focus lenses, especially if the polish is impaired, the central spot may be somewhat distorted, but its centre can be found pretty accurately by taking the semi-diameter of the outer rings. The angle between the planes may conveniently be fixed so that chord $\theta = 0{\cdot}1$ or thereabouts. This will serve for the majority of lenses, but the apparatus is so simple that several gauges of different angles may be kept ready.

I have not found any inconvenience from the use of marine glue to fasten the plates together, any variation of the angle that may result from changes of temperature being too small to affect the result appreciably.

21, Norham Road, Oxford,
January 1897.

XXXVII. *On the Passage of Waves through Apertures in Plane Screens, and Allied Problems.* *By* Lord Rayleigh, *F.R.S.**

THE waves contemplated may be either aerial waves of condensation and rarefaction, or electrical waves propagated in a dielectric. Plane waves of simple type impinge upon a parallel screen. The screen is supposed to be infinitely thin, and to be perforated by some kind of aperture. Ultimately one or both dimensions of the aperture will be regarded as infinitely small in comparison with the wave-length (λ); and the method of investigation consists in adapting to the present purpose known solutions regarding the flow of incompressible fluid.

If ϕ be a velocity-potential satisfying

$$d^2\phi/dt^2 = V^2\nabla^2\phi, \quad \ldots\ldots \quad (1)$$

where

$$\nabla^2 = d^2/dx^2 + d^2/dy^2 + d^2/dz^2,$$

the condition at the boundary may be (i.) that $d\phi/dn = 0$, or (ii.) that $\phi = 0$. The first applies directly to aerial vibrations impinging upon a fixed wall, and in this connexion has already been considered †.

If we assume that the vibration is everywhere proportional to e^{int}, (1) becomes

$$(\nabla^2 + k^2)\phi = 0, \quad \ldots\ldots \quad (2)$$

where

$$k = n/V = 2\pi/\lambda. \quad \ldots\ldots \quad (3)$$

It will conduce to brevity if we suppress the factor e^{int}.

* Communicated by the Author.

† 'Theory of Sound,' § 292.

On this understanding the equation of waves travelling parallel to x in the positive direction, and accordingly incident upon the negative side of a screen situated at $x=0$, is

$$\phi=e^{-ikx}. \quad \dots \dots \dots \quad (4)$$

When the solution is complete, the factor e^{int} is to be restored, and the imaginary part of the solution is to be rejected. The realized expression for the incident waves will therefore be

$$\phi=\cos(nt-kx). \quad \dots \dots \dots \quad (5)$$

Perforated Screen.—Boundary Condition $d\phi/dn=0$.

If the screen be complete, the reflected waves under the above condition have the expression $\phi=e^{ikx}$.

Let us divide the actual solution into two parts χ and ψ, the first the solution which would obtain were the screen complete, the second the alteration required to take account of the aperture; and let us distinguish by the suffixes m and p the values applicable upon the negative (*minus*) and upon the positive side of the screen. In the present case we have

$$\chi_m=e^{-ikx}+e^{ikx}, \qquad \chi_p=0. \quad \dots \dots \quad (6)$$

This χ-solution makes $d\chi_m/dn=0$, $d\chi_p/dn=0$ over the whole plane $x=0$, and over the same plane $\chi_m=2$, $\chi_p=0$.

For the supplementary solution, distinguished in like manner upon the two sides, we have

$$\psi_m=\iint\Psi_m\frac{e^{-ikr}}{r}\,dS, \qquad \psi_p=\iint\Psi_p\frac{e^{-ikr}}{r}\,dS, \quad \dots \quad (7)$$

where r denotes the distance of the point at which ψ is to be estimated from the element dS of the aperture, and the integration is extended over the whole of the area of aperture. Whatever functions of position Ψ_m, Ψ_p may be, these values on the two sides satisfy (2), and (as is evident from symmetry) they make $d\psi_m/dn$, $d\psi_p/dn$ vanish over the wall, viz. the unperforated part of the screen; so that the required condition over the wall for the complete solution $(\chi+\psi)$ is already satisfied. It remains to consider the further conditions that ϕ and $d\phi/dx$ shall be continuous across the *aperture*.

