreasing moduli. Let $M(r)$, $m(r)$ denote the maximum and minimum moduli of $F(z)$ on the circle $|z| = r$.

Then there exists a sequence of circles $|z| = r$, $r = r_1, r_2, \ldots$, $r_1 < r_2 < \ldots$, $\operatorname{Lt}_{s=\infty} r_s = \infty$, with the following properties.

If any positive number ϵ be assigned, however small, there is an integer μ depending on ϵ, such that, when $s > \mu$, on the circle $|z| = r_s$ we have

$$|F(z)| = \exp[\bar{P}\{1+\eta(z)\}],$$

where

$$\bar{P} = n \log r - \sum_{s=1}^{n} \log a_s,$$

n having its special meaning, and where $|\eta(z)| < \frac{1}{2}\epsilon$.

Moreover, the sequence $r_1, r_2, \ldots$ depends only on $|a_1|, |a_2|, \ldots$, and is independent of the arguments of the zeros.

Hence, if $F_1(z)$ be any integral function of zero order, the sequence of the moduli of whose zeros is the same as the corresponding sequence for $F(z)$, and for which the coefficient of z^0 in the Taylor series is C, then

$$|F_1(z)| = \exp[\bar{P}\{1+\eta_1(z)\}],$$

where $|\eta_1(z)| < \frac{1}{2}\epsilon$.

Again, when $s > \mu$, we have

$$m(r_s) > [M(r_s)]^{1-\epsilon}, \quad m_1(r_s) > [M_1(r_s)]^{1-\epsilon}, \quad m(r_s) > [M_1(r_s)]^{1-\epsilon},$$

$$m_1(r_s) > [M(r_s)]^{1-\epsilon}.$$

We shall require the two following lemmas:—

LEMMA I.—If we are given in the x-plane a circle with its centre at the origin and with radius l, and are given any m points $a_1, a_2, \ldots, a_m$ in the plane, each of which may be within, without, or on the boundary of the circle, then there exists a concentric circle of radius $\leqslant l$, such that for every point x on it,

$$\left| \prod_{s=1}^{m} (x-a_s) \right| > (l/2e)^m.$$

Moreover, such a circle can be found whose position (*i.e.*, whose radius) depends only on $|a_1| \ldots |a_m|$, and not on the arguments of the a's.

If $\quad |a_s| = a_s, \quad |x| = \rho, \quad \left| \prod_1^m (x-a_s) \right| \geqslant \left| \prod_1^m (\rho-a_s) \right|,$

whatever be the arguments of the a's. It is therefore sufficient to prove that, if $a_1, a_2, \ldots, a_m$ be any m points on the positive real axis, we can find a point ρ on this half axis such that $\rho < l$, and

$$\left| \prod_1^m (\rho - a_s) \right| \geqslant (l/2e)^m.$$

Let a be any point on the positive real axis.

Consider $$I_a \equiv \int_0^l \log | x - a | \, dx.$$

If $a \geqslant l$, $\quad I_a \geqslant \int_0^l \log (l - x) \, dx$ (algebraically)

$$\geqslant \int_0^l \log x \, dx \geqslant l \log l - l \geqslant l \log (l/e) > l \log (l/2e). \tag{1}$$

Next let $a < l$; then

$$\begin{aligned} I_a &= \int_0^a \log (a - x) \, dx + \int_a^l \log (x - a) \, dx \\ &= \left[\int_0^a + \int_0^{l-a} \right] \log x \, dx \\ &= a \log a - a + (l - x) \log (l - a) - l + a. \end{aligned}$$

Let $a = \frac{1}{2} l (1 + \beta)$, where $-1 < \beta < 1$. Then

$$\begin{aligned} I_a &= l \left[\tfrac{1}{2} (1 + \beta) \{ \log(l/2) + \log (1 + \beta) \} + \tfrac{1}{2} (1 - \beta) \{ \log (l/2) + \log(1 - \beta) \} - 1 \right] \\ &= l \log (l/2) - l + (\tfrac{1}{2} l) \left[(1 + \beta) \log (1 + \beta) + (1 - \beta) \log (1 - \beta) \right]. \end{aligned} \tag{2}$$

If $| \beta | < 1$,

$$\begin{aligned} &(1 + \beta) \log (1 + \beta) + (1 - \beta) \log (1 - \beta) \\ &= \log (1 - \beta^2) + \beta \log \frac{1 + \beta}{1 - \beta} = - \sum_1^\infty \frac{\beta^{2s}}{s} + \sum_1^\infty \frac{2\beta^{2s}}{2s - 1} = \sum_1^\infty \frac{\beta^{2s}}{s (2s - 1)} > 0. \end{aligned} \tag{3}$$

If $\beta = \pm 1$, it is easily seen that $(1 + \beta) \log (1 + \beta) + (1 - \beta) \log (1 - \beta)$ tends to the limit $2 \log 2 > 0$. (4)

From (2), (3), (4),

$$I_a \geqslant l \log (l/2) - l \geqslant l \log l/2e \;\; (a < l). \tag{5}$$

From (1) and (5) we have $I_a \geqslant l \log l/2e$ for all values of a.

Giving a the values $a_1, a_2, \ldots, a_m$ and adding, we obtain

$$\int_0^l \log | (x - a_1)(x - a_2) \ldots (x - a_m) | \, dx \geqslant ml \log (l/2e).$$

Let M be the maximum value of $|(\rho-a_1)\dots(\rho-a_m)|$ for $0 \leqslant \rho \leqslant l$.

Then $l \log M = \int_0^l \log M\,dx > \int_0^l \log|(x-a_1)\dots(x-a_m)|\,dx > ml\log(l/2e)$.

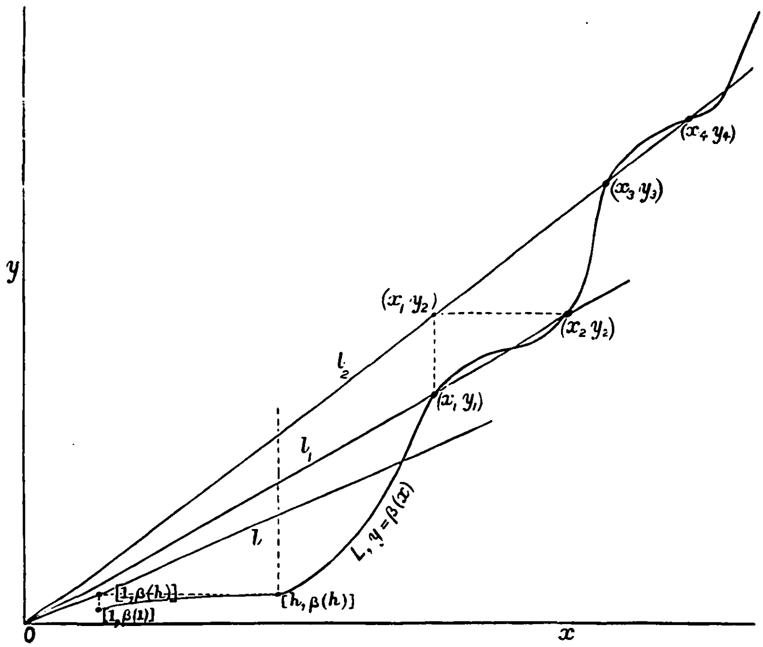
Therefore $$\log M > m\log(l/2e),$$

and $$M > (l/2e)^m.$$

Thus there exists a ρ such that

$$|(\rho-a_1)\dots(\rho-a_m)| > (l/2e)^m,$$

and the Lemma is proved.



LEMMA II.—Let $a_1, a_2, \dots$ be a sequence of real positive numbers, such that $a_{s+1} \geqslant a_s$, and such that $a_s = s^{\phi(s)}$, where $\underset{s=\infty}{\text{Lt}}\ \phi(s) = \infty$.

Let $a(x)$ be the function defined by

$$a(x) = (x-s)\,a_{s+1} + (s+1-x)\,a_s,$$

when $$s+1 \geqslant x \geqslant s.$$

Then $a(s) = a_s$, $a(x)$ is a continuous non-decreasing function of x, and $a(x) = x^{\phi(x)}$, where $\underset{x=\infty}{\text{Lt}}\ \phi(x) = \infty$.

Suppose now that we are given a number ν as large as we please. Then if any (large) number h be assigned, there exists a point (X, Y) of the curve $y = a(x)$, such that $X > h$, and such that the part of the curve C, $(y/Y)^{1/\nu} = x/X$ [which evidently passes through (X, Y)], for which $1 < x < X$, is entirely above the corresponding part of the curve $y = a(x)$, while the part of C for which* $x > X+1$ is entirely below the corresponding part of $y = a(x)$.

Effect the transformation

$$x' = x, \qquad y'^{\nu} = y.$$

The curve C becomes C', $y'/Y' = x'/X'$, *i.e.*, a straight line through the origin, and $y = a(x)$ becomes $y' = \beta(x')$, where $\beta(x) = [a(x)]^{1/\nu}$, so that $\beta(x')$ is a non-decreasing function.

If in the original plane a point A is above another point B of the same abscissa, the transformed point A' will be above B' in the transformed plane.

Hence to prove the Lemma it is sufficient to prove that a point (X', Y') exists on $y' = \beta(x')$, such that $X' > h$, and such that the part of the line C' for which $1 < x' < X'$ is entirely above, and the part of C' for which $x' > X'+1$ is entirely below, the corresponding part of $y' = \beta(x')$.

Suppose the contrary case. (1)

Call the curve† $y = \beta(x)$, L.

This curve L starts at the point $[1, \beta(1)]$.

Since $\beta(x)$ is non-decreasing it is evident from the figure that the line l joining the origin O to $[1, \beta(h)]$ does not cut the curve L in any point of abscissa $< h$.

Let l_1 be any line through O, lying between l and Oy.

Since $\beta(x) = x^{\phi(x)/\nu}$ and $\underset{x=\infty}{\text{Lt}}\ \phi(x)/\nu = \infty$, it is easily seen that every ray through the origin, lying in the first quadrant, and sufficiently near the axis Oy, but not coincident with that axis, must intersect the curve L.

Let then $(x_1 y_1)$ be that intersection of l_1 with L which is nearest to the origin.

Then, from the figure, $x_1 > h$, and the point $[h, \beta(h)]$ is below the point of l_1 whose abscissa is h. Hence the part of L for which $x < x_1$ is below the corresponding part of l_1. Again,

$$\beta(x) = x^{\phi(x)/\nu} > \frac{y_1}{x_1} x,$$

* The result can be extended to the case when this inequality is replaced by $x > X$, but the proof is somewhat troublesome, and the above form is sufficient for our purpose.

† For convenience we shall drop the accents.

when x is sufficiently large, so that the distant points of L are above the corresponding points of l_1.

Then it follows from the supposition (1) that there is an intersection $(x_2 y_2)$ of l_1 with L such that $x_2 - x_1 \geqslant 1$.

For, if this were not the case, $(x_1 y_1)$ would be a point of L satisfying all the requirements for $(X'Y')$.

Let l_2 be the line joining O and $(x_1 y_2)$, and let $(x_3 y_3)$ be that intersection of l_2 and L which is nearest to the origin. From the figure $x_3 > x_2$, and x_2 takes the place of h in the reasoning above.

Then, as before, there exists an intersection of l_2 and L such that $x_4 - x_3 > 1$. We proceed similarly with the line l_3 joining O and $(x_3 y_4)$, and so on.

Now $\dfrac{y_2}{x_2} = \dfrac{y_1}{x_1}$,

$$\frac{y_4}{x_4} = \frac{y_3}{x_3} = \frac{y_2}{x_1} = \frac{y_1}{x_1}\frac{x_2}{x_1},$$

$$\frac{y_6}{x_6} = \frac{y_5}{x_5} = \frac{y_4}{x_3} = \frac{y_1}{x_1}\frac{x_2 x_4}{x_1 x_3},$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\frac{y_{2p}}{x_{2p}} = \frac{y_1}{x_1}\frac{x_2 x_4 \dots x_{2p}}{x_1 x_3 \dots x_{2p-1}} = \frac{y_1}{x_1^2}\frac{x_2}{x_3}\frac{x_4}{x_5}\cdots\frac{x_{2p-2}}{x_{2p-1}}x_{2p} < \frac{y_1}{x_1^2}x_{2p},$$

since $x_{s+1} > x_s$. Therefore

$$y_{2p} < \left(\frac{y_1}{x_1^2}\right) x_{2p}^2 \text{ for all values of } p. \tag{2}$$

Now
$$\frac{\beta(x)}{x^2} = x^{\phi(x)/\nu - 2},$$

and tends to infinity with x. Therefore a number k exists such that, when $x > k$,

$$\frac{\beta(x)}{x^2} > \frac{y_1}{x_1^2}. \tag{3}$$

But, since $x_{2s} \geqslant x_{2s-1} + 1$, $x_{2s-1} > x_{2s-2}$, when $p > k$, we have $x_{2p} > k$. Hence, since $(x_{2p} y_{2p})$ is a point of L, (2) and (3) are incompatible. Hence the supposition (1) is false, and the Lemma is proved.

We can now proceed to the proof of the theorem.

Take for the sequence $a_1, a_2, \dots$ of Lemma II. the sequence of the moduli of the zeros of $F(z)$.

If we are given a number h, however large, and when we have assigned a ν we can, by means of Lemma II., find an $X > h$ such that the point $[X, \alpha(X)]$ has the properties of the lemma.

