On the Incorrectness of the Rules for contracting the processes of finding the Square and Cube Roots of a Number.

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I. *The rule for contracting the process of finding the Square Root.*

The rule (see *Todhunter's Algebra*, Art. 246) is this :—

When $n+1$ figures of a square root have been obtained by the ordinary method, n more may be obtained by division only, supposing $2n+1$ to be the whole number.

It will be shown in this paper that the rule cannot always be extended to cases in which the $2n+1$ figures are followed by other figures, as is usually assumed.

Let the number be denoted by N ; let its square root consist of $2n+1$ figures followed by a decimal. Let the first $n+1$ figures followed by n zeros be denoted by a . Let the remaining n figures be denoted by b , and the decimal part by c .

Therefore $\sqrt{N} = a + b + c$,

therefore
$$\frac{N - a^2}{2a} = b + c + \frac{(b+c)^2}{2a}.$$

If it can be shown that $c + \frac{(b+c)^2}{2a}$ may be greater than 1, but not so great as 2, the result of the division $\frac{N - a^2}{2a}$ may have an integral part $b+1$, and not b ; and in such cases the rule will fail.

Putting $u = c + \frac{(b+c)^2}{2a}$; since a, b, c are all positive, it follows that u increases when a diminishes, but when b and c increase.

Now the least value of a is 10^{2n} ; and the greatest value of b is when it consists of n nines, and is therefore $10^n - 1$. The greatest value c can have must be less than 1: let this be denoted by $\dot{9}$. Consequently, the greatest value of u is

$$\dot{9} + \frac{(10^n - 1 + \dot{9})^2}{2 \cdot 10^{2n}},$$

which approaches indefinitely close to $1\frac{1}{2}$.

Consequently the rule does not always furnish a correct result.

Two Propositions (A.) and (B.) will now be proved.

(A.) If the nearest integer to $\sqrt{N-a}$ is b , then $\frac{N-a^2}{2a}$ is $< b+1$, so that the rule is correct in this case.

But the nearest integer to $\frac{N-a^2}{2a}$ may be

$$(1) \ b \dots\dots\dots [\text{Case A.1}],$$

or $(2) \ b+1 \dots\dots\dots [\text{Case A.2}].$

In this proposition

$$0 < c < \frac{1}{2},$$

therefore

$$\begin{aligned} 0 < c + \frac{(b+c)^2}{2a} &< \frac{1}{2} + \frac{(b+\frac{1}{2})^2}{2a} \\ &< \frac{1}{2} + \frac{(10^n - 1 + \frac{1}{2})^2}{2 \cdot 10^{2n}} \\ &< \frac{1}{2} + \frac{1}{2} \left(1 - \frac{1}{2} \cdot \frac{1}{10^n}\right)^2 \\ &< \frac{1}{2} + \frac{1}{2} \\ &< 1. \end{aligned}$$

Consequently $c + \frac{(b+c)^2}{2a}$ can have values ranging between 0 and 1. It can reach the lower limit when $b = 0$, $c = 0$; but it cannot come very close to the upper limit unless n be large.

All that is necessary to observe is that $c + \frac{(b+c)^2}{2a}$ may be either less or greater than $\frac{1}{2}$, so that the nearest integer to $\frac{N-a^2}{2a}$ may be b or $b+1$.

Numerical Examples.

Case A.1.:

$$N = 11884,$$

$$\sqrt{N} = 109.01\dots,$$

$$a = 100,$$

$$b = 9,$$

$$\sqrt{N-a} = 9.01\dots,$$

$$\frac{N-a^2}{2a} = 9.42.$$

The nearest integer to $\sqrt{N-a}$ is in this case b ;
 the integral part of $\frac{N-a^2}{2a}$ is also b ;
 and the nearest integer to $\frac{N-a^2}{2a}$ is also b .

Case A.2: $N = 11903,$
 $\sqrt{N} = 109.1\dots,$
 $a = 100,$
 $b = 9,$
 $\sqrt{N-a} = 9.1\dots,$
 $\frac{N-a^2}{2a} = 9.515.$

The nearest integer to $\sqrt{N-a}$ is in this case b ;
 the integral part of $\frac{N-a^2}{2a}$ is also b ;
 but the nearest integer to $\frac{N-a^2}{2a}$ is now $b+1$.

(B.) If the nearest integer to $\sqrt{N-a}$ is $b+1$,
 then the nearest integer to $\frac{N-a^2}{2a}$ is also $b+1$;
 but now the integral part of $\frac{N-a^2}{2a}$ may be

- (1) b [Case B.1],
 or (2) $b+1$ [Case B.2],

so that in the case B.2 the ordinary rule is incorrect.

In this proposition $\frac{1}{2} < c < 1$,

therefore $\frac{1}{2} < c + \frac{(b+c)^2}{2a} < \cdot 9 + \frac{(10^n - 1 + \cdot 9)^2}{2 \cdot 10^{2n}},$
 $< 1\frac{1}{2}.$

When c is $\frac{1}{2}$, b zero, and n large so that a is large, $c + \frac{(b+c)^2}{2a}$ approaches close to the lower limit. Also it may approach very close to the upper limit $1\frac{1}{2}$.