These conditions require that on the aperture

$$2+\psi_m=\psi_p, \qquad d\psi_m/dx=d\psi_p/dx. \quad \dots \quad (8)*$$

* The use of dx implies that the variation is in a fixed direction, while dn may be supposed to be drawn outwards from the screen in both cases.

The second is satisfied if $\Psi = -\Psi_m$; so that

$$\psi_m = \iint \Psi_m \frac{e^{-ikr}}{r} dS, \qquad \psi_p = -\iint \Psi_m \frac{e^{-ikr}}{r} dS, \quad . \quad (9)$$

making the values of ψ_m and ψ_p equal and opposite at all corresponding points, viz. points which are images of one another in the plane $x=0$. In order further to satisfy the first condition it suffices that over the area of aperture

$$\psi_m = -1, \qquad \psi_p = 1, \quad . \quad . \quad . \quad . \quad (10)$$

and the remainder of the problem consists in so determining Ψ_m that this shall be the case.

In this part of the problem we limit ourselves to the supposition that all the dimensions of the aperture are small in comparison with λ. For points at a distance from the aperture e^{-ikr}/r may then be removed from under the sign of integration, so that (9) becomes

$$\psi = \frac{e^{-ikr}}{r} \iint \Psi_m dS, \qquad \psi_p = -\frac{e^{-ikr}}{r} \iint \Psi_m dS. \quad . \quad (11)$$

The significance of $\iint \Psi_m dS$ is readily understood from an electrical interpretation. For in its application to a point, itself situated upon the area of aperture, e^{-ikr} in (9) may be identified with unity, so that ψ_m is the potential of a distribution of density Ψ_m on S. But by (10) this potential must have the constant value -1; so that $-\iint \Psi_m dS$, or $\iint \Psi_p dS$, represents the electrical *capacity* of a conducting disk having the size and shape of the aperture, and situated at a distance from all other electrified bodies. If we denote this by M, the solution applicable to points at a distance from the aperture may be written

$$\psi_m = -M\frac{e^{-ikr}}{r}, \qquad \psi_p = M\frac{e^{-ikr}}{r}. \quad . \quad . \quad (12)$$

To these are to be added the values of χ in (6). The realized solutions are accordingly

$$\phi_m = 2\cos nt \cos kx - M\frac{\cos(nt-kr)}{r}, \quad . \quad . \quad (13)$$

$$\phi_p = M\frac{\cos(nt-kr)}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The value of M may be expressed* for an ellipse of semi-

* 'Theory of Sound,' §§ 292, 306, where is given a discussion of the effect of ellipticity when area is given.

major axis a and eccentricity e. We have

$$M=\frac{a}{F(e)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

F being the symbol of the complete elliptic function of the first kind. When $e=0$, $F(e)=\frac{1}{2}\pi$; so that for a circle $M=2a/\pi$.

It should be remarked that Ψ in (9) is closely connected with the normal velocity at dS. In general,

$$\frac{d\psi}{dx}=\iint \Psi \frac{d}{dx}\left(\frac{e^{-ikr}}{r}\right)dS. \quad . \quad . \quad . \quad . \quad (16)$$

At a point (x) infinitely close to the surface, the neighbouring elements only contribute to the integral, and the factor e^{-ikr} may be omitted. Thus

$$\frac{d\psi}{dx}=-\iint \Psi \frac{x}{r^3}dS=-2\pi x\int_x^\infty \Psi \frac{rdr}{r^3}=-2\pi\Psi ;$$

or

$$\Psi=-\frac{1}{2\pi}\frac{d\psi}{dn}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$d\psi/dn$ being the normal velocity at the point of the surface in question.

Boundary Condition $\phi=0$.

We will now suppose that the condition to be satisfied on the walls is $\phi=0$, although this case has no simple application to aerial vibrations. Using a similar notation to that previously employed, we have as the expression for the principal solution

$$\chi_m=e^{-ikx}-e^{ikx}, \qquad \chi_p=0, \quad . \quad . \quad . \quad (18)$$

giving over the whole plane $(x=0)$, $\chi_m=0$, $\chi_p=0$, $d\chi_m/dx=-2ik$, $d\chi_p/dx=0$.