Let m be the greatest integer contained in X. By Lemma I. we can find a number $r \leqslant \frac{1}{2}\alpha(X)$ such that, for all points z on the circle $|z| = r$,

$$\left|\prod_1^m (a_s+z)\right| > \left[\frac{\alpha(X)}{4e}\right]^m,$$

and such that this inequality obtains without alteration of r, when the arguments of the a's are varied; so that

$$\left|\prod_1^m \left(1+\frac{z}{a_s}\right)\right| > \frac{[\alpha(X)/4e]^m}{a_1 a_2 \dots a_m}$$

and
$$\log\left|\prod_1^m \left(1+\frac{z}{a_s}\right)\right| > P_1 - m\log(4e) \tag{1}$$

where
$$P_1 = m\log\alpha(X) - \log(a_1 a_2 \dots a_m). \tag{2}$$

Again,
$$\log\left|\prod_1^m \left(1+\frac{z}{a_s}\right)\right| < \log\prod_1^m\left[\frac{a(X)+\alpha(X)}{a_s}\right] < P_1 + m\log 2. \tag{3}$$

From (1) and (3) it follows that

$$\log\left|\prod_1^m \left(1+\frac{z}{a_s}\right)\right| = P_1 + \theta m \quad \text{where } |\theta| \text{ is finite.} \tag{4}$$

When $|x| \geqslant \frac{1}{2}$, $\quad |[\log|1+x|]| < 2|x|.$

Therefore, since $|z| < \frac{1}{2}\alpha(X)$, so that $\left|\frac{z}{a_{m+s}}\right| \leqslant \frac{1}{2}$ $(s \geqslant 1)$,

$$\left|\left[\log\left|\prod_1^\infty \left(1+\frac{z}{a_{m+s}}\right)\right|\right]\right| \leqslant \sum_1^\infty \left|\left[\log\left|1+\frac{z}{a_{m+s}}\right|\right]\right| < 2\sum_1^\infty\left|\frac{z}{a_{m+s}}\right|$$
$$< \alpha(X)\sum_1^\infty \frac{1}{a_{m+s}}. \tag{5}$$

Now by the property of Lemma II., when $m+s > X+1$, *i.e.*, when $s > 1$,

$a_{m+s} >$ the ordinate of $[y/\alpha(X)]^{1/\nu} = x/X$ corresponding to $x = m+s$

$> \frac{(m+s)^\nu}{X^\nu}\alpha(X).$

Hence

$$\alpha(X)\sum_1^\infty \frac{1}{a_{m+s}} < 1+\alpha(X)\sum_2^\infty \frac{1}{a_{m+s}} < 1+X^\nu \sum_2^\infty \frac{1}{(m+s)^\nu}$$
$$< 1+X^\nu\int_1^\infty \frac{dt}{(m+t)^\nu} < 1+X^\nu\frac{1}{(\nu-1)(m+1)^{\nu-1}} < 1+\frac{m+1}{\nu-1}.$$

Hence, from (5), when $m > \nu > 2$,

$$\left|\left[\log\left|\prod_1^\infty\left(1+\frac{z}{a_{m+s}}\right)\right|\right]\right| < 1+\frac{m+1}{\nu-1} = \theta' m, \quad \text{where } |\theta'| < \frac{3}{\nu-1} < 3. \tag{6}$$

Next consider P_1. We have

$$P_1 = m\log\frac{a(X)}{(a_1 a_2 \dots a_m)^{1/m}} > m\log\left[\frac{m\,a(X)}{a_1+a_2+\dots+a_m}\right], \tag{7}$$

since the arithmetic mean is greater than the geometric mean. Now by the property of Lemma II., when $1 < s < X$,

$$a_s < \text{the ordinate of } [y/a(X)]^{1/\nu} = x/X \text{ corresponding to } x = s,$$

$$< a(X)\frac{s^\nu}{X^\nu}.$$

Hence $\displaystyle\sum_1^{m-1} a_s < \frac{a(X)}{X^\nu}\sum_1^{m-1} s^\nu < \frac{a(X)}{X^\nu}\int_0^m t^\nu\,dt < \frac{a(X)}{X^\nu}\frac{m^{\nu+1}}{\nu+1} < \frac{a(X)}{\nu}m.$

Therefore $$\left[\sum_1^m a_s\right]\Big/[ma(X)] < \frac{1}{m}+\frac{1}{\nu}.$$

Choose ν so large that $1/\nu < \frac{1}{2}\eta$, where $\eta = e^{-1/\epsilon_1}$ and ϵ_1 will presently be chosen, and choose $h > \nu$ so that $m > \nu$, $1/m < \frac{1}{2}\eta$. Then

$$\left[\sum_1^m a_s\right]\Big/[ma(X)] < \eta,$$

and therefore $$\log\left\{\frac{ma(X)}{\sum_1^m a_s}\right\} > \log\frac{1}{\eta} > \frac{1}{\epsilon_1}.$$

Then, from (7), $$P_1 > m/\epsilon_1. \tag{8}$$

From (4) and (6), $\log|F(z)| = P_1+(\theta+\theta')m+\log C.$

Now

$$\left|\frac{(\theta+\theta')m+\log C}{P_1}\right| < \frac{m[|\theta|+|\theta'|+|\log C|]}{m/\epsilon_1} < \epsilon_1[|\theta|+|\theta'|+|\log C|.$$

Choose ϵ_1 so that this last expression is less than $\frac{1}{8}\epsilon$. Then

$$\log|F(z)| = P_1[1+\eta_1(z)], \quad \text{where } |\eta_1(z)| < \tfrac{1}{8}\epsilon \tag{9}$$

for every point of the circle $|z| = r$.

By choosing h sufficiently large, we can make m as large as we please, and hence P_1, which is certainly positive, as large as we please.

Now r cannot be equal to a_s for any value of s, for then $\log|F(z)|$ would be $-\infty$ for $z = -a_s$. Hence there is an n such that $a_n < r < a_{n+1}$, both limits excluded.

Then $$\log|F(z)| = \bar{P}+\log Q, \tag{10}$$

where $$\bar{P} = n\log r - \sum_1^n \log a_s$$

and $$\log Q = \log\left|\left[\prod_1^n\left(1+\frac{a_s}{z}\right)\prod_1^\infty\left(1+\frac{z}{a_{n+s}}\right)\right]\right|.$$

We shall show that there is a point z_1 on $|z| = r$ for which $Q \leqslant 1$; so that $[\log Q]_{z_1} \leqslant 0$ algebraically.

Suppose the contrary case. Then, since Q is continuous for points z on the circle, and, since a continuous function attains its limits, there is a number $\lambda > 1$ such that $Q > \lambda$ for every point z of the circle.

[λ may depend on n, and tend to zero with $1/n$, but we only require λ to be independent of the argument of z.]

Hence, if p be any integer, and z' any point on the circle,

$$\prod_{t=0}^{p-1}\left[Q\left(z'e^{2\pi t\iota/p}\right)\right] > \lambda^{p+1}. \tag{11}$$

But, since $\prod_1^\infty\left(1+\frac{z}{a_{n+s}}\right)$ is absolutely convergent and $\prod_1^n\left(1+\frac{a_s}{z}\right)$ is a finite product,

$$\begin{aligned}\prod_{t=0}^{p-1}\left[Q\left(z'e^{2\pi t\iota/p}\right)\right] &= \prod_{s=1}^n\left|\left[\prod_{t=0}^{p-1}\left\{1+\frac{a_s}{z'e^{2\pi t\iota/p}}\right\}\right]\right|\prod_{s=1}^\infty\left|\left[\prod_{t=0}^{p-1}\left\{1+\frac{z'e^{2\pi t\iota/p}}{a_{n+s}}\right\}\right]\right| \\ &= \prod_{s=1}^n\left|\left[1+\left(\frac{a_s}{z'}\right)^p\right]\right|\prod_{s=1}^\infty\left|\left[1+\left(\frac{z'}{a_{n+s}}\right)^p\right]\right| \\ &< \prod_{s=1}^n\left[1+\left(\frac{a_s}{r}\right)^p\right]\prod_{s=1}^\infty\left[1+\left(\frac{r}{a_{n+s}}\right)^p\right]^* \\ &< \prod_{s=1}^n\left(1+\frac{a_s}{r}\right)\prod_{s=1}^\infty\left(1+\frac{r}{a_{n+s}}\right) \\ &< U, \text{ where } U \text{ is independent of } p, \text{ though not of } n. \end{aligned} \tag{12}$$

But, if p be taken sufficiently large, $\lambda^{p+1} > U$.

Hence (11) and (12) are incompatible, our supposition was false, and it follows that there exists a point z_1 on $|z| = r$ such that $Q(z_1) \leqslant 1$ and $[\log Q]_{z_1} \leqslant 0$.

In a similar manner we can prove that there exists a point z_2 such that $[\log Q]_{z_2} \geqslant 0$.

The number $\lambda > 1$ is replaced by $\mu < 1$. We consider the same product of Q's as before, and the inequality indicated by the star is replaced by

$$\begin{aligned}\prod_{t=0}^{p-1}\left[Q\left(z'e^{2\pi t\iota/p}\right)\right] &> \prod_{s=1}^n\left[1-\left(\frac{a_s}{r}\right)^p\right]\prod_{s=1}^\infty\left[1-\left(\frac{r}{a_{n+s}}\right)^p\right] \\ &> \prod_{s=1}^n\left(1-\frac{a_s}{r}\right)\prod_{s=1}^\infty\left(1-\frac{r}{a_{n+s}}\right) > V.\end{aligned}$$

We thus obtain an inequality incompatible with $\prod_{t=0}^{p-1}[Q(z'e^{2\pi t i/p})] < \mu^{p+1}$ when p is sufficiently large.

Now $\bar{P}$ is independent of the argument of z, and is the same for z_1 and z_2. The same thing is true for P_1.

Now $$[\log F(z) - \bar{P}]_{z=z_1} \leqslant 0 \quad \text{(algebraically)}.$$

From (9), $$[\log F(z) - (1-\tfrac{1}{8}\epsilon)P_1]_{z=z_1} > 0.$$

Hence, subtracting, $$(1-\tfrac{1}{8}\epsilon)P_1 - \bar{P} < 0.$$

Similarly, by putting $z = z_2$, we obtain
$$(1+\tfrac{1}{8}\epsilon)P_1 - \bar{P} > 0,$$
whence $P_1 = \bar{P}/(1+\frac{1}{8}\epsilon\kappa_z)$, where $|\kappa_z| < 1$ for all points z of the circle, and
$$\log|F(z)| = P_1[1+\eta_1(z)] = \bar{P}\left[\frac{1+\eta_1(z)}{1+\frac{1}{8}\epsilon\kappa_z}\right].$$

Now $$\left|1 - \frac{1+\eta_1(z)}{1+\frac{1}{8}\epsilon\kappa_z}\right| < (\tfrac{1}{8}\epsilon+\tfrac{1}{8}\epsilon)/(1-\tfrac{1}{8}\epsilon) < \tfrac{1}{2}\epsilon, \quad \text{when } \epsilon < 4.$$

Then $$\log|F(z)| = \bar{P}[1+\eta(z)], \quad \text{where } |\eta(z)| < \tfrac{1}{2}\epsilon. \tag{13}$$

The complete form of the theorem is established by the following considerations :—

By choosing h sufficiently large, we can ensure that m is as large as we please, and that therefore X, therefore $a(X)$, therefore P_1, therefore $M(r)$, and therefore r or n, is as large as we please.

We have then determined an r such that
$$|F(z)| = \exp[\bar{P}\{1+\eta(z)\}], \qquad |\eta(z)| < \tfrac{1}{2}\epsilon.$$

Moreover, $P = n\log r - \sum_1^n \log a_s$ is evidently independent of the arguments of z and of the zeros. In the course of the work r was determined (by means of Lemma I.) solely by means of the moduli $a_1, a_2, \ldots$, and is independent of the arguments of the a's ; moreover, the analysis throughout assumes these arguments to be arbitrary.

Thus on the circle $|z| = r$
$$|F_1(z)| = \exp[\bar{P}\{1+\eta_1(z)\}], \qquad |\eta_1(z)| < \tfrac{1}{2}\epsilon.$$

We determine the sequence $r_1, r_2, \ldots$ as follows :—

Let r_1 be a value of r which has the above properties when $\epsilon = \frac{1}{2}$.

Let r_2 be the least possible value of r such that $r_2 > r_1+1$, and such that r_2 has the above properties for $\epsilon = 1/2^2$.

Similarly, let r_3 be the least possible $r > r_2+1$ and corresponding to $\epsilon = 1/2^3$, and so on. We evidently have $\operatorname{Lt}_{s=\infty} r_s = \infty$. Then, when an

ϵ is assigned, if μ be the least integer such that $2^{-\mu} < \epsilon$, when $s > \mu$, on the circle $|z| = r_s$, we have

$$|F(z)| = \exp[\bar{P}\{1+\eta(z)\}],$$

where
$$|\eta(z)| < \frac{1}{2^{s+1}} < \tfrac{1}{2}\epsilon.$$

A similar result evidently obtains for $F_1(z)$.

Finally, we have to show that $m(r_s) > [M(r_s)]^{1-\epsilon}$, ... We have

$$\log m(r_s) > \bar{P}(1-\tfrac{1}{2}\epsilon) > \bar{P}(1+\tfrac{1}{2}\epsilon)(1-\epsilon) > [\log M(r_s)](1-\epsilon).$$

Therefore
$$m(r_s) > [M(r_s)]^{1-\epsilon}.$$

Similarly, the other inequalities of the theorem are proved.

5. The theorem of the preceding article does not in general hold for functions of finite or infinite order, and it does not hold for any of the ordinary functions of finite order which occur in analysis, *e.g.*, e^z, $\sin z$, $1/\Gamma(z)$, $\sin z$.

It does not hold, moreover, for the functions* $P_\rho(z)$ of Dr. Barnes, which may be regarded as the standard type of functions of finite order.

The most precise completely general theorem † which is known concerning the relations of $m(r)$ and $M(r)$ for functions of finite order ρ, is that, on certain circles $|z| = r$, where r is as large as we please, $m(r) > e^{-r^{\rho+\epsilon}}$, where ϵ has its usual meaning. It seems certain, moreover, that when $\rho > 1$ we cannot find a lower limit for $m(r)$ which is of higher order than is implied in this inequality, unless we make further assumptions ‡ as to the nature of the zeros of the function.

Again, on the circles $|z| = r$ of § 4, $\log|F(z)|$ has a dominant term § $\bar{P}$ which is independent at once of the argument of z and of the arguments of the zeros, and $|F(z)|$ has a dominant factor $\exp(\bar{P})$ which is similarly independent.

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* Barnes, *Proc. London Math. Soc.*, Ser. 2, Vol. 3, Part 4.

† [*Note added August* 17*th*.—I have proved the following theorem :—If $\rho < \frac{1}{2}$, a sequence of circles exists for which $m(r) > [M(r)]^{\cos 2\pi\rho - \epsilon}$, which is a generalisation of the theorem of § 4. I hope to publish the proof in another paper.]

‡ Cf. Lindelöf, *Acta Soc. Fenn.*, Vol. XXXI. (1), 1902, pp. 1–79, where an extensive theory of this kind is elaborated.

§ In saying that P is the dominant term of $f(z)$, it is meant that $f(z) = P + Q$, where $\underset{|z|=\infty}{\text{Lt}} \left[\left|\frac{Q}{P}\right|\right] = 0$. Similarly, P is a dominant factor of $f(z)$ when $f(z) = P \cdot Q$, where

$$\underset{z=\infty}{\text{Lt}} \frac{|Q|\pm 1}{|P|} = 0.$$

This again is not true for the standard functions of finite order.

Consider the function

$$F(z) = z\prod_1^\infty\left[\left(1-\frac{z}{n^2}\right)\left(1+\frac{z}{n^2}\right)\right] = \frac{\sin(\pi z^{\frac{1}{2}})\sinh(\pi z^{\frac{1}{2}})}{\pi^2},$$

whose order is $\frac{1}{2}$.

If $z^{\frac{1}{2}} = \xi+\iota\eta$, the dominant term of $\log|F(z)|$ is $\pi(|\xi|+|\eta|)$. This evidently depends on the argument of z, and varies from $\sqrt{2}\pi r^{\frac{1}{2}}$ to $\pi r^{\frac{1}{2}}$. But even the maximum value of the dominant term can be changed by a change in the argument of the zeros.

Consider

$$F_1(z) = z\left[\prod_1^\infty\left(1+\frac{z}{n^2}\right)\prod_1^\infty\left(1+\frac{z}{n^2}\right)\right],$$

whose zeros have the same moduli as those of $F(z)$.

The maximum value of the dominant term of $\log|F_1(z)|$ is $2\pi r^{\frac{1}{2}}$, which is different from the corresponding maximum $\sqrt{2}\pi r^{\frac{1}{2}}$ for $F(z)$.

We have chosen a function of order less than unity because of the exponential factors which may occur in the product form of a function of order greater than or equal to unity. In the case of functions of order unity, a more striking example is afforded by the pair of integral functions* $\sin\pi z$ and $z^{-1}[1/\Gamma(z)]^2$. The moduli of the zeros of these two functions are the same in each case, but the maximum values of the dominant terms of their logarithms are respectively πr and $2r\log r$.

It is possible† to construct functions of finite‡ and infinite (including transfinite) orders for which $\log F(z)$ has a dominant term independent of the arguments of z and of the zeros, but these are exceptional cases.