Consequently the nearest integer to $\frac{N-a^2}{2a}$ is $b+1$ in this case; but

the integral part of $\frac{N-a^3}{2a}$ is sometimes b and sometimes $b+1$.

The mistake which is made in applying the ordinary proof to cases in which the $2n+1$ figures are followed by a decimal, is that this is equivalent to omitting the quantity called c in this paper. Further, on taking account of this quantity c , which can approach very close to unity, it is at once seen that it is not possible to make the rule correct by requiring a to contain more than $n+1$ figures, b containing n as before, for cases can still be found in which $c + \frac{(b+c)^3}{2a} > 1$.

Numerical Examples.

Case B.1:

$$N = 11992,$$

$$\sqrt{N} = 109.507\dots,$$

$$a = 100,$$

$$b = 9,$$

$$\sqrt{N-a} = 9.507\dots,$$

$$\frac{N-a^3}{2a} = 9.96.$$

The nearest integer to $\sqrt{N-a}$ is in this case $b+1$;
the nearest integer to $\frac{N-a^3}{2a}$ is also $b+1$;
but the integral part of $\frac{N-a^3}{2a}$ is b .

Case B.2:

$$N = 11880;$$

$$\sqrt{N} = 108.9\dots,$$

$$a = 100,$$

$$b = 8,$$

$$\sqrt{N-a} = 8.9\dots,$$

$$\frac{N-a^3}{2a} = 9.4.$$

The nearest integer to $\sqrt{N-a}$ is $b+1$ in this case;
the nearest integer to $\frac{N-a^3}{2a}$ is $b+1$;
the integral part of $\frac{N-a^3}{2a}$ is $b+1$;
so that the ordinary rule fails.

Another example will perhaps be a little more striking, as in it the integral part of $\frac{N-a^2}{2a}$ consists of $n+1$ and not n figures.

$$N = 12099,$$

$$\sqrt{N} = 109.995 \dots,$$

$$a = 100,$$

$$b = 9,$$

$$\sqrt{N-a} = 9.995 \dots,$$

$$\frac{N-a^2}{2a} = 10.495.$$

The nearest integer to $\sqrt{N-a}$ is $b+1$;
 the nearest integer to $\frac{N-a^2}{2a}$ is $b+1$;
 the integral part of $\frac{N-a^2}{2a}$ is $b+1$, and consists of two, *i.e.*, $n+1$ figures, for here $n = 1$.

Todhunter gives no general proof that it is possible to extend the rule to cases in which there is a decimal following the $2n+1$ figures. He takes, in effect, the number 12 followed by 8 zeros as an illustration. If, now, the number 12 followed by 3 zeros be taken, then

$$N = 12000,$$

$$\sqrt{N} = 109.5 \dots,$$

$$a = 100,$$

$$b = 9,$$

$$\sqrt{N-a} = 9.5 \dots,$$

$$\frac{N-a^2}{2a} = 10.$$

So that the rule fails in this case.

Another example of the failure of the rule is

$$N = 14399,$$

$$\sqrt{N} = 119.9 \dots,$$

$$a = 110,$$

$$b = 9,$$

$$\sqrt{N-a} = 9\cdot9 \dots,$$

$$\frac{N-a^2}{2a} = 10\cdot45.$$

The following scheme states concisely the four cases considered above, A.1, A.2, B.1, B.2:—

Case.	Integral part of $\sqrt{N-a}$.	Nearest integer to $\sqrt{N-a}$.	Integral part of $\frac{N-a^2}{2a}$.	Nearest integer to $\frac{N-a^2}{2a}$.
A.1	b	b	b	b
A.2	b	b	b	$b+1$
B.1	b	$b+1$	b	$b+1$
B.2	b	$b+1$	$b+1$	$b+1$

It shows that the four following questions (one of these questions being answered wrongly in the affirmative by the ordinary rule), which form a symmetrical scheme, must all be answered in the negative:—

1. Is the integral part of $\sqrt{N-a}$ always the same as the integral part of $\frac{N-a^2}{2a}$?

2. Is the nearest integer to $\sqrt{N-a}$ always the same as the integral part of $\frac{N-a^2}{2a}$?

3. Is the integral part of $\sqrt{N-a}$ always the same as the nearest integer to $\frac{N-a^2}{2a}$?

4. Is the nearest integer to $\sqrt{N-a}$ always the same as the nearest integer to $\frac{N-a^2}{2a}$?

Question 1 is answered negatively by Case B.2,

„	2	„	„	Case B.1,
„	3	„	„	Cases A.2, B.1, and B.2,
„	4	„	„	Case A.2.