The supplementary solutions now take the form

$$\psi_m=\iint\frac{d}{dx}\left(\frac{e^{-ikr}}{r}\right)\Psi_m\, dS, \quad \psi_p=\iint\frac{d}{dx}\left(\frac{e^{-ikr}}{r}\right)\Psi_p\, dS. \quad (19)$$

These give on the *walls* $\psi_m=\psi_p=0$, and so do not disturb the condition of evanescence already satisfied by χ. It remains to satisfy over the *aperture*

$$\psi_m=\psi_p, \quad -2ik+d\psi_m/dx=d\psi_p/dx. \quad . \quad . \quad (20)$$

The first of these is satisfied if $\Psi_m=-\Psi_p$, so that ψ_m and ψ_p are equal at any pair of corresponding points upon the two sides. The values of $d\psi_m/dx$, $d\psi_p/dx$ are then opposite, and the remaining condition is also satisfied if

$$d\psi_m/dx = ik, \qquad d\psi_p/dx = -ik. \quad . \quad . \quad . \quad (21)$$

Thus Ψ_m is to be such as to make $d\psi_m/dx = ik$; and, as in the proof of (17), it is easy to show that in (19)

$$\Psi_m = \psi_m/2\pi, \qquad \Psi_p = -\psi_p/2\pi, \quad . \quad . \quad . \quad (22)$$

where ψ_m, ψ_p are the (equal) surface-values at dS.

When all the dimensions of S are small in comparison with the wave-length, (19) in its application to points at a sufficient distance from S assumes the form

$$\psi_p = \frac{ikx\, e^{-ikr}}{2\pi r^2} \iint \psi_p\, dS, \quad . \quad . \quad . \quad . \quad (23)$$

and it only remains to find what is the value of $\iint \psi_p\, dS$ which corresponds to $d\psi_p/dx = -ik$.

Now this correspondence is ultimately the same as if we were dealing with an absolutely incompressible fluid. If we imagine a rigid and infinitely thin plate (having the form of the aperture) to move normally through unlimited fluid with velocity u, the condition is satisfied that over the remainder of the plane the velocity-potential ψ vanishes. In this case the values of ψ at corresponding points upon the two sides are opposite ; but if we limit our attention to the positive side, the conditions are the same as in the present problem. The kinetic energy of the motion is proportional to u^2, and we will suppose that twice the energy upon one side is hu^2. By Green's theorem this is equal to $-\iint \psi \,.\, d\psi/dn \,.\, dS$, or $-u\iint \psi dS$; so that $\iint \psi dS = -hu$. In the present application $u = -ik$, so that the corresponding value of $\iint \psi_p dS$ is ihk. Thus (23) becomes

$$\psi_p = -\frac{hk^2 x\, e^{-ikr}}{2\pi r^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

The same algebraic expression gives ψ_m, if the *minus* sign be omitted ; for as x itself changes sign in passing from one side to the other, the values of ψ_m and ψ_p at corresponding points are then equal.

The value of h can be determined in certain cases. For a circle* of radius c

$$h = \frac{4c^3}{3} ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

* Lamb's 'Hydrodynamics,' § 105.

so that for a circular aperture the realized solution is

$$\phi_p = -\frac{8\pi c^3}{3\lambda^2} \frac{x}{r^2} \cos(nt - kr), \quad . \quad . \quad . \quad . \quad (27)$$

$$\phi_m = 2 \sin nt \sin kx + \frac{8\pi c^3}{3\lambda^2} \frac{x}{r^2} \cos(nt - kr). \quad . \quad . \quad . \quad (28)$$

It will be remarked that while in the first problem the wave (ψ) divergent from the aperture is proportional to the first power of the linear dimension, in the present case the amplitude is very much less, being proportional to the cube of that quantity.

The solution for an elliptic aperture is deducible from the general theory of the motion of an ellipsoid (a, b, c) through incompressible fluid*, by supposing $a=0$, while b and c remain finite and unequal; but the general expression does not appear to have been worked out. When the eccentricity of the residual ellipse is small, I find that

$$h = \tfrac{4}{3}(bc)^{\frac{3}{2}}(1 - \tfrac{3}{64}e^4), \quad . \quad . \quad . \quad . \quad . \quad (29)$$

showing that the effect of moderate ellipticity is very small when the *area* is given.