It is clear, then, that functions of zero order behave at infinity in a manner quite different from that of functions of finite order, and that they deserve to be studied by special methods, not applicable, in general, to the theory of finite order.

In the general theorem of § 4, we have no means of determining the precise positions of the circles $|z| = r_s$. We may, however, expect that, in the case of functions of standard type, the results of this theorem will hold for circles in a part of the plane which can be determined.

In the following articles we shall undertake the approximation to functions of standard type.

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* This example was given to me by one of the referees. It is due to Borel.

† This is an example of a theory which I propose to develop in another paper.

‡ Mr. Hardy has given examples of functions of finite order with the above property, *Proc. London. Math. Soc.*, Ser. 2, Vol. 2, 1904, pp. 332–339.

6. We proceed to establish the following theorem for functions of standard type:—

Let $F(z) = \prod_{1}^{\infty} (1+z/a_s)$ be any function of standard type, so that $\phi(s)$ increases with s after some finite value N of s.

We define the region R of the z plane as follows:—Assign any positive number k, which may be chosen as large as we please.

Let C_s be the circle $|z+a_s| = |a_s| \exp[-\chi(s)]$, where $\chi(s)$ is the greater of the two numbers k and $k[s/\phi(s)]$. Then R is the region of the plane which is exterior to all the circles C_s.

Then, when z is restricted to lie in the region R, a constant K exists such that

$$\log F(z) = \bar{P}+Q,$$

where

$$\bar{P} = n \log r - \sum_{s=1}^{n} \log a_s,$$

n having its usual meaning, and where

$$|Q| < \text{the greater of the numbers } K \text{ and } K[n/\phi(n)].$$

As for $\bar{P}$, if any positive number $\lambda < 1$ be assigned, then, when r is sufficiently great,

$$\bar{P} > \lambda n \phi(n).$$

Further, if the circle $|z| = r$ do not cut any of the circles C_s, then, when r is sufficiently great,

$$m(r) > [M(r)]^{1-\epsilon},$$

where ϵ is any assigned positive number.

We have (§ 3)

$$\log F(z) = \bar{P}+\bar{R}+\bar{S}+\log\left|1+\frac{z}{a_{n+1}}\right|+\log\left|1+\frac{a_n}{z}\right|+\iota\gamma_z,$$

where $|\gamma_z| < \pi$. We shall show that $\bar{P}$ is of order not less than that of $n\phi(n)$, while

$$\left|\bar{R}+\bar{S}+\log\left|1+\frac{z}{a_{n+1}}\right|+\log\left|1+\frac{a_n}{z}\right|+\iota\gamma_z\right|$$

is of order not greater than that of the greater of K and $K[n/\phi(n)]$, and we prove the latter result by considering separately the order of each of the five terms $\bar{R}$, $\bar{S}$, ...

First let us prove the result concerning $\bar{P}$.

We have $\bar{P} = \mathfrak{R}P = \mathfrak{R}\left\{\log \prod_1^n \frac{z}{a_s}\right\} = n \log r - \sum_1^n \log a_s$

$$> n \log a_n - \sum_1^n \log a_s$$

$$> n\phi(n) \log n - \log a_1 - \sum_2^n \phi(s) \log s.$$

Let N be the value of s after which $\phi(s)$ increases, and let

$$\sum_1^N \log a_s = C,$$

where C is a finite constant. Then

$$\bar{P} > n\phi(n) \log n - C - \sum_{N+1}^n \phi(s) \log s > n\phi(n) \log n - \phi(n) \sum_{N+1}^n \log s - C.$$

Now $\sum_{N+1}^n \log s \leqslant \sum_1^n \log s \leqslant \log(n!) \leqslant n \log n - n + \log \sqrt{2\pi}\, n + \epsilon_n$,

where ϵ_n vanishes when n becomes infinite (Stirling's formula). Therefore

$$\bar{P} > n\phi(n) \log n - \phi(n)\{n \log n - n + \tfrac{1}{2}\log(2\pi) + \tfrac{1}{2}\log n + \epsilon_n\} - C$$

$$> \phi(n)(n - \tfrac{1}{2}\log n - \tfrac{1}{2}\log(2\pi) - C - \epsilon_n).$$

Now, when n is sufficiently large, *i.e.*, when r is sufficiently large,

$$\tfrac{1}{2}\log n + \tfrac{1}{2}\log(2\pi) + C + \epsilon_n < (1-\lambda)\, n,$$

where λ is any assigned number < 1.

Therefore, when r is sufficiently large,

$$\bar{P} > \lambda n \phi(n), \tag{1}$$

where λ is any assigned number < 1.

We notice, in passing, an upper limit for $\bar{P}$,

$$\bar{P} < n \log r < n \log a_{n+1} < n\phi(n+1) \log(n+1). \tag{2}$$

Next consider the term $\bar{R}$.

We notice once for all the two following inequalities.

If $|x| < 1$, then $\mathfrak{R} \log(1+x) < |x|$, algebraically,

and $$|\mathfrak{R} \log(1+x)| \leqslant -\log(1-|x|).$$

For the first $\mathfrak{R} \log(1+x) = \log|1+x| \leqslant \log(1+|x|) \leqslant |x|$.

For the second $|\mathfrak{R} \log(1+x)| = \left|\mathfrak{R}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots\right)\right|$

$$\leqslant |x|+\frac{|x|^2}{2}+\ldots \leqslant -\log(1-|x|).$$

We shall frequently have occasion to use these inequalities in the course of the present paper, and shall hereafter use them without further remark.

We have
$$\bar{R} = \sum_{s=2}^{\infty} \mathfrak{R} \log\left(1+\frac{z}{a_{n+s}}\right).$$

We divide .his sum into two parts as follows.

Let λ* bt a constant which we shall choose later to be greater than a certain finite number.

Let η be the greater of the numbers λ/n, $\lambda/\phi(n)$. Then, by choosing λ sufficiently large, we can ensure that ηn is greater than any assigned number, and, since $\phi(n)$ tends to infinity with n, by choosing n sufficiently large we can make η as small as we please. We suppose for the present that n is so large that $\eta < 1$.

We divide the sum $\bar{R}$ into

$$\sigma_1 = \sum_{s=2}^{[\eta n]} \log|1+z/a_{n+s}|, \quad \text{and} \quad \sigma_2 = \sum_{[\eta n]+1}^{\infty} \log|1+z/a_{n+s}|.$$

We shall prove successively that σ_1 and σ_2 are of order not greater than the greater of $Kn/\phi(n)$ and K.

First consider σ_1. We have

$$|\sigma_1| \leqslant -\sum_{s=2}^{[\eta n]} \log\left(1-\frac{r}{a_{n+s}}\right) \leqslant -\sum_{2}^{[\eta n]} \log\left(1-\frac{a_{n+1}}{a_{n+s}}\right)$$

$$\leqslant -\sum_{2}^{[\eta n]} \log\left[1-\frac{(n+1)^{\phi(n+1)}}{(n+s)^{\phi(n+s)}}\right]$$

$$\leqslant -\sum_{2}^{[\eta n]} \log\left[1-\left(\frac{n+1}{n+s}\right)^{\phi(n+1)}\right]$$

$$[\text{since } \phi(n+s) \geqslant \phi(n+1)]$$

$$\leqslant -\sum_{2}^{[\eta n]} \log\left[1-\left(\frac{n+1}{n+s}\right)^{\lambda/\eta}\right] \tag{3}$$

$$[\text{since } \lambda/\eta = \text{the lesser of } n \text{ and } \phi(n) \leqslant \phi(n+1)].$$

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* This λ is not the λ appearing in (1).

Now $$\log\left[\left(\frac{n+s}{n+1}\right)^{\lambda/\eta}\right] = -\frac{\lambda}{\eta}\log\frac{n+1}{n+s} = -\frac{\lambda}{\eta}\log\left(1-\frac{s-1}{n+s}\right)$$

$$> \frac{\lambda}{\eta}\,\frac{s-1}{n+s}$$

$$> \frac{\lambda}{\eta}\,\frac{s-1}{n+\eta n} > \frac{\lambda}{2\eta}\,\frac{s-1}{n} \quad (\text{since } \eta < 1).$$

Hence $$1\bigg/\left[1-\left(\frac{n+1}{n+s}\right)^{\lambda/\eta}\right] < 1\bigg/\left[1-\exp\left(-\frac{\lambda}{2\eta}\,\frac{s-1}{n}\right)\right]. \tag{4}$$

Now, when $x < \frac{1}{2}$,

$$e^{-x} < 1-x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots < 1-x+\frac{x}{2}\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots\right) < 1-\frac{x}{2}.$$

Hence, when $$\eta < \tfrac{1}{4}\lambda, \tag{5}$$

so that $$\frac{\lambda}{2\eta}\,\frac{s-1}{n} < \frac{\lambda}{2\eta}\,\frac{n}{n} < \frac{1}{2},$$

$$1-\exp\left[-\frac{\lambda}{2\eta}\,\frac{s-1}{n}\right] > 1-\frac{1}{2}\,\frac{\lambda}{2\eta}\,\frac{s-1}{n},$$

and, from (4),

$$1\bigg/\left[1-\left(\frac{n+1}{n+s}\right)^{\lambda/\eta}\right] < 1\bigg/\frac{1}{2}\,\frac{\lambda}{2\eta}\,\frac{s-1}{n} < \frac{4\eta}{\lambda}\,\frac{n}{s-1}$$

$$< \frac{4([\eta n]+1)}{\lambda(s-1)} < \frac{8[\eta n]}{\lambda(s-1)},$$

provided $\lambda > 2$, so that $[\eta n] \geqslant \eta n - 1 \geqslant \lambda - 1 > 1$. Then, from (3),

$$|\sigma_1| \leqslant \sum_2^{[\eta n]} \log\left(\frac{8[\eta n]}{\lambda(s-1)}\right) \leqslant \sum_2^{[\eta n]} \log\left(\frac{16[\eta n]}{\lambda s}\right) \quad (\text{since } s \geqslant 2)$$

$$< [\eta n]\log\left(\frac{16[\eta n]}{\lambda}\right) - \log\Gamma(1+[\eta n]).$$

Using the asymptotic formula for $\log\Gamma(x+1)$ (Stirling's formula), we obtain

$$|\sigma_1| < [\eta n]\log\left(\frac{16}{\lambda}\right) + [\eta n]\log([\eta n]) - [\eta n]\log[\eta n] + [\eta n]$$
$$-\tfrac{1}{2}\log[\eta n] + A(n),$$

where $A(n)$ is finite for all values of n,

$$< [\eta n]\left(\log\frac{16}{\lambda}+1\right)+A(n) \quad (\text{since } [\eta n] > 1)$$

$$< \eta n\left(\log\frac{16}{\lambda}+1\right)+A(n).$$

Let $\frac{1}{2}K_1$ be the greater of (1°), $\lambda\left(\log\frac{16}{\lambda}+1\right)$, and (2°) the maximum value of $|A(n)|$. Then, if $\eta = \lambda/n$,

$$|\sigma_1| < \tfrac{1}{2}K_1+\tfrac{1}{2}K_1 < K_1.$$

If $\eta = \lambda/\phi(n)$, so that $\phi(n) > n$,

$$|\sigma_1| < \tfrac{1}{2}K_1\frac{n}{\phi(n)}+\tfrac{1}{2}K_1 < K_1\frac{n}{\phi(n)}.$$

Hence, in any case, $|\sigma_1| <$ the greater of K_1 and $K_1\dfrac{n}{\phi(n)}$. (6)

Now consider σ_2. When $s \geqslant [\eta n]+1 \geqslant \eta n$,

$$\frac{r}{a_{n+s}} \leqslant \frac{a_{n+1}}{a_{n+s}} \leqslant \frac{(n+1)^{\phi(n+1)}}{(n+s)^{\phi(n+s)}} \leqslant \left(\frac{n+1}{n+s}\right)^{\phi(n+1)} \leqslant \left(\frac{n+1}{n+s}\right)^{\phi(n)}$$

$$< \left[\frac{n+1}{n+1+\eta(n+1)-2}\right]^{\phi(n)} < \left\{1\Big/\left[1+\eta-\frac{2}{n+1}\right]\right\}^{\phi(n)}.$$

Choose $\lambda > 4$; then $\dfrac{2}{n+1} < \dfrac{1}{2}$, $\dfrac{4}{n} < \frac{1}{2}\eta$, and

$$\frac{r}{a_{n+s}} < (1+\tfrac{1}{2}\eta)^{-\phi(n)} < \left[1+\tfrac{1}{2}\frac{\lambda}{\phi(n)}\right]^{-\phi(n)} < \exp\left[-\phi(n)\log\left\{1+\tfrac{1}{2}\frac{\lambda}{\phi(n)}\right\}\right]$$

$$< \exp\left[-\phi(n)\tfrac{1}{4}\frac{\lambda}{\phi(n)}\right] \quad [\text{if } \phi(n) > \lambda]^*$$

$$< \exp(-\tfrac{1}{4}\lambda).$$

Now, when $0 < x < 1$, $\dfrac{-\log(1-x)}{x} = 1+\dfrac{x}{2}+\dfrac{x^2}{3}+\dots$ increases as x increases.

Hence, when $s > \eta n$,

$$-\log\left(1-\frac{r}{a_{n+s}}\right)\Big/\frac{r}{a_{n+s}} < \frac{e^{-\frac{1}{4}\lambda}}{\log(1-e^{-\frac{1}{4}\lambda})} < \text{a finite } l.$$

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* If $0 < x < \frac{1}{2}$, $\log(1+x) = x-\dfrac{x^2}{2}+\dots > x\left[1-\tfrac{1}{2}\left(\dfrac{1}{2}+\dfrac{1}{2^2}+\dots\right)\right] > \tfrac{1}{2}x$.

Then $$|\sigma_2| \leqslant -\sum_{[\eta n]+1}^{\infty} \log\left(1-\frac{r}{a_{n+s}}\right) < l.\sum_{[\eta n]+1}^{\infty} \frac{r}{a_{n+s}}$$

$$< la_{n+1}\sum_{2}^{\infty}\frac{1}{a_{n+s}} < la_{n+1}\sum_{2}^{\infty}(n+s)^{-\phi(n+s)}$$

$$< la_{n+1}\sum_{2}^{\infty}(n+s)^{-\phi(n+1)} < l(n+1)^{-\phi(n+1)}\int_{1}^{\infty}(n+x)^{-\phi(n+1)}dx$$

$$< l(n+1)^{-\phi(n+1)}\frac{(n+1)^{1-\phi(n+1)}}{\phi(n+1)-1}$$

$$< l\frac{n+1}{\phi(n+1)-1} < \frac{4ln}{\phi(n+1)},$$

when $$\phi(n+1) > 2.$$

Hence $$|\sigma_2| < \frac{4ln}{\phi(n)}. \tag{7}$$

Next let us consider the term $|\bar{S}|$.

The analysis is somewhat similar to the above, and I shall abbreviate it where this is possible.

We divide the sum $$\bar{S} = \sum_{1}^{n-1}\log|1+a_s/z|,$$

which may be written $$\bar{S} = \sum_{1}^{n-1}\log|1+a_{n-s}/z|,$$

into two parts,

$$\sigma_3 = \sum_{1}^{[\eta n]}\log|1+a_{n-s}/z|, \quad \text{and} \quad \sigma_4 = \sum_{[\eta n]+1}^{n-1}\log|1+a_{n-s}/z|,$$

where η is the number defined above.