II. *The rule for contracting the process of finding the Cube Root.*

The rule (see Todhunter's *Algebra*, Art. 257) is this:—

When $n+2$ figures of a cube root have been obtained by the ordinary method, n more may be obtained by division only, supposing $2n+2$ to be the whole number.

To show that this rule cannot be always extended to cases in which there is a decimal following the $2n+2$ figures, take

$$\sqrt[3]{N} = a + b + c,$$

where a denotes the first $n+2$ figures followed by n zeros, b the remaining n figures, and c the decimal.

$$\text{Then} \quad \frac{N-a^3}{3a^2} = b + c + \frac{(b+c)^2}{a} + \frac{(b+c)^3}{3a^3}.$$

$$\text{In this case, put} \quad u = c + \frac{(b+c)^2}{a} + \frac{(b+c)^3}{3a^3},$$

so that u is greatest when a is least and b and c are greatest.

The least value of a is 10^{2n+1} .

The greatest value of b is $10^n - 1$.

The greatest value of c is $\cdot 9$.

Therefore the greatest value of u is

$$\cdot 9 + \frac{(10^n - 1 + \cdot 9)^2}{10^{2n+1}} + \frac{(10^n - 1 + \cdot 9)^3}{3 \cdot 10^{4n+2}},$$

which approaches indefinitely close to

$$1 + \frac{1}{10} + \frac{1}{3} \frac{1}{10^{n+2}}.$$

This is greatest when n is least. Regarding $n = 1$ as the least value of n for which the rule is interpretable, the greatest value of u is

$$1 + \frac{1}{10} + \frac{1}{3000} = 1.100\dot{9}.$$

178 *On the processes of finding Square and Cube Roots.* [Jan. 13,

To construct a case in which the rule fails, put $n = 1$, $a = 1000$, $b = 9$, $c = a$ decimal; then

$$a + b + c < 1010.$$

Take, therefore, $N = (1010)^3 - 1 = 1030300999$,

$$N = 1030300999,$$

$$\sqrt[3]{N} = 1009.9 \dots,$$

$$a = 1000,$$

$$b = 9,$$

$$\sqrt[3]{N} - a = 9.9 \dots,$$

$$\frac{N - a^3}{3a^2} = 10.100333.$$

The ordinary rule would make the integral part of $\frac{N - a^3}{3a^2}$ equal to 9, and not 10 as it really is. $\frac{N - a^3}{3a^2} - b$ in this case is 1.100333, which is less than the maximum excess 1.1005.

If it be considered possible to interpret the rule for the case $n = 0$, then the cube root contains two figures, which belong wholly to the first part of the root; this can be made as small as possible by putting it = 10; in this case b must be considered to vanish, and c will be nearly = 1. Take, therefore,

$$N = 11^3 - 1 = 1330,$$

$$N = 1330,$$

$$\sqrt[3]{N} = 10.9 \dots,$$

$$a = 10,$$

$$b = 0$$

[b is considered to vanish if rule be interpretable],

$$\sqrt[3]{N} - a = 0.9 \dots,$$

$$\frac{N - a^3}{3a^2} = 1.1.$$

The integral part of $\sqrt[3]{N} - a$ is 0, but that of $\frac{N - a^3}{3a^2}$ is 1; and the rule, if interpretable, fails:

February 10th, 1887.

Sir JAMES COCKLE, F.R.S., President, in the Chair.

Mr. J. R. Holt, B.A., T.C.D., J.P. Banda, India, was elected a Member.

The following communications were made :—

On the Equation of Riccati: the President (Prof. Hart taking the chair *pro tem.*).

The "Orthocentroidal" Triangle: R. Tucker.

On Polygons inscribed in a Quadric and circumscribed about two Confocal Quadrics: R. A. Roberts.

On the Binomial Equation $x^p - 1 = 0$, Quinquisection: Prof. Lloyd Tanner.

Symmetrical Determinant Formulæ in Elliptic Functions: L. J. Rogers.

A Note on Curves: H. M. Taylor.

Some Generalisations of Formulæ connected with the change of the Independent Variable in a Differential Expression, with application to a new class of Reciprocants: C. Leudesdorf.

On the initiative of Prof. Sylvester, the wish was strongly expressed by the members present that, either authors should endeavour to be present to make their communications *viva voce*, or, failing their ability to be present, should send concise but sufficiently full abstracts of their papers, so as to enable the members to appreciate the gist of the communications.

The following presents were received :—

"Proceedings of the Royal Society," Vol. xli., Nos. 248 and 249.

"Educational Times," for February.

"Proceedings of the Cambridge Philosophical Society," Vol. v., Part vi.; Camb., 1886.

"Bulletin des Sciences Mathématiques," Tome xi., Jan., 1887.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. xi., St. 1.

"Bollettino delle Pubblicazioni Italiane," ricevute per diritto di Stampa, Nos. 25 and 26.

"Sitzungsberichte der Physikalisch-medizinischen Societät zu Erlangen," 18 Heft, 1886.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. ii., Fasc. 12; Roma, 1886.