From the solutions already obtained it is possible to derive others by differentiation. If, for example, we take the value of ϕ in the first problem and differentiate it with respect to x, we obtain a function which satisfies (2), which includes plane waves and their reflexion on the negative side, and which satisfies over the wall the condition of evanescence. It would seem at first sight as if this could be no other than the solution of the second problem, but the manner in which the linear dimension of the aperture enters suffices to show that it is not so. The fact is that although the proposed function vanishes over the plane part of the wall, it becomes infinite at the *edge*, and thus includes the action of *sources* there distributed. A similar remark applies to the solutions that might be obtained by differentiation of the second solution with respect to y or z, the coordinates measured parallel to the plane of the screen.

Reflecting Plate.—$d\phi/dn=0$.

We now pass to the consideration of allied problems in which the transparent and opaque parts of the screen are interchanged. Under the above-written boundary condition

* Lamb's 'Hydrodynamics,' § 111.

the case is that of plane aerial waves incident upon a parallel infinitely thin plate, whose dimensions are ultimately supposed to be small in comparison with λ. The analytical process of solution may be illustrated by the following argument. Suppose a motion communicated to the plate identical with that which the air at that place would execute were the plate absent. It is evident that the propagation of the primary wave will then be undisturbed. The supplementary solution, representing the disturbance due to the plate, must then correspond to the reduction of the plate to rest, that is to a motion of the plate equal and opposite to that just imagined. The supplementary solution is accordingly analogous to that which occurs in the *second* of the problems already treated.

Using a similar notation, we have for the principal solution upon the two sides

$$\chi_m=\chi_p=e^{-ikx}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

giving when $x=0$

$$\chi_m=\chi_p=1, \qquad d\chi_m/dx=d\chi_p/dx=-ik.$$

The supplementary solution is of the form (19), and gives upon the aperture, viz. the part of the plane $x=0$ unoccupied by the plate, $\psi_m=\psi_p=0$, and so does not disturb the continuity of ϕ. But in order that the continuity of $d\phi/dx$ may be maintained it is necessary that $\Psi_p=\Psi_m$; and then the values of ψ_m and ψ_p are *opposite* at any pair of corresponding points upon the two sides.

It remains to satisfy the necessary conditions at the plate itself. These are

$$\frac{d\chi_m}{dx}+\frac{d\psi_m}{dx}=0, \qquad \frac{d\chi_p}{dx}+\frac{d\psi_p}{dx}=0\;;$$

or, since $d\psi_m/dx$, $d\psi_p/dx$ are equal,

$$d\psi_m/dx=d\psi_p/dx=ik. \quad . \quad . \quad . \quad . \quad . \quad (31)$$

It follows that ψ_p has the opposite value to that expressed in (25); and the realized solution for a circular plate of radius c becomes

$$\phi_p=\cos(nt-kx)+\frac{8\pi c^3}{3\lambda^2}\frac{x}{r^2}\cos(nt-kr), \quad . \quad . \quad . \quad . \quad (32)$$

$$\phi_m=\cos(nt-kx)+\frac{8\pi c^3}{3\lambda^2}\frac{x}{r^2}\cos(nt-kr), \quad . \quad . \quad . \quad . \quad (33)$$

the analytical form being the same in the two cases.

It is important to notice that the reflexion from the plate is utterly different from the transmission by a corresponding aperture in an opaque screen, as given in (14), the former varying as the cube of the linear dimension, and the latter as the first power simply.

Reflecting Plate.—$\phi=0$.

For the sake of completeness it may be well to indicate the solution of a fourth problem defined by the above heading. This has an affinity with the *first* problem, analogous to that of the third with the second. The form of χ is the same as in (30), and those for ψ_m, ψ_p the same as in (7). These make $d\psi_m/dx$, $d\psi_p/dx$ vanish on the aperture, and so do not disturb the continuity of $d\phi/dx$. But in order that the continuity of ϕ may also be maintained, we must have $\Psi_m=\Psi_p$, and not as in (9) $\Psi_m=-\Psi_p$. On the plate itself we must have

$$\psi_m=\psi_p=-1.$$

Accordingly ψ_m is the same as in (12), while ψ_p in (12) must have its sign reversed. The realized solution is

$$\phi_p=\phi_m=\cos(nt-kx)-\mathrm{M}\frac{\cos(nt-kr)}{r}. \quad . \quad (34)$$

Two-dimensional Vibrations.