Consider σ_3. We have

$$|\sigma_3| \leqslant -\sum_{1}^{[\eta n]}\log\left[1-\frac{a_{n-s}}{r}\right] \leqslant \sum_{1}^{[\eta n]}\log\left[1\Big/\left(1-\frac{a_{n-s}}{a_n}\right)\right]$$

$$< \sum_{1}^{[\eta n]}\log\left[1\Big/\left(1-\frac{(n-s)^{\phi(n-s)}}{n^{\phi(n)}}\right)\right]$$

$$< \sum_{1}^{[\eta n]}\log\left[1\Big/\left(1-\frac{(n-s)^{\phi(n)}}{n^{\phi(n)}}\right)\right] \quad [\text{since } \phi(n) > \phi(n-s)]$$

$$< \sum_{1}^{[\eta n]}\log\left[1\Big/\left\{1-\left(\frac{n-s}{n}\right)^{\lambda/\eta}\right\}\right] \quad [\text{since } \phi(n) \geqslant \lambda/\eta].$$

Now $$\log\left[\left(\frac{n}{n-s}\right)^{\lambda/\eta}\right] = -\frac{\lambda}{\eta}\log\left(1-\frac{s}{n}\right) > \frac{\lambda s}{\eta n} > \frac{\lambda s}{2[\eta n]},$$

when $\lambda > 2$ and $\eta n > 2$.

Hence, reasoning as in the case above, we have

$$1\Big/\left[1-\left(\frac{n-s}{n}\right)^{\lambda/\eta}\right] < \frac{8[\eta n]}{\lambda s} < \frac{16[\eta n]}{\lambda s}.$$

Then
$$|\sigma_3| < \sum_{1}^{[\eta n]} \log\left[\frac{16[\eta n]}{\lambda(s-1)}\right].$$

This is the same inequality as we obtained for $|\sigma_1|$.

Hence, by repeating the reasoning for that case, we have

$$|\sigma_3| < \text{the greater of } K_1 \text{ and } K_1 n/\phi(n). \tag{8}$$

Now consider σ_4. We have

$$\sigma_4 = \sum_{s=[\eta n]+1}^{n-N-1} \log|1+a_{n-1}/z| + \sum_{s=1}^{N} \log|1+a_s/z|,$$

where N is the least integer such that $\phi(s+1) \geqslant \phi(s)$ when $s \geqslant N$.

If n be so large that $r \geqslant a_n > 2a_N$,

$$\left|\sum_{1}^{N} \log|1+a_s/z|\right| < \sum_{1}^{N} |\log \tfrac{1}{2}| < N \log 2. \tag{9}$$

Again, when $s \geqslant [\eta n]+1 \geqslant \eta n$,

$$\frac{a_{n-s}}{r} \leqslant \frac{a_{n-s}}{a_n} \leqslant \frac{(n-s)^{\phi(n-s)}}{n^{\phi(n)}} \leqslant \left(\frac{n-s}{n}\right)^{\phi(n)} \leqslant \left(\frac{n-\eta n}{n}\right)^{\phi(n)} \leqslant (1-\eta)^{\phi(n)}$$

$$\leqslant (1-\eta)^{\lambda/\eta} \leqslant \exp\left[\frac{\lambda}{\eta}\log(1-\eta)\right] < \exp(-\lambda).$$

Hence, by reasoning similar to that used in the case of σ_2,

$$-\log\left(1-\frac{a_{n-s}}{r}\right) < l'\frac{a_{n-s}}{r},$$

where l' is a finite constant. Then

$$\left[\sum_{[\eta n]+1}^{n-N-1} \log\left|1+\frac{a_{n-s}}{z}\right|\right] \leqslant -\sum_{[\eta n]+1}^{n-N-1} \log\left(1-\frac{a_{n-s}}{r}\right) < \frac{l'}{r}\sum_{[\eta n]+1}^{n-N-1} a_{n-s}$$

$$< \frac{l'}{a_n}\sum_{1}^{n-N-1} a_{n-s} < l'n^{-\phi(n)}\sum_{1}^{n-N-1}(n-s)^{\phi(n-s)}$$

$$< l'n^{-\phi(n)}\sum_{1}^{n-N-1}(n-s)^{\phi(n)}$$

$$< l'n^{-\phi(n)}\int_0^n (n-x)^{\phi(n)}\,dx$$

$$< l'n^{-\phi(n)}\frac{n^{\phi(n)+1}}{\phi(n)+1} < l'\frac{n}{\phi(n)+1} < l'\frac{n}{\phi(n)}. \tag{10}$$

Adding (9) and (10), if $\frac{1}{2}K_2$ be the greater of $N\log 2$ and l', we see that

$$|\sigma_4| < \text{the greater of } K_2 \text{ and } K_2 n/\phi(n). \tag{11}$$

Next consider the term

$$T_1 = \log|1+z/a_{n+1}|.$$

$|[\log|1+z/a_{n+1}|]|$ is a maximum for a given n when $|z+a_{n+1}|$ is a maximum or minimum.

Now, by the definitions of the number n, $|z+a_{n+1}|$ has its maximum when $z=a_{n+1}$. We then have

$$|T_1| = \log 2.$$

Again, by the definition of the region R, $|z+a_{n+1}|$ has its minimum value when z lies on the circumference of the circle C_{n+1}, and this value is $\exp[-\chi(n+1)]$.

Then $|T_1| = \chi(n+1) = \text{the greater of } k \text{ and } k(n+1)/\phi(n+1)$

$$< \text{the greater of } 2k \text{ and } 2k/\phi(n). \tag{12}$$

We proceed similarly with $T_2 = \log|1+a_n/z|$. $|T_2|$ is a maximum when $|1/z+1/a_n|$ is a maximum or a minimum.

The first case occurs when $z=a_n$, and we then have $|T_2| = \log 2$.

The second case occurs when z is the point on the circumference of C_n which is at the greatest distance from the origin.

In this case, since

$$\left|\frac{z+a_n}{a_n}\right| = \exp[-\chi(n)] \leqslant e^{-k},$$

so that

$$|z| \leqslant |a_n|(1+e^{-k}),$$

$$|T_2| = \left|\left[\log|1+a_n/z|\right]\right| \leqslant \left|\log\frac{\exp[-\chi(n)]}{(1+e^{-k})}\right|$$

$$\leqslant \chi(n)+\log(1+e^{-k})$$

$$\leqslant \text{the greater of } 2k+\log(1+e^{-k}) \text{ and } [2k+\log(1+e^{-k})]\,n/\phi(n). \tag{13}$$

Finally, $|\gamma_z\iota| \leqslant \pi < \text{the greater of } 2\pi \text{ and } 2\pi n/\phi(n).$ (14)

Now, adding the inequalities (6), (7), (8), (11), (12), (13), and (14), we have, if

$$K = K_1+4l+K_1+K_2+2k+[2k+\log(1+e^{-k})],$$

$$|\bar{R}+\bar{S}+\log|1+z/a_{n+1}|+\log|1+a_n/z|+\iota\gamma_z|$$

$$\leqslant |\sigma_1|+|\sigma_2|+|\sigma_3|+|\sigma_4|+|T_1|+|T_2|+|\gamma_z|$$

$$< \text{the greater of } K \text{ and } K n/\phi(n).$$

Then the first part of the theorem is proved.

To prove the second part we recall that, if λ be any assigned positive number < 1, then, when n is sufficiently large, $\bar{P} > \lambda n \phi(n)$.

Thus on any circle $|z| = r$ which does not cut any circle C_s, *i.e.*, which lies entirely within the region R,

$$\log F(z) = \bar{P} + Q,$$

where $\qquad |Q| <$ the greater of K and $Kn/\phi(n)$.

Now, when n, or r, is sufficiently large,

$$\frac{K}{\lambda n \phi(n)} < \tfrac{1}{4}\epsilon, \qquad \frac{Kn}{\phi(n)}\Big/[\lambda n \phi(n)] < \tfrac{1}{4}\epsilon.$$

It follows at once that, when r is sufficiently large,

$$\log m(r) > (1-\epsilon) \log M(r),$$

and therefore $\qquad m(r) > [M(r)]^{1-\epsilon}.$

7. The formula $\log F(z) = \bar{P} + Q$ of the last article may be used to approximate to the logarithm of any function of standard type.

In practice it is necessary to approximate to the finite sum $\sum_1^n \log a_s$ which occurs in $\bar{P}$, in terms of n, which is supposed large, to approximate to n as a function of r, and to effect the substitution of n in terms of r.

Both of the above approximations should be carried just so far, when this is possible, that the final approximation for $\bar{P}$ has an error of the order of $n/\phi(n)$, or a finite error, according as we have the inequality $|Q| < Kn/\phi(n)$ or $|Q| < K$.

The order of the terms in r thus obtained ranges from that of K or $Kn/\phi(n)$ as a lower limit (which is not attained), to the order of $\bar{P}$, which varies with different functions from $n\phi(n)$ to $n \log n \phi(n)$ [cf. the inequalities (1), (2), of § 6].

Thus, provided we can effect the approximations spoken of above, we can find some of the higher terms in the asymptotic expression of $\log F(z)$.

It may be remarked that in the case of some functions the expression Q is actually of order $n/\phi(n)$, so that the inequality for Q which constitutes the theorem of the last article is the most precise general inequality which can be obtained on our lines. In the case of certain functions for which $\phi(n)$ is of less order than n, so that $|Q| < K n/\phi(n)$, Q is of slightly less order than $n/\phi(n)$. The difference, however, is usually irrelevant in practice, so far as finding more terms of $\log F(z)$ is concerned.

Since $\bar{P}$ is independent of the arguments of z and the zeros, the terms in r which we obtain are similarly independent, so that our method gives the same approximation for all functions whose constant term in the Taylor series is 1, and whose zeros have the same sequence of moduli.

The region R, however, to which z is confined, varies with the arguments of the a's. It contains, however, the region R' which is the part of the plane exterior to every annulus A_s, whose boundaries are the circles with centre at the origin, touching the circle C_s internally and externally.

The region R' is independent of the arguments of the a's. Hence, when z is confined to R', a change in the arguments of the zeros can only affect terms of $\log F(z)$ whose order is not greater than the greater of K and $K\, n/\phi(n)$.

It is clear, from the alternative inequalities which occur in § 6, that there is a change in the behaviour of $\log F(z)$ when $\phi(n)$ is approximately n.

When $\phi(n)$ is of order not less than $n/\log n$, we can obtain a better approximation than that afforded by the theorem of § 6, and we shall in the following three articles consider the approximation to classes of functions of this type. The analysis in these cases is fortunately far simpler than that of § 6.

The theorem of § 6, then, is more particularly appropriate to functions for which $\phi(n)$ is of order less than $n/\log n$.

It is necessary, however, for the cases when $\phi(n)$, although not decreasing, is of irregular growth. It is possible,* *e.g.*, for $\phi(n)$, while being an increasing function, to be of order $n^{\frac{1}{2}}$ for an infinite set of values of n, and to be of order n^2 for another infinite set of values. In this case we should have the inequality $|Q| < K$ for certain ranges of z [*i.e.*, those ranges for which $\phi(n)$ is of order n^2], and for other ranges the inequality $|Q| < Kn^{\frac{1}{2}}$.

We shall conclude this article with some remarks on the regions R and R'.

We suppose $\phi(s)$ is of less order than $s/\log s$, and that it is of regular growth (in a rough sense).

The radius ρ_s of the circle C_s is $a_s \exp[-ks/\phi(s)]$.

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* Borel, *Leçons sur les Fonctions entières*. Borel constructs a function which we may call $f(x)$, which is of order comparable with e^x for an infinity of values of x as large as we please, and of order comparable with e^{x^2} for another similar range of values. If we take

$$\phi(n) = \frac{1}{n^{\frac{1}{2}}} \log f(n^{\frac{3}{2}}),$$

we have the case above.

The distance of $-a_s$ from the nearest other zero (usually)

$$\geqslant a_s - a_{s-1} \geqslant a_s \left[1-\left(\frac{s-1}{s}\right)^{\phi(s)}\right] \geqslant a_s \frac{\phi(s)}{s} \text{ (approximately).}$$

Then the ratio of ρ_s to the distance from $-a_s$ to the nearest other zero $< \frac{s}{\phi(s)} \exp\left(-k\frac{s}{\phi(s)}\right)$ (approximately).

This tends to zero with $1/s$, and is very small if $\phi(s)$ increases slowly with s.

Thus the region R covers by far the greater part of the plane, and since the circles C_s do not intersect when s is large, as is evident from the above, the point z can move in the region R from any point in R to infinity.

The region R' covers the greater part of the plane, but consists of disconnected annuli.

8. THEOREM.—Let $F(z)$ be a function such that after a certain value N of s,

$$\frac{a_s}{a_{s+1}} \leqslant \beta < 1; \tag{1}$$

but no other restriction is placed upon $\phi(s)$ than that implied by (1).

Assign any positive number k, which may be as large as we please.

Let R_1 be the region exterior to every circle C_s,

$$|z+a_s| = e^{-k}|a_s|.$$

Then, when z is confined to the region R_1, there is a number K such that

$$\log F(z) = \bar{P}+Q,$$

where

$$|Q| < K.$$

We notice firstly that the class of functions defined by (1) includes functions other than of standard type, and secondly that this theorem gives a better approximation than the theorem of § 6, for functions for which $\phi(n)$ is of order not lower than $n/\log n$, and lower than n.

We have

$$\log F(z) = \bar{P}+\bar{R}+\bar{S}+\log\left|1+\frac{z}{a_{n+1}}\right|+\log\left|1+\frac{a_n}{z}\right|+\iota\gamma_z.$$

Consider $\bar{R}$. We have

$$|\bar{R}| = \left|\left[\sum_{s=2}^{\infty} \log\left|1+\frac{z}{a_{n+s}}\right|\right]\right| \leqslant -\sum_{s=2}^{\infty} \log\left(1-\frac{r}{a_{n+s}}\right)$$

$$\leqslant -\sum_{s=2}^{\infty} \log\left(1-\frac{a_{n+1}}{a_{n+s}}\right)$$

$$\leqslant -\sum_{s=2}^{\infty} \log(1-\beta^{s-1}) \quad \text{(when } n > N\text{)}. \quad (2)$$

Now $\Sigma\beta^{s-1}$ is convergent. Therefore $\Pi(1-\beta^{s-1})$ is finite and not zero, and therefore $-\sum_{s=2}^{\infty} \log(1-\beta^{s-1})$ is finite and $< K_1$.