In the class of problems before us the velocity-potential of a point-source, viz. e^{-ikr}/r, is replaced by that of a linear source; and this in general is much more complicated. If we denote it by $\mathrm{D}(kr)$, the expressions are*

$$\mathrm{D}(kr)=-\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}}e^{-ikr}\left\{1-\frac{1^2}{1.8ikr}+\frac{1^2.3^2}{1.2.(8ikr)^2}-\ldots\right\}$$
$$=\left(\gamma+\log\frac{ikr}{2}\right)\left\{1-\frac{k^2r^2}{2^2}+\frac{k^4r^4}{2^2.4^2}-\ldots\right\}$$
$$+\frac{k^2r^2}{2^2}\mathrm{S}_1-\frac{k^4r^4}{2^2.4^2}\mathrm{S}_2+\frac{k^6r^6}{2^2.4^2.6^2}\mathrm{S}_3-\ldots, \quad . \quad (35)$$

where γ is Euler's constant (·5772 . . .), and

$$\mathrm{S}_m=1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+1/m.$$

Of these the first is "semiconvergent," and is applicable when kr is large; the second is fully convergent and gives the form of the function when kr is small.

Since the complete analytical theory is rather complicated, it may be convenient to give a comparatively simple deriva-

* See for example 'Theory of Sound,' § 341.

tion of the extreme forms, which includes all that is required for our present purpose, starting from the conception of a linear source as composed of distributed point-sources. If ρ be the distance of any element dx of the linear source from O, the point at which the potential is to be estimated, and r be the smallest value of ρ, so that $\rho^2=r^2+x^2$, we may take as the potential, constant factors being omitted,

$$\psi=-\int_0^\infty \frac{e^{-ik\rho}\,dx}{\rho}=-\int_r^\infty \frac{e^{-ik\rho}\,d\rho}{\sqrt{(\rho^2-r^2)}}. \quad . \quad (36)$$

We have now to trace the form of (36) when kr is very great, and also when kr is very small. For the former case we replace ρ by $r+y$, thus obtaining

$$\psi=-\int_0^\infty \frac{e^{-ikr}\,e^{-iky}\,dy}{\sqrt{y}\,.\,\sqrt{(2r+y)}}. \quad . \quad . \quad . \quad (37)$$

When kr is very great, the approximate value of the integral in (37) may be obtained by neglecting the variation of $\sqrt{(2r+y)}$, since on account of the rapid fluctuation of sign caused by the factor e^{-iky} we need attend only to small values of y. Now, as is known,

$$\int_0^\infty \frac{\cos x\,dx}{\sqrt{x}}=\int_0^\infty \frac{\sin x\,dx}{\sqrt{x}}=\sqrt{\left(\frac{\pi}{2}\right)},$$

so that in the limit

$$\psi=-(1-i)\sqrt{\left(\frac{\pi}{2kr}\right)}e^{-ikr}=-\sqrt{\left(\frac{\pi}{2ikr}\right)}e^{-ikr}, \quad (38)$$

in agreement with (35).

We have next to deduce the limiting form of (36) when kr is very small. For this purpose we may write it in the form

$$\psi=-\int_r^\infty \frac{e^{-ik\rho}\,d\rho}{\rho}-\int_r^\infty e^{-ik\rho}\left\{\frac{1}{\sqrt{(\rho^2-r^2)}}-\frac{1}{\rho}\right\}d\rho. \quad (39)$$

The first integral in (39) is well known. We have

$$-\int_r^\infty \frac{e^{-ik\rho}\,d\rho}{\rho}=\text{Ci}(kr)-i\{\tfrac{1}{2}\pi+\text{Si}(kr)\}$$

$$=\gamma+\log kr-\frac{k^2r^2}{2^2}+\frac{k^4r^4}{2\,.\,3\,.\,4^2}-\ldots$$

$$+i\left\{\frac{\pi}{2}-kr+\frac{k^3r^3}{2\,.\,3^2}-\ldots\right\}.$$

In the second integral of (39) the function to be integrated vanishes when ρ is great compared to r, and when ρ is not great in comparison with r, $k\rho$ is small and $e^{-ik\rho}$ may be identified with unity. Thus in the limit

$$\int_r^\infty e^{-ik\rho}\left\{\frac{1}{\sqrt{(\rho^2-r^2)}}-\frac{1}{\rho}\right\}d\rho=\left[\log\frac{\rho+\sqrt{(\rho^2-r^2)}}{\rho}\right]_r^\infty=\log 2;$$

and (39) becomes

$$\psi=\gamma+\log kr+\tfrac{1}{2}i\pi-\log 2=\gamma+\log(\tfrac{1}{2}ikr), \quad . \quad (40)$$

in agreement with (35).