Hence $$|\bar{R}| < K_1. \quad (3)$$

Again, $$|\bar{S}| = \left|\left[\sum_{s=1}^{n-1} \log\left(1+\frac{a_s}{z}\right)\right]\right| \leqslant -\sum_{s=1}^{n-1} \log\left(1-\frac{a_s}{r}\right)$$

$$\leqslant -\sum_{s=1}^{n-1} \log\left(1-\frac{a_s}{a_n}\right)$$

$$\leqslant -\sum_{s=1}^{N} \log\left(1-\frac{a_s}{a_n}\right) - \sum_{s=N+1}^{n-1} \log\left(1-\frac{a_s}{a_n}\right).$$

Choose r so large that $r \geqslant a_n > 2a_N$.
Then, if $s < N$,

$$1-\frac{a_s}{a_n} > \tfrac{1}{2}, \quad \text{and} \quad -\sum_{s=1}^{N} \log\left(1-\frac{a_s}{a_n}\right) < N\log 2.$$

Then

$$|\bar{S}| \leqslant N\log 2 - \sum_{s=N+1}^{n-1} \log(1-\beta^{n-s}) \leqslant N\log 2 - \sum_{s=1}^{\infty} \log(1-\beta^{s})$$

$$< \text{a finite constant } K_2. \quad (4)$$

We prove, by reasoning as in § 6, that

$$|T_1| = |\log(1+z/a_{n+1})|, \quad \text{and} \quad |T_2| = \log(1+a_n/z)|,$$

are less than finite constants H_1 and H_2. (5)

Finally, $$|\iota\gamma_z| \leqslant \pi. \quad (6)$$

From (3), (4), (5), (6), we have, if $K_3 = K_1+K_2+H_1+H_2+\pi$,

$$|Q| = |\bar{R}+\bar{S}+T_1+T_2+\iota\gamma_z| < K_3.$$

We have so far assumed that n is greater than some finite value, N' suppose.

Let K_4 be the maximum value of Q for all values of z belonging to R, and such that $n < N'$. Let K be greater than K_3 and K_4.

Then $|Q| < K$, and the theorem is proved.

Taking the exponential of each side of the formula

$$\log F(z) = \bar{P} + Q,$$

we have

$$F(z) = C(z)\, e^{\bar{P}},$$

where

$$e^K > |C(z)| > e^{-K}.$$

Let R_1' be the region derived from R_1 in the same manner that R' was derived from R. Then R_1' is independent of the arguments of the zeros.

Hence, if $F(z) = \Pi\,(1+z/a_s)$, $F_1(z) = \Pi\,(1+z/a_s')$ be two functions of the class defined in the theorem, for which the sequence of the moduli of the zeros is the same, then, if $|z| = r$ lies within R_1',

$$e^{2K} > |F_1(z)/F(z)| > e^{-2K}.$$

Thus $F(z)$ and $F_1(z)$ only differ by a finite factor in the region R_1'.

If we can express $\bar{P}$ in terms of r, with a finite error, we can find all the large terms in the asymptotic expression for $\log F(z)$.

The ratio of ρ_s, the radius of C_s, to the least possible distance d_s from $-a_s$ to the nearest other zero is greater than in the case considered in § 7.

We have
$$a_s \geqslant \beta^{-s}.$$

Then (usually)
$$d_s = a_s - a_{s-1} \geqslant (1/\beta - 1)\,\beta^{-s},$$

$$\rho_s = e^{-k} a_s, \qquad \rho_s/d_s \leqslant e^{-k}/(e^{1/\beta} - 1),$$

and is finite, though by choosing k sufficiently large we can make it as large as we please. We can thus ensure that the circles C do not intersect, and that R_1' exists.

9. THEOREM.—Let $F(z) = \prod_{s=1}^{\infty} (1+z/a_s)$ be a function such that

$$\operatorname*{Lt}_{s=\infty} a_s/a_{s+1} = 0,$$

but no further restriction is placed upon $\phi(s)$.

Let $R_2(\eta)$ be the region of the z-plane which consists of all the annuli $\eta a_{s+1} > |z| \geqslant a_s/\eta$, where s is such that $\eta a_{s+1} > a_s/\eta$, and where η is a positive number < 1.

Then, if any positive number ϵ be assigned, there is a corresponding non-zero positive number η_ϵ, such that when z is confined to the region $R_2(\eta_\epsilon)$, and when moreover $|z| > K_\epsilon$, where K_ϵ is a positive number depending on ϵ, we have

$$\log F(z) = P+Q,$$

where
$$P = n \log z - \sum_{s=1}^{n} \log a_s,$$

and where
$$|Q| < \epsilon.$$

The class of functions considered is evidently contained in the class considered in the last article.

It is evident, however, since we propose in this case to obtain the inequality $|Q| < \epsilon$, that we must take count of the imaginary part of $\log F(z)$, so that $\bar{P}$ will no longer serve as an approximation with the accuracy which we require.

Again, $\log(1+z/a_{n+1})$ is not small when $|z/a_{n+1}|$ is nearly unity, so that the theorem requires a new kind of region R.

We have

$$\log F(z) = P+R+S+\log(1+z/a_{n+1})+\log(1+a_n/z),$$

where the logarithms in R, S, and in the last two terms are supposed to have their principal values.

Let ϵ_1, which for the present we suppose less than $\frac{1}{2}$, be a positive number, which we shall presently choose suitably.

Then there exists a ν such that when $s > \nu$,

$$\frac{a_s}{a_{s+1}} < \epsilon_1^2. \tag{1}$$

Then, if z be any point of the region $R(\epsilon_1)$, there is an n such that

$$r/a_{n+1} < \epsilon_1, \qquad a_n/r < \epsilon_1, \tag{2}$$

and such that n has its usual meaning.

Now, if $|x| < \frac{1}{2}$, and if $\log(1+x)$ has its principal value,

$$|\log(1+x)| < 2|x|.*$$

Moreover
$$\left|\frac{z}{a_{n+s}}\right| = \frac{r}{a_{n+1}} \frac{a_{n+1}}{a_{n+2}} \cdots \frac{a_{n+s-1}}{a_{n+s}} < \epsilon_1^{2s-1} < \tfrac{1}{2},$$

provided $n > \nu$.

Hence
$$|\log(1+z/a_{n+s})| < 2\epsilon_1^{2s-1} \quad (s = 1, 2, \ldots),$$

* $|\log(1+x)| = \left|x - \frac{x^2}{2} + \ldots\right| \leqslant |x| + \frac{|x|^2}{2} + \ldots < |x| + |x|^2 \ldots < \frac{|x|}{1-|x|} < 2|x|.$

and therefore

$$\left|\log\left(1+\frac{z}{a_{n+1}}\right)+R\right| \leqslant \sum_{s=1}^{\infty}\left|\log\left(1+\frac{z}{a_{n+s}}\right)\right|$$

$$< 2\sum_{s=1}^{\infty}\epsilon_1^{2s-1} < \frac{2\epsilon_1}{1-\epsilon_1^2} < \tfrac{3}{2}\epsilon_1. \tag{3}$$

Again, if $n > \nu+1$,

$$\left|\log\left(1+\frac{a_n}{z}\right)+S\right| \leqslant \sum_{s=1}^{n}\left|\log\left(1+\frac{a_s}{z}\right)\right|$$

$$\leqslant \sum_{s=1}^{\nu}\left|\log\left(1+\frac{a_s}{z}\right)\right| + \sum_{s=\nu+1}^{n}\left|\log\left(1+\frac{a_s}{z}\right)\right|. \tag{4}$$

When $s > \nu$, $\left|\frac{a_s}{z}\right| = \frac{a_s}{a_{s+1}} \cdots \frac{a_{n-1}}{a_n}\frac{a_n}{r} < \epsilon_1^{2(n-s)}\epsilon_1 < \tfrac{1}{2}$.

When $s \leqslant \nu$, $\left|\frac{a_s}{z}\right| < \epsilon_1^{2(n-N)}\epsilon_1 < \tfrac{1}{2}$.

Then, from (4),

$$\left|\log\left(1+\frac{a_n}{z}\right)+S\right| < 2\left[\nu\epsilon_1^{2(n-N)+1} + \sum_{s=\nu+1}^{n}\epsilon_1^{2(n-s)+1}\right]$$

$$< \epsilon_1\left[2\nu+\sum_{s=0}^{\infty}\epsilon_1^{2s}\right] < \epsilon_1\left[2\nu+\frac{1}{1-\epsilon_1^2}\right] < \epsilon_1[2\nu+\tfrac{3}{2}]. \tag{5}$$

Hence, from (3) and (5),

$$|Q| = \left|R+S+\log\left(1+\frac{z}{a_{n+1}}\right)+\log\left(1+\frac{a_n}{z}\right)\right| < \epsilon_1[\tfrac{3}{2}+2\nu+\tfrac{3}{2}],$$

when $n > \nu+1$, or when $|z| >$ a finite K'_{ϵ_1} depending on ϵ_1. Choose

$$\epsilon_1 = \epsilon/(3+2\nu), \qquad K_\epsilon = K'_{\epsilon_1}, \qquad \eta_\epsilon = \epsilon_1.$$

Then, when z is confined to $R(\eta_\epsilon)$ and $|z| > K_\epsilon$, we have $|Q| < \epsilon$, which proves the theorem.

Since $\text{Lt}\, a_s/a_{s+1} = 0$, when s is sufficiently large, we have $\eta_\epsilon a_{s+1} > a_s/\eta_\epsilon$, so that the region $R_z(\eta_\epsilon)$ exists.

The breadth of the annulus which passes between the points a_s and a_{s+1} is large compared with a_s, but small compared with a_{s+1}.

Taking the exponential of each side of $\log F(z) = P+Q$, we have, for large values of z within $R_z(\eta_\epsilon)$,

$$F(z) = [1+\epsilon(z)]\frac{z^n}{a_1 a_2 \ldots a_n},$$

where $$|\epsilon(z)| < \exp(\epsilon) - 1 < 2\epsilon.$$

If we can approximate to P as a function of z, with an error which tends to zero with $1/r$, this approximation gives all the large* and finite terms of $\log F(z)$.

10. THEOREM. — Let $F(z) = \prod_{s=1}^{\infty} (1+z/a_s)$ be a function, such that $\chi(s) = \phi(s)/s$ is an increasing function after some finite value N of s, and such that
$$\operatorname*{Lt}_{s=\infty} \chi(s) = \infty.$$

We define the region R_3 as follows.

Assign a positive number k such that $0 < k < \frac{1}{2}$, and where k may be taken as small as we please.

Then R_3 is the region of the plane which consists of all the annuli
$$(s+k)^{\phi(s)} < |z| < (s+1-k)^{\phi(s+1)}.$$

Then when z is confined to the region R_3, and when $r >$ some finite constant H,
$$\log F(z) = P+Q,$$
where
$$|Q| < K \exp\left[-\tfrac{1}{2}k\,\chi(n)\right],$$
K being a finite constant.

Since z lies in R_3, there is an integer n, such that $r = (n+\theta)^{\phi(n)}$ $(\theta > k)$ and $r = (n+\theta')^{\phi(n+1)}$ $(\theta' < 1-k)$. Moreover, n evidently has its proper meaning.

We suppose r so large that $n > N$. Then
$$\phi(n+s) \geqslant \frac{n+s}{n+s-1}\,\phi(n+s-1) > \phi(n+s-1).$$
We have
$$\frac{r}{a_{n+1}} = \left(\frac{n+\theta'}{n+1}\right)^{\phi(n+1)} < \left(\frac{n+1-k}{n+1}\right)^{\phi(n+1)} < \left[\left(1-\frac{k}{n+1}\right)^{n+1}\right]^{[\phi(n+1)]/(n+1)}$$
$$< \left[e^{-k}\right]^{\chi(n+1)}\dagger < e^{-k\chi(n)} < e^{-\frac{1}{2}k\chi(n)}. \tag{1}$$

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* I shall use the phrase "large terms" to mean "terms tending to infinity with $|z|$," "small terms" to mean "terms tending to zero with $1/|z|$," and "finite terms" to mean "terms whose moduli lie between fixed finite limits."

† We have $\log\left(1-\frac{k}{n+1}\right)^{n+1} = (n+1)\log\left(1-\frac{k}{n+1}\right) < -k.$

Similarly

$$\frac{a_n}{r} = \left(\frac{n}{n+\theta}\right)^{\phi(n)} < \left(\frac{n}{n+k}\right)^{\phi(n)} < \left[\left(1+\frac{k}{n}\right)^n\right]^{-\chi(n)} < (e^{\frac{1}{2}k})^{-\chi(n)}* < e^{-\frac{1}{2}k\chi(n)}. \tag{2}$$

Let N_1 be the integer such that $\chi(N_1+s) > k$ for all values of s, and N' the greater of N and N_1.

Then, when $s > N'$,

$$\frac{a_s}{a_{s+1}} < \frac{s^{\phi(s)}}{(s+1)^{\phi(s+1)}} < \left(\frac{s}{s+1}\right)^{\phi(s)} < \left[\left(1+\frac{1}{s}\right)^s\right]^{-\chi(s)} < (e^{\frac{1}{2}})^{-k} < e^{-\frac{1}{2}k}. \tag{3}$$

Suppose $n > N'$, and, further, so large that $e^{-\frac{1}{2}k\chi(n)} < \frac{1}{2}$. Then, from (1), (2), and (3), if $\lambda = \exp(-\frac{1}{2}k)$,

$$\frac{r}{a_{n+s}} < \lambda^{\chi(n)+s} < \tfrac{1}{2};$$

and, when $s > N'$, $\qquad \frac{a_s}{r} < \lambda^{\chi(n)+(n-s)} < \frac{1}{2}.$

Also when $s \leqslant N'$, $\qquad \frac{a_s}{r} < \frac{a_{N'}}{r} < \lambda^{\chi(n)+(n-N')}.$

Hence $\qquad \left|\log\left(1+\frac{z}{a_{n+s}}\right)\right| < 2\left|\frac{r}{a_{n+s}}\right| < 2\lambda^{\chi(n)+s};$

when $s > N'$, $\qquad \left|\log\left(1+\frac{a_s}{z}\right)\right| < 2\left|\frac{a_s}{z}\right| < 2\lambda^{\chi(n)+(n-s)};$

and, when $s < N'$, $\left|\log\left(1+\frac{a_s}{z}\right)\right| < 2\left|\frac{a_s}{z}\right| < 2\lambda^{\chi(n)+(n-N)}.$

$$\begin{aligned} \text{Then} \quad |Q| &= \left|\left\{\log\left(1+\frac{z}{a_{n+1}}\right)+R\right\}+\left\{\log\left(1+\frac{a_n}{z}\right)+S\right\}\right| \\ &\leqslant \sum_{s=1}^{\infty}\left|\log\left(1+\frac{z}{a_{n+s}}\right)\right| + \sum_{s=1}^{n}\left|\log\left(1+\frac{a_s}{z}\right)\right| \\ &< 2\lambda^{\chi(n)}\left[\sum_{s=1}^{\infty}\lambda^s + \sum_{s=1}^{n-N'}\lambda^s + N'\lambda^{n-N'}\right] \\ &< 2\left[\frac{2}{1-\lambda}+N\lambda^{-N}\right]\lambda^{\chi(n)} < Ke^{-\frac{1}{2}k\chi(n)}, \end{aligned}$$

where $$K = 2\left[\frac{2}{1-\lambda}+N'\lambda^{-N'}\right].$$

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* We have $\log\left(1+\frac{k}{n}\right)^n > n\log\left(1+\frac{k}{n}\right) > n\left(\frac{1}{2}\frac{k}{n}\right)$, since $\frac{k}{n} < \frac{1}{2n} < \frac{1}{2}$.

We have supposed $n > N'$: this is equivalent to supposing r greater than a finite constant.

Thus the theorem is proved.

It is possible, by finding closer inequalities than (1) and (2), to show that

$$|Q| < K' \exp[-k(1-\epsilon)\chi(n)]$$

when r is sufficiently large, where ϵ is as large as we please. The difference is unimportant in practice.

If we can calculate P in terms of z, we have an expression which represents $\log F(z)$ asymptotically with an error of order $e^{-\chi(n)}$.