When kr is extremely small (40) may be considered for some purposes to reduce to $\log kr$; but the term $\frac{1}{2}i\pi$ is required in order to represent the equality of work done in the neighbourhood of the linear source and at a great distance from it.

We may now proceed to solve four problems relative to narrow slits and reflecting blades analogous to the four already considered in which the aperture or the reflecting plate was small in *both* its dimensions in comparison with the wave-length.

Narrow Slit.—Boundary Condition $d\phi/dn=0$.

As in the former problem the principal solution is

$$\chi_m=e^{-ikx}+e^{ikx}, \qquad \chi_p=0, \quad . \quad . \quad . \quad . \quad (41)$$

making $d\chi_m/dn$, $d\chi_p/dn$ vanish over the whole plane $x=0$ and over the same plane $\chi_m=2$, $\chi_p=0$. The supplementary solution, which represents the effect of the slit, may be written

$$\psi_m=\int\Psi_m D(kr)dy, \qquad \psi_p=\int\Psi_p D(kr)dy, \quad . \quad (42)$$

Ψ_m, Ψ_p being certain functions of y to be determined, and the integration extending over the width of the slit from $y=-b$ to $y=+b$.

These additions do not disturb the condition to be satisfied over the wall. On the aperture continuity requires, as in (8), that

$$2+\psi_m=\psi_p, \qquad d\psi_m/dx=d\psi_p/dx.$$

The second of these is satisfied by taking $\Psi_p=-\Psi_m$, so that at all corresponding pairs of points $\psi_m=-\psi_p$. It remains to determine Ψ_m so that on the aperture $\psi_m=-1$; and then by what has been said $\psi_p=+1$.

At a sufficient distance from the slit, supposed to be very narrow, D(kr) may be removed from under the integral sign and also be replaced by its limiting form given in (35). Thus

$$\psi_m=-\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}}e^{-ikr}\int\Psi_m dy. \quad . \quad . \quad . \quad (43)$$

The condition by which Ψ_m is determined is that for all points upon the aperture

$$\int_{-b}^{+b} \Psi_m D(kr) dy = -1, \quad . \quad . \quad . \quad . \quad (44)$$

where, since kr is small throughout, the second limiting form given in (35) may be introduced.

From the known solution for the flow of incompressible fluid through a slit in an infinite plane we may infer that Ψ_m will be of the form $A(b^2-y^2)^{-\frac{1}{2}}$, where A is some constant. Thus (44) becomes

$$A\left[(\gamma+\log \tfrac{1}{2}ik)\pi + \int_{-b}^{+b} \frac{\log(r)dy}{\sqrt{(b^2-y^2)}}\right] = -1. \quad . \quad (45)$$

In this equation the first integral is obviously independent of the position of the point chosen, and if the form of Ψ_m has been rightly taken the second integral must also be independent of it. If its coordinate be η, lying between $\pm b$,

$$\int_{-b}^{+b} \frac{\log r\,dy}{\sqrt{(b^2-y^2)}} = \int_{-b}^{\eta} \frac{\log(\eta-y)dy}{\sqrt{(b^2-y^2)}} + \int_{\eta}^{b} \frac{\log(y-\eta)dy}{\sqrt{(b^2-y^2)}},$$

and must be independent of η. This can be verified without much difficulty by assuming $\eta = b\sin\alpha$, $y = b\sin\theta$; but merely to determine A in (45) it suffices to consider the particular case of $\eta=0$. Here

$$\int_{-b}^{+b} \frac{\log r\,dy}{\sqrt{(b^2-y^2)}} = 2\int_0^b \frac{\log y\,dy}{\sqrt{(b^2-y^2)}}$$

$$= 2\int_0^{\frac{1}{2}\pi} \log(b\sin\theta)\,d\theta = \pi\log(\tfrac{1}{2}b).$$

Thus

$$A(\gamma+\log\tfrac{1}{4}ikb)\pi = -1,$$

and

$$\int_{-}^{+b} \Psi_m dy = A\int_{-b}^{+b} \frac{dy}{\sqrt{(b^2-y^2)}} = \pi A;$$

so that (43) becomes

$$\psi_m = \frac{e^{-ikr}}{\gamma+\log(\frac{1}{4}ikb)}\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}}. \quad . \quad . \quad (46)$$

From this ψ_p is derived by simply prefixing a negative sign.