It may happen that P can be asymptotically expanded as in a divergent series of descending powers of some function of r, and that the error committed by stopping at any particular term is always of greater order than $e^{-\chi(n)}$. In this case, the asymptotic expansion for P obtains for $\log F(z)$. Such an example occurs in § 15.

11. It will have been observed that in §§ 6, 8, 9, 10 the terms

$$\left|\log\left(1+\frac{z}{a_{n+1}}\right)\right|, \qquad \left|\log\left(1+\frac{a_n}{z}\right)\right|$$

satisfy an inequality precisely similar to that satisfied by $|Q|$; indeed, the various regions R and R' are defined so that this may be the case.

It is easily seen that in the case of any function $F(z)$, which belongs to one of the classes considered in the above articles, we have, in the cases of §§ 6 and 8,

$$\log F(z) = \bar{P} + \log\left|1+\frac{z}{a_{n+1}}\right| + \log\left|1+\frac{a_n}{z}\right| + Q_1,$$

where Q_1 satisfies the same inequality* as Q, and where z may have any value whatever; and, in the cases of §§ 9 and 10,

$$\log F(z) = P + \log\left(1+\frac{z}{a_{n-1}}\right) + \log\left(1+\frac{a_n}{z}\right) + Q_1,$$

where Q_1 satisfies the same inequalities as Q, and z may have any value (subject only, in the case of § 9, to the condition that $|z|$ is greater than a certain value).

These formulæ enable us to study the behaviour of $F(z)$ near its zeros

* That is, of course, when K is suitably rechosen.

and, in the cases of the functions of §§ 9 and 10, near the circles with their centres at the origin, and passing through the zeros: *e.g.*, if $F(z)$ be one of the functions of § 8, we have, near the zero $-a_n$,

$$\log F(z) = \bar{P} + \log\left(1+\frac{a_n}{z}\right) + Q_1,$$

where $|Q_1| <$ a finite K.

12. We shall now apply our general methods and results to the investigation of some particular functions.

The simplest function of zero order is

$$F(z) = \prod_{s=1}^{\infty}\left(1+\frac{z}{e^{s\omega}}\right) \quad (\mathbf{R}\omega > 0).$$

$F(z)$ evidently belongs to the class of functions considered in § 8.

This function has been considered by M. Mellin,* Dr. Barnes,† and Mr. Hardy.‡

M. Mellin obtains a result, which, written in our notation, is as follows :—

$$\log F(z) + \log\left[\left(1+\frac{1}{z}\right)F\left(\frac{1}{z}\right)\right]$$

$$= \frac{(\log z)^2}{2\omega} - \tfrac{1}{2}\log z + \left(\frac{\pi^2}{6\omega}+\frac{\omega}{12}\right) - \sum_{m=1}^{\infty}\frac{z^{2m\pi i/\omega}+z^{-2m\pi i/\omega}}{2m\sinh\left(\dfrac{2m\pi^2}{\omega}\right)}. \quad (1)$$

When $|z| > 1$, $\log\left[\left(1+\frac{1}{z}\right)F\left(\frac{1}{z}\right)\right] = -\sum_{s=1}^{\infty}\frac{(-)^s z^{-s}}{s(1-e^{-s\omega})}$,

and we have

$$\log F(z)$$

$$= \frac{(\log z)^2}{2\omega} - \tfrac{1}{2}\log z + \left(\frac{\pi^2}{6\omega}+\frac{\omega}{12}\right) - \left\{\sum_{m=1}^{\infty}\frac{z^{2m\pi i/\omega}+z^{-2m\pi i/\omega}}{2m\sinh\dfrac{2m\pi^2}{\omega}}\right\} + \sum_{s=1}^{\infty}\frac{(-)^s z^{-s}}{s(1-e^{-s\omega})}. \quad (2)$$

When ω is real, the term in crooked brackets

$$= \sum_{m=1}^{\infty}\frac{\cos\left(\dfrac{2m\pi\log z}{\omega}\right)}{2m\sinh\dfrac{2m\pi^2}{\omega}},$$

and is finite.

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* *Acta Soc., Fenn.*, t. XXIX.

† *Phil. Trans.*, Ser. A, Vol. 199; *Camb. Phil. Trans.*, Vol. XIX., pp. 333-335 and 433-435.

‡ *Quarterly Journal of Mathematics*, 1905.

Hence, when ω is real, (2) affords a complete (convergent) asymptotic expansion.

It happens in the case of this function (with ω complex), that our methods enable us to find a complete asymptotic expansion.

The case is quite exceptional, but it exemplifies to some extent the general procedure.

Let $\omega = \omega_1 + \iota\omega_2$. n is the first integer for which $|z^{-1}e^{(n+1)\omega}| > 1$, or for which $(n+1)\omega_1 > \log r$.

$$\text{Let} \qquad (n+1)\omega_1 = \log r + \beta\omega_1, \quad \text{where } 0 < \beta < 1. \tag{4}$$

$$\text{Then} \qquad (n+1)\omega = \log r + \beta\omega_1 + (n+1)\omega_2\iota = \log z + a\omega_1,$$

where
$$a = \beta + \frac{(n+1)\omega_2 - \theta}{\omega_1}\iota \qquad (\theta = \arg z)$$

and
$$z = e^{(n+1)\omega - a\omega_1}. \tag{5}$$

One value of $\log F(z)$

$$= \left\{n \log z - \sum_1^n s\omega\right\} + \sum_1^\infty \log\left(1 + \frac{z}{e^{(n+s)\omega}}\right) + \sum_1^n \log\left(1 + \frac{e^{s\omega}}{z}\right). \tag{6}$$

$$\text{Now} \quad \sum_1^\infty \log\left(1 + \frac{z}{e^{(n+s)\omega}}\right) = \sum_1^\infty \log\left(1 + \frac{e^{\omega - a\omega_1}}{e^{s\omega}}\right) = \log F(e^{\omega - a\omega_1}), \tag{7}$$

$$\sum_1^n \log\left(1 + \frac{e^{s\omega}}{z}\right) = \sum_1^n \log\left(1 + \frac{e^{a\omega_1}}{e^{(n-s+1)\omega}}\right) = \sum_1^n \log\left(1 + \frac{e^{a\omega_1}}{e^{s\omega}}\right)$$

$$= \sum_1^\infty \log\left(1 + \frac{e^{a\omega_1}}{e^{s\omega}}\right) - \sum_{n+1}^\infty \log\left(1 + \frac{e^{a\omega_1}}{e^{s\omega}}\right)$$

$$= \log F(e^{a\omega_1}) - \sum_1^\infty \log\left(1 + \frac{1}{e^{(n+1)\omega - a\omega_1}e^{(s-1)\omega}}\right)$$

$$= \log F(e^{a\omega_1}) - \sum_0^\infty \log\left(1 + \frac{1}{ze^{s\omega}}\right). \tag{8}$$

$$P = \left\{n \log z - \sum_1^n s\omega\right\} = n \log z - \tfrac{1}{2}\omega n(n+1)$$

$$= \frac{\log z}{\omega}(\log z + a\omega_1 - \omega) - \frac{1}{2\omega}(\log z + a\omega_1)(\log z + a\omega_1 - \omega)$$

[on substituting for n from (5)]

$$= \frac{1}{2\omega}(\log z)^2 - \tfrac{1}{2}\log z - \frac{a\omega_1}{2\omega}(a\omega_1 - \omega). \tag{9}$$

We have for a,

$$a = \beta - \frac{\theta\iota}{\omega_1} + \frac{\omega_2\iota}{\omega_1^2}\,\omega_1(n+1) = \beta - \frac{\theta\iota}{\omega_1} + \frac{\omega_2\iota}{\omega_1^2}\,(\log r + \beta\omega_1)$$

$$= \frac{\iota\omega_2}{\omega_1^2}\log z + \beta - \frac{\theta\iota}{\omega_1} + \frac{\omega_2\iota}{\omega_1}\beta + \frac{\omega_2}{\omega_1^2}\,\theta,$$

or

$$a = \frac{\iota\omega_2}{\omega_1^2}\log z + \frac{\omega\beta}{\omega_1} - \frac{\iota\omega}{\omega_1^2}\,\theta, \tag{10}$$

where $0 < \beta < 1$.

Adding (6), (7), and (8), we have

$$\log F(z) = \left[\frac{1}{2\omega}(\log z)^2 - \tfrac{1}{2}\log z - \frac{a\omega_1}{2\omega}(a\omega_1 - \omega)\right]$$
$$+ \log F(e^{a\omega_1}) + \log F(e^{\omega - a\omega_1}) + \sum_1^\infty \frac{(-)^{s-1}}{s(1-e^{-s\omega})}\frac{1}{z^s}, \tag{11}$$

on substituting for $\sum\limits_0^\infty \log(1+z^{-1}e^{-s\omega})$ its expansion in descending powers of z. a is given as a function of $\log z_1$, and of β and θ (which are finite) by equation (10).

Now $Ra = \beta$. Hence $|e^{a\omega_1}|$ and $|e^{\omega - a\omega_1}|$ are equal to $e^{\beta\omega_1}$ and $e^{(1-\beta)\omega_1}$ respectively. If $0 < \beta < 1$, these are both $< e^{\omega_1} <$ | the first zero of $F(z)$ |.

If, then, $0 < \beta < 1$, both limits being excluded, $F(e^{a\omega_1})$ and $F(e^{\omega - a\omega_1})$ are finite both ways. Then their logarithms are finite.

Thus in equation (11) the terms in square brackets give the large terms, and the other terms are finite or small.

Substituting for a from (10), we obtain, from (11),

$$\log F(z) = \frac{\omega_1^2 + \omega_2^2}{2\omega_1^2\omega}(\log z)^2 - \frac{1}{2\omega_1}\left\{\omega_1 + \iota\omega_2(2\beta - 1) + \frac{2\theta\omega_2}{\omega_1}\right\}(\log z)$$
$$+ \left[\frac{1}{2}\left(\beta - \frac{\iota\theta}{\omega_1}\right)^2 - \frac{1}{2\omega}\left(\beta - \frac{\iota\theta}{\omega_1}\right)^2 + \log F(e^{a\omega_1}) + \log F(e^{\omega - a\omega_1})\right]$$
$$+ \sum_1^\infty \frac{(-)^{s-1}}{s(1-e^{-s\omega})}\frac{1}{z^s}, \tag{12}$$

where the terms in square brackets are the finite terms.

When ω is real, *i.e.*, $\omega_2 = 0$, this expression simplifies considerably.

Put $\omega = \omega_1 = 1$, $\omega_2 = 0$.

$$\log F(z)$$
$$= \tfrac{1}{2}(\log z)^2 - \tfrac{1}{2}\log z + \left[\tfrac{1}{2}(\beta - \iota\theta) - \tfrac{1}{2}(\beta - \iota\theta)^2 + \log F(e^{\beta - \iota\theta}) + \log F(e^{(1-\beta) - \iota\theta})\right]$$
$$+ \sum_1^\infty \frac{(-)^{s-1}}{s(1-e^{-s})}\frac{1}{z^s}. \tag{13}$$

Comparing this with (3), we see that the large and the small terms agree.

We have chosen a different determination of the logarithm of $F(z)$ from that of M. Mellin.

M. Mellin's $\log F(z)$ is equal to $\sum_1^\infty \log(1+ze^{-s\omega})$, each logarithm on the right-hand side having its principal value (*i.e.*, having the modulus of its imaginary part less than π).

Our determination is that implied by equation (6.) Thus the finite terms in (3) and (13) may differ by some multiple of $2\pi\iota$. We may notice that it was quite possible, *a priori*, that the large terms should differ in the same way.

We have expressed $\log F(z)$, in the general case of ω complex, in terms of $\log z$ rather than in terms of $\log r$, in order to exhibit the correspondence between our form and M. Mellin's.

Written in this way, the large terms have imaginary parts which do not really count as large terms.

Let us find the expression in terms of $\log r$.

We have

$$P = n\log z - \tfrac{1}{2}\omega n(n+1)$$

$$= \frac{1}{\omega_1}\left[\log r-(1-\beta)\omega_1\right](\log r+\iota\theta) - \frac{\omega_1+\iota\omega_2}{2\omega_1^2}(\log r+\beta\omega_1)(\log r+\beta\omega_1-\omega_1),$$

which reduces to

$$P = \frac{1}{2\omega_1}(\log r)^2 - \tfrac{1}{2}\log r$$

$$+\left[\frac{\beta(1-\beta)\omega_1}{2} + \frac{\iota}{2\omega_1^2}\left\{[\log r+(\beta-1)\omega_1](2\theta\omega_1-\beta\omega_1\omega_2-\omega_2\log r)\right\}\right], \qquad (15)$$

where the terms in square brackets count as finite. The remaining terms of $\log F(z)$ may be written as before, and we have

$$\log F(z) = P + \log\left\{F(e^{a\omega_1})\,F(e^{\omega-a\omega_1})\right\} + \sum_1^\infty \frac{(-)^{s-1}}{s(1-e^{-s\omega})z^s}, \qquad (16)$$

where P is given by equation (15).

Thus we see that the large terms of $\log F(z)$ are $(\log r)^2/(2\omega_1) - \tfrac{1}{2}\log r$.

Finally, suppose $|z|$ tends either to $|e^{n\omega}|$ or to $|e^{(n+1)\omega}|$. Then β tends to 1 or 0. But, if z tends to a point on the circle $|z| = |e^{n\omega}|$ other than the zero $-e^{n\omega}$, the right-hand sides of (12) and (15) are continuous, and so are their left-hand sides. Hence the equations (12) and (15) are valid when $\beta = 1$ or 0, provided z be not actually at the corresponding zero $-e^{n\omega}$ or $-e^{(n+1)\omega}$.

13. The comparison of the equation (11) of the last article with M. Mellin's result (1), leads to certain other formulæ. We have

$$\left(\frac{\pi^2}{6\omega}+\frac{\omega}{12}\right)-\sum_{m=1}^{\infty}\frac{z^{2m\pi\iota/\omega}+z^{-2m\pi\iota/\omega}}{2m\sinh\left(\frac{2m\pi^2}{\omega}\right)}$$

$$=\log\left[F(e^{a\omega_1})\,F(e^{\omega-a\omega_1})\right]-\frac{a\omega_1}{2\omega}\,(a\omega_1-\omega), \quad (1)$$

where the $2k\pi\iota$ is supposed absorbed into the logarithm, and where $z^{2m\pi\iota/\omega}$ is interpreted to mean $\exp(2m\pi\iota\,\omega^{-1}\log z)$, where

$$|\arg z|<\pi, \qquad \log z=\log|z|+\iota\arg z.$$

Let $$(1+1/z)\,F(z)\,F(1/z)=\phi(z), \quad \phi(e^{\iota\theta})=\psi(\theta);$$

so that $$\phi(1/z)=z\phi(z).$$

Retaining the usual signification for n, we have

$$a=\beta+\frac{(n+1)\,\omega_2-\theta}{\omega_1}\,\iota,$$

$$e^{\omega-a\omega_1}=\exp\left[(1-\beta)\,\omega_1-(\theta-n\omega_2)\,\iota\right],$$

$$e^{a\omega_1}=e^{\omega}\exp\left[-(1-\beta)\,\omega_1+(\theta-n\omega_2)\,\iota\right].$$

Then

$$F(e^{a\omega_1})\,F(e^{\omega-a\omega_1})=\phi\left[e^{(1-\beta)\omega_1-(\theta-n\omega_2)\iota}\right]=e^{-(1-\beta)\omega_1+(\theta-n\omega_2)\iota}\,\phi\left[e^{-(1-\beta)\omega_1+(\theta-n\omega_2)\iota}\right].$$

Substituting in the last term of the right-hand side of (1) the value of a in terms of $\log z$, *i.e.*, $\iota\omega_2/\omega_1^2\log z+[\omega(\beta\omega_1-\iota\theta)]/\omega_1^2$, and using (2), we obtain, after some reduction,

$$\sum_{m=1}^{\infty}\frac{z^{2m\pi\iota/\omega}+z^{-2m\pi\iota/\omega}}{2m\sinh\frac{2m\pi^2}{\omega}}$$

$$=-\frac{\omega_2^2}{2\omega\,\omega_1^2}\,(\log z)^2+\frac{\iota\omega_2}{2\omega_1^2}\,\{(2\beta+1)\,\omega_1-2\iota\theta\}\log z$$

$$+\left[\frac{\omega}{2\omega_1^2}\,(\beta\omega_1-\iota\theta)\,[(\beta-1)\,\omega_1-\iota\theta]-(1-\beta)\,\omega\right.$$

$$\left.-\log\phi\{e^{-(1-\beta)\omega_1+(\theta-n\omega_2)\iota}\}+\left(\frac{\pi^2}{6\omega}+\frac{\omega}{12}\right)\right], \quad (3)$$

where the term in square brackets is finite, and n and β are determined by $0<\beta<1$, $\log|z|=\omega_1(n+1-\beta)$.