The realized solution is obtained from (46) by omitting the imaginary part after introduction of the suppressed factor e^{int}. If the imaginary part of $\log(\frac{1}{4}ikb)$ be neglected, the result is

$$\psi_m = \left(\frac{\pi}{2kr}\right)^{\frac{1}{2}} \frac{\cos(nt-kr-\frac{1}{4}\pi)}{\gamma+\log(\frac{1}{4}kb)}, \quad . \quad (47)$$

corresponding to

$$\chi_m = 2 \cos nt \cos kx. \quad . \quad . \quad . \quad . \quad . \quad (48)$$

The solution (47) applies directly to aerial vibrations incident upon a perforated wall, and to an electrical problem which will be specified later. Perhaps the most remarkable feature of it is the very limited dependence of the transmitted vibration on the *width* (2*b*) of the aperture.

Narrow Slit.—Boundary Condition $\phi = 0$.

The principal solution is the same as in (18); and the conditions for the supplementary solution, to be satisfied over the aperture, are those expressed in (21). In place of (19)

$$\psi_m = -\int \frac{d\mathrm{D}}{dx} \Psi_p \, dy, \qquad \psi_p = \int \frac{d\mathrm{D}}{dx} \Psi_p \, dy; \quad . \quad (49)$$

the values of Ψ_m and Ψ_p being opposite, and those of ψ_m and ψ_p equal at corresponding points. At a distance we have

$$\psi_p = \frac{d\mathrm{D}}{dx} \int_{-b}^{+b} \Psi_p \, dy, \quad . \quad . \quad . \quad . \quad . \quad (50)$$

in which

$$\frac{d\mathrm{D}}{dx} = \frac{ikx}{r} \left(\frac{\pi}{2ikr} \right)^{\frac{1}{2}} e^{-ikr}. \quad . \quad . \quad . \quad . \quad (51)$$

There is a simple relation between the value of Ψ_p at any point of the aperture and that of ψ_p at the same point. For in the application of (49) to any point of the narrow aperture, $d\mathrm{D}/dx = x/r^2$, showing that only those elements of the integral are sensible which lie infinitely near the point where ψ_p is to be estimated. The evaluation is effected by considering in the first instance a point for which x is finite, and afterwards passing to the limit. Thus

$$\psi_p = \int \frac{x}{x^2 + y^2} \Psi_p \, dy = \Psi_p \left[\tan^{-1} \frac{y}{x} \right]_{-b}^{+b} = \pi \Psi_p;$$

so that (50) becomes

$$\psi_p = \frac{1}{\pi} \frac{d\mathrm{D}}{dx} \int_{-b}^{+b} \psi_p \, dy. \quad . \quad . \quad . \quad . \quad (52)$$

It remains only to express the connexion between $\int \psi_p \, dy$ and the constant value of $d\psi_p/dx$ on the area of the aperture; and this is effected by the known solution for an incompressible fluid moving under similar conditions. The argument is the same as in the corresponding problem where the perforation is circular. In the motion (u) of a lamina of width ($2b$) through infinite fluid, the whole kinetic energy per unit of length may be denoted by hu^2, and it appears from Green's theorem that $\int \psi_p dy = ihk$. The value of h * is $\frac{1}{2}\pi b^2$; so that

$$\psi_p = -\frac{k^2 b^2 x}{2r}\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} e^{-ikr}. \quad . \quad . \quad . \quad . \quad (53)$$

The same algebraic expression gives ψ_m, if the *minus* sign be omitted.

The realized solution from (53) is

$$\psi_p = -\frac{k^2 b^2 x}{2r}\left(\frac{\pi}{2kr}\right)^{\frac{1}{2}} \cos\left(nt - kr - \tfrac{1}{4}\pi\right), \quad . \quad . \quad (54)$$

corresponding to

$$\chi_m = 2 \sin nt \sin kx. \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

Reflecting Blade.—Boundary Condition $d\phi/dn = 0$.