The equation (3) gives an asymptotic expression for the left-hand side. We notice that the real part of the right-hand side tends to $-\infty$ as $|z|$ tends to ∞.

Next suppose that ω is real and $|z| = e^{(n+1)\omega}$. Then $\omega = \omega_1$, $\omega_2 = 0$, $\beta = 1$. The formula (3) is still valid, provided $|\theta| < \pi$.

We have

$$z^{2m\pi\iota/\omega} = \exp\left\{\frac{2m\pi\iota}{\omega}\log z\right\} = \exp\left[\frac{2m\pi\iota}{\omega}\{(n+1)\omega+\iota\theta\}\right] = \exp\left(-\frac{2m\pi\theta}{\omega}\right),$$

$\frac{1}{2}(z^{2m\pi\iota/\omega}+z^{-2m\pi\iota/\omega}) = \cosh\left(\frac{2m\pi\theta}{\omega}\right)$, and $\phi\{e^{-(1-\beta)\omega_1+(\theta-n\omega_2)\iota}\}$ becomes $\phi(e^{\iota\theta})$ or $\psi(\theta)$. Thus (3) becomes

$$\sum_{m=1}^{\infty}\frac{\cosh\left(\frac{2m\pi\theta}{\omega}\right)}{m\sinh\left(\frac{2m\pi}{\omega}\right)} = -\frac{\theta^2}{2\omega}+\left(\frac{\pi^2}{6\omega}+\frac{\omega}{12}\right)-\left[\log\psi(\theta)+\frac{\iota\theta}{2}\right]. \tag{4}$$

Now $$e^{\frac{1}{2}\iota\theta}\psi(\theta) = e^{\frac{1}{2}\iota\theta}(1+e^{-\iota\theta})\prod_1^{\infty}\left\{\left(1+\frac{e^{\iota\theta}}{e^{s\omega}}\right)\left(1+\frac{e^{-\iota\theta}}{e^{s\omega}}\right)\right\}$$
$$= 2\cos\tfrac{1}{2}\theta\prod_1^{\infty}(1+2q^{2s}\cos\theta+q^{4s}),$$

on putting $q = e^{-\frac{1}{2}\omega}$.

In the notation of MM. Tannery and Molk,*

$$q_0 = \prod_1^{\infty}(1-q^{2s}),$$
$$\vartheta_2(x) = 2q_0q^{\frac{1}{4}}\cos\pi x\prod_1^{\infty}\{1+2q^{2s}\cos 2\pi x+q^{4s}\}.$$

Then $$\vartheta_2(x) = q_0q^{\frac{1}{4}}e^{\iota\pi x}\psi(2\pi x), \qquad \log q_0 = -\sum_{s=1}^{\infty}\frac{q^{2s}}{s(1-q^{2s})}.$$

Then, from (4), we obtain

$$\log\vartheta_2(x) = \left(\frac{\pi^2}{6\omega}-\frac{\omega}{24}\right)-\sum_{s=1}^{\infty}\frac{q^{2s}}{s(1-q^{2s})}-\frac{2\pi^2x^2}{\omega}-\sum_{m=1}^{\infty}\frac{\cosh\left(\frac{4m\pi^2x}{\omega}\right)}{\sinh\left(\frac{2m\pi^2}{\omega}\right)}, \tag{5}$$

where $\omega = 2\log(1/q)$.

This expansion is valid when x and q are real, $0 < q < 1$, $|x| < \frac{1}{2}$. The series involving x is highly convergent when ω is small, *i.e.*, when q is nearly unity.

When $|x| < \frac{1}{2}$ and q is nearly unity, *i.e.*, ω is small, we have

$$\log[\vartheta_2(x)/q_0] = \left(\frac{\pi^2}{12}-\pi^2x^2\right)\frac{1}{\log(1/q)}+\epsilon(q),$$

where $\epsilon(q)$ tends to 0 as q tends to 1.

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* *Eléments de la Theorie des Fonctions elliptiques*, t. II.

14. In the case of functions $F(z)$, such that $\operatorname*{Lt}_{s=\infty} a_s/a_{s+1} = e^{-\omega}$, $\mathfrak{R}\omega > 0$, we can express the finite term of $\log F(z)$ by means of the function $\phi(x)$ defined in the last article.

THEOREM.—Let $F(z) = \prod_1^\infty (1+z/a_s)$, and let $\operatorname*{Lt}_{s=\infty} a_s/a_{s+1} = e^{-\omega}$, and suppose z is confined to the region R.

Then, when any positive ϵ is assigned, there is a constant K_ϵ, such that, when $|z| > K_\epsilon$, we have

$$\log F(z) = P + \log \phi(z/a_n) + Q',$$

where
$$|Q'| < \epsilon.$$

Let η be a small positive number, which we shall presently choose suitably. Then there is a number N, such that, when $s > N$,

$$\frac{a_s}{a_{s+1}} = e^{-\omega}(1+\eta_s),$$

where
$$|\eta_s| < \eta.$$

Suppose r so large that $n > N$, and let $\omega = \omega_1 + \iota\omega_2$. Then, if $s \geqslant 1$,

$$\left| \left\{ \left(1+\frac{z}{a_n}e^{-s\omega}\right) \Big/ \left(1+\frac{z}{a_{n+s}}\right) \right\} - 1 \right| = \left| \frac{\left(e^{-s\omega}\dfrac{z}{a_n}\right)\left(1-\dfrac{a_n e^{s\omega}}{a_{n+s}}\right)}{1+z/a_{n+s}} \right|$$

$$\leqslant \left[e^{-s\omega_1} \left| \frac{a_{n+1}}{a_n} \right| \right] \frac{|1-(1+\eta_n)(1+\eta_{n+1})\dots(1+\eta_{n+s-1})|}{e^{-k}}$$

$$< e^{-s\omega_1} e^{\omega_1}(1+\eta) \frac{|1-(1+\eta)^s|}{e^{-k}}. \tag{1}$$

Now $(1+\eta)[(1+\eta)^s - 1] = (1+\eta)[1+(1+\eta)+\dots+(1+\eta)^{s-1}](1+\eta-1)$

$$< s\eta(1+\eta)^s < s\eta \exp[s\log(1+\eta)] < s\eta\exp(s\eta).$$

Choose $\eta < \frac{1}{2}\omega_1$. Then, from (1),

$$\left| \left(1+\frac{z}{a_n}e^{-s\omega}\right) \Big/ \left(1+\frac{z}{a_{n+s}}\right) - 1 \right| < \eta\, e^{k+\omega_1} s e^{-\frac{1}{2}s\omega_1}. \tag{2}$$

Now $\operatorname*{Lt}_{s=\infty} se^{-\frac{1}{2}s\omega_1} = 0$, so that $se^{-\frac{1}{2}s\omega_1}$ has a maximum h.

Choose η so that $\qquad \eta \,.\, he^{k+\omega_1} < \frac{1}{2}.$

Then, from (2),

$$\left|\log\left(1+\frac{z}{a_n}e^{-s\omega}\right)-\log\left(1+\frac{z}{a_{n+s}}\right)\right| < |\log(1-\eta\, e^{k+\omega_1}\, s e^{-\frac{1}{2}s\omega_1})|$$

$$< 2\eta\, e^{k+\omega_1} s e^{-\frac{1}{2}s\omega_1};$$

and therefore

$$\left|\sum_{s=1}^{\infty}\log\left(1+\frac{z}{a_{n+s}}\right)-\sum_{s=1}^{\infty}\log\left(1+\frac{z}{a_n}\right)\right| < 2\eta\, e^{k+\omega_1}\sum_{s=1}^{\infty} s e^{-\frac{1}{2}s\omega_1} < K'\eta, \quad (3)$$

since both series on the left-hand side are uniformly convergent, and since $\Sigma s e^{-\frac{1}{2}s\omega_1}$ is convergent.

Again, we can show in a similar manner that

$$\left|\sum_{s=0}^{n-N}\log\left(1+\frac{a_{n-s}}{z}\right)-\sum_{s=0}^{n-N}\log\left(1+\frac{a_n}{z}e^{-s\omega}\right)\right| < K''\eta. \quad (4)$$

Now choose n so large that

$$\left|\frac{a_N}{z}\right| < \eta < \tfrac{1}{2}, \qquad e^{-(n-N)\omega_1} < \eta < \tfrac{1}{2}.$$

Then, when $s \geqslant n-N$,

$$\left|\log\left(1+\frac{a_{n-s}}{z}\right)\right| \leqslant \left|\log\left(1-\frac{a_N}{r}\right)\right| < 2\frac{a_N}{r} < 2\eta;$$

and, similarly, $$\left|\log\left(1+\frac{a_n}{z}e^{-s\omega}\right)\right| < 2\eta.$$

Hence

$$\left|\sum_{s=n-N+1}^{n-1}\log\left(1+\frac{a_{n-s}}{z}\right)-\sum_{s=n-N+1}^{n-1}\log\left(1+\frac{a_n}{z}e^{-s\omega}\right)\right|$$

$$< \sum_{s=n-N+1}^{n-1}(2\eta+2\eta) < 4N.\eta. \quad (5)$$

Finally, when $s > n$,

$$\log\left(1+\frac{a_n}{z}e^{-s\omega}\right) < 2\left|\frac{a_n}{z}e^{-s\omega}\right| < 2e^{-n\omega_1}e^{-(s-n)\omega_1},$$

so that $$\left|\sum_{s=n}^{\infty}\log\left(1+\frac{a_n}{z}e^{-s\omega}\right)\right| < 2e^{-(n-1)\omega_1}\sum_{s=n+1}^{\infty}e^{-(s-n+1)\omega_1} < \frac{2e^{-n\omega_1}}{1-e^{-\omega_1}}.$$

Choose n so large that this last expression is less than η. Then

$$\left| \sum_{s=n}^{\infty} [0] - \sum_{s=n}^{\infty} \log\left(1+\frac{a_n}{z} e^{-s\omega}\right) \right| < \eta. \tag{6}$$

Adding (3), (4), (5), and (6), we have

$$\left| \left[\sum_{s=1}^{\infty} \log\left(1+\frac{z}{a_{n+s}}\right) + \sum_{s=0}^{n-1} \log\left(1+\frac{a_{n-s}}{z}\right) \right] \right.$$

$$\left. - \left[\sum_{s=1}^{\infty} \log\left(1+\frac{z}{a_n} e^{-s\omega}\right) + \sum_{s=0}^{\infty} \left(1+\frac{a_n}{z} e^{-s\omega}\right) \right] \right|$$

$$< (K' + K'' + 4N + 1)\,\eta < K\eta,$$

or

$$\left| [\log F(z) - P] - [\log \phi(z/a_n)] \right| < K\eta.$$

Choose $\eta = \epsilon/K$. Then, when n is greater than some finite number depending on η, or when r is greater than some number K_ϵ,

$$\log F(z) = P + \log \phi(z/a_n) + Q',$$

where

$$|Q'| < \epsilon.$$

As an example, consider

$$F(z) = \prod_{s=1}^{\infty} \left[1 + \frac{z}{(a+sv)^k e^{s\omega}} \right],$$

where k is real, $\mathfrak{R}\omega > 0$, and a and v are any complex numbers.

We evidently have $\operatorname*{Lt}_{s=\infty} a_s/a_{s+1} = e^{-\omega}$, so that the formula of the theorem holds in this case.

n is determined by

$$\log |[a+(n+1)v]^k e^{(n+1)\omega}| > \log r \geqslant \log |(a+nv)^k e^{n\omega}|,$$

whence n is determined as a function of r by the relation

$$n\omega_1 + k \log |a+vn| - \log r = -\alpha\omega_1, \tag{1}$$

where

$$0 \leqslant \alpha < 1 + \frac{k}{\omega_1} \left| \frac{va}{v+a} \right| \frac{1}{n}, \tag{2}$$

so that α is a function of r.

We approximate to P as a function of n, retaining terms of order not less than $1/n$, and to n as a function of r and α, to the same order of approximation.

Substituting from the second approximation in the first, we obtain

$\log F(z)$

$$= \left[\frac{1}{2\omega_1}(\log r)^2 - \frac{k}{\omega_1}\log r \log_2 r + (c-\tfrac{1}{2})\log r + \frac{k^2}{2\omega_1}(\log_2 r)^2 - k(b+c)\log_2 r\right]$$

$$+ [\omega_1\{c^2 - c + a(1-a)\} + k\{\log \Gamma(1+a/v) + 1 - \tfrac{1}{2}\log 2\pi\}]$$

$$-\iota\left\{\frac{\omega_2}{2\omega_1^2}\Big[(\log r)^2 - 2k\log r\log_2 r + \omega_1(2c+1-2a)\log r + k^2(\log_2 r)\right.$$

$$+k[2k-\omega_1(2c+1-2a)](\log_2 r)$$

$$+\{\omega_1^2[c^2+(1-2a)c+a(1-a)] - 2k(b-c+a)\}\Big]$$

$$\left.+(k\gamma-\theta)[\log r - k\log_2 r + \omega_1(c-a)] + kb'\log_2 r\right\}$$

$$+\log[\phi(z/a_n)] + Q',$$

where $|Q'| < \epsilon$, when r is sufficiently large, and where

$$\theta = \arg z, \quad \log_2 r = \log\log r, \quad a/v = b + \iota b', \quad \gamma = \arg v$$

and
$$c = (k/\omega_1)(\log \omega_1 - \log |v|).$$

The third term of the above expression is purely imaginary and counts as a finite term.

Hence, when z is not near a zero, so that $\phi(z/a_n)$ is finite, the first term gives the large terms.

It will be noticed that no terms in a occur in these large terms, which are expressed by means of the ordinary functions of analysis.

If ω is real, so that $\omega_2 = 0$, the above expression simplifies considerably, the greater part of the third term disappearing.

The comparative simplicity of the large terms is somewhat remarkable, as will be seen on working through the rather complicated analysis.

If we only wish to obtain the large terms, we have the following result:—

If z be confined to the region R defined in § 8, we have

$$\log F(z) = \frac{1}{2\omega_1}(\log r)^2 - \frac{k}{\omega_1}\log r\log_2 r + (c-\tfrac{1}{2})\log r + \frac{k}{2\omega_1^2}(\log_2 r)^2$$

$$-k(b+c)\log_2 r + Q,$$

where
$$|Q| < K.$$

We notice that the dominant term, $(\log r)^2/(2\omega_1)$, is independent of k. Thus, multiplying the n-th zero $-e^{n\omega}$ by n^k does not affect the dominant term of $\log F(z)$.

If $k = 0$, we have the function of the last article, and the large terms reduce, as they should, to $(\log r)^2/(2\omega_1) - \frac{1}{2}\log r$.

15. We shall work out one other particular case. Consider the function

$$F(z) = \prod_{s=1}^{\infty}\left[1 + \frac{z}{e^{(a+s\omega)^k}}\right],$$

k real and positive, $\mathfrak{R}\omega^k > 0$.

The two cases, $k > 1$, and $k < 1$, present very different characteristics, and are separated by the case $k = 1$, which has intermediate characteristics, as we shall explain later.

We shall consider in detail the case when $k > 1$. We have

$$\phi(s) = \mathfrak{R}\,\frac{(a+s\omega)^k}{\log s},$$

and is an increasing function of the same value of s. Also

$$\operatorname*{Lt}_{s=\infty} s^{-1}\phi(s) = \infty.$$

Hence $F(z)$ belongs to the class of functions considered in § 10. The number n is determined by the relation

$$|\exp[a+(n+1)\omega]^k| > r \geqslant |\exp(a+n\omega)^k|$$

or

$$\mathfrak{R}[a+(n+1)\omega]^k > \log r \geqslant \mathfrak{R}(a+n\omega)^k$$

or

$$\mathfrak{R}\left\{\omega^k(n+1)^k\left[1+\frac{ka}{\omega}\,\frac{1}{n+1}+\frac{A_n}{(n+1)^2}\right]\right\}$$

$$> \log r > \mathfrak{R}\left\{\omega^k n^k\left[1+\frac{ka}{\omega}\,\frac{1}{n}+\frac{A'_n}{n}\right]\right\},$$

where A_n, A'_n are finite.

Put $\mathfrak{R}\omega^k = \gamma^k$, $\mathfrak{R}\omega^{k-1}a = \gamma^{k-1}\gamma_1$.

Then

$$\gamma(n+1)\left[1+\frac{\gamma_1}{\gamma}\,\frac{1}{n+1}+\frac{B_n}{(n+1)^2}\right] > (\log r)^{1/k} > \gamma n\left[1+\frac{\gamma_1}{\gamma}\,\frac{1}{n}+\frac{B'_n}{n^2}\right],$$

and

$$n = \frac{1}{\gamma}\left[(\log r)^{1/k} - \gamma_1 - a\right], \tag{1}$$

where a lies between $-c/n$ and $\gamma + c'/n$, for some (positive) values of c and c'. When z is confined to the region R_3, we have (§ 10)

$$|Q| < K\exp[-\tfrac{1}{2}k\phi(n)/n], \quad\text{or}\quad |Q| < K\exp[-\lambda n^{k-1}/\log n],$$

for some non-zero value of λ. (2)

Again, we have

$$P = n\log z - \sum_{1}^{n}(a+s\omega)^k. \tag{3}$$

Now, when n is large, we have the asymptotic expansion*

$$\sum_{1}^{n} (a+s\omega)^k = \zeta(-k \mid a+\omega) + \sum_{p=0}^{l} \frac{S_p'(a+\omega)}{p!} \left[\left(\frac{d}{dx}\right)^p \frac{x^{k+1}}{k+1} \right]_{x=n\omega} + J_l, \quad (4)$$

where l is any positive integer, and $|J_l n^{l-k-1}|$ tends to zero when n tends to infinity.

Now $$\underset{n=\infty}{\mathrm{Lt}} |Q n^p| \leqslant \underset{n=\infty}{\mathrm{Lt}} |K e^{-\lambda n^{k-1}/\log n} n^p| = 0.$$

Hence, substituting from (2) in P, we may take l to be any integer we please, and we obtain

$$\log F(z) = n \log z - \frac{S_0'(a+\omega)}{k+1} (\omega n)^{k+1} - S_1'(a+\omega)(\omega n)^k$$

$$- \sum_{p=2}^{l} \frac{S_p'(a+\omega)}{p!} k(k-1)\dots(k-p+2)(\omega n)^{k-p+1} - \zeta(-k \mid a+\omega) + K_l, \quad (5)$$

where $$K_l = -J_l + Q,$$

so that $$\underset{n=\infty}{\mathrm{Lt}} K_l n^{l-k-1} = 0.$$

Then $$\underset{r=\infty}{\mathrm{Lt}} |K_l (\log r)^{(l-k-1)/k}| = 0.$$

Thus, on substituting $n = (1/\gamma)[(\log r)^{1/k} - \gamma_1 - a]$, (5) provides an asymptotic expansion for $\log F(z)$ in descending powers of $[(\log r)^{1/k} - \gamma_1 - a]$.

On arranging the right-hand side in terms whose order in r decrease, and regarding pure imaginaries as finite terms, we obtain

$$\log F(z) = \frac{k}{(k+1)\gamma} (\log r)^{(k+1)/k} - (\gamma_1/\gamma + \tfrac{1}{2}) \log r$$

$$- \frac{k}{2\gamma} [a(a-\gamma) - \gamma_1^2 + \gamma_2 \gamma + \tfrac{1}{6}\gamma^2] (\log r)^{(k-1)/k} + \text{lower terms}, \quad (6)$$

where $$\gamma = [\Re \omega^k]^{1/k},$$

$$\gamma_1 = [\Re a \omega^{k-1}]/[\Re \omega^k]^{(k-1)/k}, \qquad \gamma_2 = [\Re a^2 \omega^{k-2}]/[\Re \omega^k]^{(k-2)/k}.$$

The first two terms involve only known analytic functions.

We may give some idea of the nature of the region R_3, by observing that the restriction of z to R_3 is equivalent to restricting a to lie between $A/\log n$ and $1 - B/\log n$, where A and B are finite constants depending on k which can be made as small as we please by taking k sufficiently small.

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* Barnes, *Proc. London Math. Soc.*, Ser. 2, Vol. 3, Part 4, p. 262. The above formula is obtained by changing a into $a+\omega$.

As the possible range of a is from $-c/n$ to $1+c'/n$, we see that the restricted range differs but slightly from the total possible range.

In the case when $k<1$, $\phi(n)$ is of less order than n, and we cannot find all the large terms of $F(z)$. We restrict z to lie in the region R defined in § 6. Then we may retain terms of order greater than that of $n/\phi(n)$, or $n^{1-k}\log n$, or $(\log r)^{1/k-1}\log\log r$. Thus, when $1>k>\frac{1}{2}$, so that $\log r>(\log r)^{1/k-1}$, we have

$$\log F(z)=\frac{k}{(k+1)\gamma}(\log r)^{(k+1)/k}-\left(\frac{\gamma_1}{\gamma}+\frac{1}{2}\right)\log r$$

$$+\text{terms of order not greater than } (\log r)^{1/k-1}\log\log r,^{*} \quad (7)$$

and, when $k\leqslant\frac{1}{2}$,

$$\log F(z)=\frac{k}{(k+1)\gamma}(\log r)^{(k+1)/k}$$

$$+\text{terms of order not greater than } (\log r)^{1/k-1}\log\log r. \quad (8)$$

The ratio ρ_s/a_s, where ρ_s is the radius of the circle C_s, is in this case of order $e^{-s^{1-k}\log s}$, while the ratio d_s/a_s, where d_s is the lower limit of the distance between $-a_s$ and the nearest other zero, is of order $1/s^{1-k}$.

Hence ρ_s/d_s is of order $\exp(-s^{1-k}\log s)$, which is very small.

16. The two examples which we have worked out in detail suffice to show how, in practice, the value of P is approximately calculated in terms of r. In particular cases, various special artifices may be used to facilitate the approximation, such as replacing a sum by an integral in cases when the consequent error is of an order which we are to neglect.

We give the following summary of particular cases :—

Call, for convenience of reference, the case of the function $\prod_{s=1}^{\infty}(1+ze^{-s\omega})$, considered in § 12, I., the case $\prod_{s=1}^{\infty}\left[1+\frac{z}{(a+sv)^k e^{s\omega}}\right]$, II., the case $\prod_{s=1}^{\infty}\left[1+\frac{z}{e^{(a+s\omega)^k}}\right]$, with $k>1$, III., and the case of the same form, with $k<1$, IV.

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* It can be shown, by a closer study of $\bar{R}$ and $\bar{S}$, that this error is actually of order $(\log r)^{1/k-1}$, except near the circles C_s.

V. $$F(z) = \prod_{s=2}^{\infty} \left[1 + \frac{z}{\exp\{(\log s)^{1+k}\}}\right],$$

k real and positive, z confined to the region R;

$$\log F(z) = \exp\left[(\log r)^{1/(1+k)}\right]\left[\frac{1}{k+1}(\log r)^{k/(k+1)} - \frac{1}{(k+1)k}(\log r)^{(k-1)/(k+1)} + \dots \pm \frac{1}{(k+1)k \dots (k-l+1)}(\log r)^{(k-l)/(k+1)}\right],$$

with an error of order $\exp\left[(\log r)^{1/(1+k)}\right](\log r)^{-k/(k+1)}$, where l is the greatest integer less than $2k$, and where the series in square brackets stops at the term $\pm \dfrac{1}{(k+1)k \dots 1}(\log r)^{(k-k)/(k+1)}$ when k is an integer.

VI. $$F(z) = \prod_{s=1}^{\infty} \left[1 + \frac{z}{e^{\omega e^{as}}}\right],$$

a real and positive, $\omega = \omega_1 + \iota\omega_2$, $\omega_1 > 0$;

$$\log F(z) = \frac{1}{a}\left[\log_2 r - \log \omega_1 + \log(1+\alpha) - \alpha\right] \log z - \frac{\omega}{e^{\alpha}-1}\frac{1+\alpha}{\omega_1}\log r + \frac{\omega e^{\alpha}}{e^{\alpha}-1} + (\text{an expression of order } r^{-h} \text{ where } h > 0),$$

where $[1 - \{\log(1+\alpha)\}/a]$ is the fractional part of $(\log\log r - \log \omega_1)/a$, and z is supposed restricted to the part of the plane defined by

$$k < \alpha < (1-k)(e^{a}-1),$$

where k is a constant as small as we please. The total possible range of α is from 0 to $e^{a}-1$.

The constant h depends on k, and tends to zero with k.

17. We shall conclude by some remarks concerning certain general facts of which the above six cases afford illustration.

Of the six cases, IV. and V. are cases in which $\text{Lt}\, a_{s+1}/a_s = 1$, or a_s increases less rapidly than $e^{s\omega}$. In these two cases, and under the various restrictions placed upon z, we have been able to find, in terms of known analytic functions, an expression which proceeds to precisely that order of approximation which our theory allows as valid.

III. and VI. are cases in which $\text{Lt}\, a_{s+1}/a_s = \infty$, or in which a_s increases more rapidly than $e^{s\omega}$. These two cases are distinguished by the presence of terms of the expansion of $\log F(z)$ involving α, where α is determined by means of the fractional part of some analytic function of r. Now such a function of r as a cannot possible be expressed in terms of the ordinary known analytic functions, and I think it is obvious from the manner in which a enters into the expansions in cases III. and VI., that the whole expansion cannot possibly be expressed in terms of known analytic functions.*

It is easy to multiply instances of both types, and we are led to these conclusions: If a_s increases less rapidly than $e^{s\omega}$ (for all positive values of ω), and is expressed in terms of known analytic functions, then the terms in $\bar{P}$ which our theory permits us to retain as an approximation for $\log F(z)$, can be expressed in terms of known analytic functions of n and r, and, moreover, the problem of finding these terms is practically soluble, however complicated a function of s a_s may be. Further, provided we can invert the equation $a_n = t$, into an equation of the form $n = \chi(t) + \tau$, where $\chi(t)$ is expressed in finite terms, and in terms of known functions of t, and where $|\tau|$ is finite for all values of n, then we can express the above expression (for $\bar{P}$ in terms of n) in terms of known functions of r.

If, on the other hand, a_s increases more rapidly than $e^{s\omega}$, then there will be *large* terms in the expansion for $\log F(z)$ which involve a number a analogous to the a's in cases III. and VI., and consequently the complete expansion cannot be effected in terms of known functions of r.

Again, when $\mathrm{Lt}\, a_{s+1}/a_s = \infty$, it is not always possible in practice to find all the terms of P which are admissible, even by the introduction of a number a, nor is it always possible even to find all the large terms of P. For example, if $a_s = \exp[\exp(\exp s)]$, to determine all the large terms of P, we should have to sum the series $\sum_1^n \exp(e^s)$ with an error only finite, which is impossible by any known analysis.

It is found that the more nearly the rate of increase of a_s approaches that of $e^{s\omega}$ (while remaining greater than that rate) the smaller is the order of the highest term of $\log F(z)$ which involves a, in comparison with the order of the dominant term. In consideration of this fact, and of the behaviour of functions for which $\mathrm{Lt}\, a_{s+1}/a_s = 1$, we might expect in the case when $\mathrm{Lt}\, a_{s+1}/a_s =$ a finite number greater than 1, which separates the two cases which we have been considering, all the large terms of $\log F(z)$ are sometimes expressible in terms of known functions, while the finite part is not so expressible. This we see to be the case in cases I. and II., and instances can be multiplied.

It will be noticed that our conclusions confirm those of Dr. Barnes †

* It is easy to see that the occurrence of a is not due to the fact that we have expressed the various terms of the expansion in terms of r, instead of in terms of z.

† Cf. § 1.

as to the impossibility of solving, by the method of contour integration, the problem of the complete asymptotic expansion of $\log F(z)$ when a_s is of high order in s.

We might expect, however, that when $\underset{s=\infty}{\text{Lt}}\, a_s/a_{s+1} = 1$, this method might, in certain cases, be applied to obtain a complete expansion.

This is, as a matter of fact, the case, and it is possible to obtain such an expansion for the function IV. I shall reserve this theory for another paper.

It may be remarked that the property of requiring non-analytic functions for the complete asymptotic expansion, is not confined to functions of zero order, and, indeed, has no real connection with the order of the maximum modulus.

The same property belongs to certain functions* defined by a product form of the type $\Pi[1+(z/a_s)^{f(s)}]$, where $f(s)$ becomes infinite with s, and $|a_1|, |a_2|, \ldots$ tend in value to infinity, or to a finite limit.

These functions may be integral functions of any order—zero, finite, infinite, or transfinite. They may also be functions whose Taylor series has a finite radius of convergence.

The property, however, does appear to be intimately connected with the rapidity of the convergence of the product form. In the particular case of functions of zero order of standard type, the general condition for the property is equivalent to the condition that a_s should increase as rapidly or more rapidly than $e^{s\omega}$, and this condition, again, can be expressed in terms of the order of $|F(z)|$ *qua* function of $|z|$, but the question of *order* is really quite irrelevant.

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* I hope, in another paper, to develop a theory of these functions, which is somewhat analogous to that of the present paper. It is easy, in this theory, to give a multitude of particular examples, which bear out the rule governing the property.