We have now to consider two problems which differ from the last in that the opaque and transparent parts of the screen are interchanged. As in the case of the circular aperture, we shall find that the correspondence lies between the reflecting blade under the condition $d\phi/dn = 0$ and the transmitting aperture under the condition $\phi = 0$, and reciprocally.

The principal solution remains as in (30). The supplementary solution must satisfy (31), where

$$\psi_m = \int \frac{dD}{dx} \Psi_p dy, \qquad \psi_p = \int \frac{dD}{dx} \Psi_p dy, \quad . \quad . \quad (56)$$

since Ψ_m and Ψ_p must be equal in order that the continuity of $d\phi/dx$ over the aperture may be maintained. Thus ψ_m and ψ_p have opposite values at any pair of corresponding points.

If we compare these conditions with those by which (53) was determined, we see that ψ_m has the same value as in that case, but that the sign of ψ_p must be reversed. Thus in the present problem

$$\psi_m = \psi_p = \frac{k^2 b^2 x}{2r}\left(\frac{\pi}{2kr}\right)^{\frac{1}{2}} \cos\left(nt - kr - \tfrac{1}{4}\pi\right), \quad . \quad (57)$$

corresponding to

$$\chi_m = \chi_p = \cos(nt - kx). \quad . \quad . \quad . \quad . \quad (58)$$

Reflecting Blade.—Boundary Condition $\phi = 0$.

In this case χ still remains as in (30). The general forms for ψ_m, ψ_p are as in (42), which secure that $d\psi_m/dx$, $d\psi_p/dx$ shall vanish on the aperture (*i. e.* the part of the plane $x = 0$ unoccupied by the blade). But in order that the continuity of ϕ may also be maintained over that area we must have $\Psi_m = \Psi_p$. Thus ψ_m, ψ_p have equal values at corresponding points. On the blade itself $\psi_m = \psi_p = -1$.

* Lamb's 'Hydrodynamics,' § 71.

A comparison of these conditions with those by which (46) was determined shows that in the present case

$$\psi_m = \psi_p = \frac{e^{-ikr}}{\gamma + \log\left(\frac{1}{4}ikb\right)} \left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}}. \quad . \quad . \quad (59)$$

When $\log i$ in the denominator of (59) may be omitted, the realized form is that expressed by (47), and this corresponds to

$$\chi_m = \chi_p = \cos(nt - kx). \quad . \quad . \quad . \quad . \quad (60)$$

Various Applications.

Of the eight problems, whose solutions have now been given, four have an immediate application to aerial vibrations, viz. those in which the condition on the walls is $d\phi/dn = 0$. The symbol ϕ then denotes the velocity-potential, and the condition expresses simply that the fluid does not penetrate the boundary. The four problems relating to two dimensions have also a direct application to electrical vibrations, if we suppose that the thin material constituting the screen (or the blade) is a perfect conductor. For if R denote the electromotive intensity parallel to z, the condition at the face of the conductor is $R = 0$; so that if R be written for ψ in (53), (59), we have the solutions for a narrow aperture in an infinite screen, and for a narrow reflecting blade respectively, corresponding to the incident wave $R = e^{-ikx}$. A narrow aperture parallel to the electric vibrations transmits very much less than is reflected by a conductor elongated in the same direction.

The two other solutions relative to two dimensions find electrical application if we identify ϕ with c, the component of magnetic intensity parallel to z. For when the other components a and b are zero, the condition to be satisfied at the face of a conductor is $dc/dn = 0$. Thus (46), (57) apply to incident vibrations represented by $c = e^{-ikx}$. In this case the slit transmits much more than the blade reflects.

It may be remarked that in general problems of electrical vibration in *two dimensions* have simple acoustical analogues*. As an example we may refer to the reflexion of plane electric waves incident perpendicularly upon a *corrugated* surface, the acoustical analogue of which is treated in 'Theory of Sound,' 2nd ed. § 272 *a*, and to the reflexion of electric waves from a conducting cylinder (§ 343).

* The comparison is not limited to the case of perfect conductors, but applies also when the obstacles, being non-conductors, differ from the surrounding medium in specific inductive capacity, or in magnetic permeability, or in both properties.