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ON THE MAGNETIC AND ELECTRIC FIELDS WHICH SPONTANEOUSLY ARISE IN A ROTATING CONDUCTING SPHERE.

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INTRODUCTION.

There are several ways in which small magnetic and electric fields may be conceived as arising in a rotating conducting body. Thus, centrifugal force will make the free electrons move away from the axis of rotation, until the electrostatic forces brought about as a result of this phenomenon restore equilibrium. The rotation of a body, a sphere for example, may accordingly, from this cause alone, be expected to result in the generation of magnetic and electric fields. Again, in so far as gravity pulls the electrons, they will tend to move towards the center of the sphere and will do so until the electrostatic forces, etc., resulting from this action restore equilibrium. This effect also provides a reason for the existence of a magnetic field. Analogous remarks result from a consideration of the potential-gradient which arises in a sphere, from causes of the nature of the Thomson effect, when the sphere is hot at the center and cold at the surface.

A very little consideration of the order of magnitude of effects such as those cited above will show, without exact mathematical analysis, that they can play no appreciable part in the explanation of the Earth's magnetism; but it is perhaps nevertheless not without interest, as a problem in pure physics, to calculate the expressions for the fields arising in this way.

The primary influences which we shall consider are those cited above, viz.:

- (1) Effects of centrifugal force on the electrons;
- (2) Effects of gravity on the electrons; and
- (3) Effects of temperature-gradient in the sphere.

It is by no means implied that these are the only agencies which may be invoked to account for the existence of a magnetic field in a rotating sphere; and the only reason for concentrating attention on these influences in particular is that they are of a type for which a calculation of the effects may be made without the introduction of much of a speculative nature. The problem will first be discussed for a case where the specific inductive capacity of the sphere is unity, and the solutions will afterwards be extended to correspond to an arbitrary specific inductive capacity.

The law to be obeyed by the free charge arising in the sphere as a result of the above influences is that expressed by Poisson's equation, which, in spherical polar coordinates and in electromagnetic units takes, for this case, the form

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) = -4\pi r^2 c^2 \rho$$

c being the velocity of light, ρ the charge density, V the potential, and r and θ the polar coordinates of a point. In addition to this, the potential must vary throughout the sphere in such a way that the electrostatic force on a free electron at any point in the sphere just balances the forces represented by the phenomena (1), (2), and (3).

Now the force on an electron as represented by (1) is independent of the distribution of charge in the sphere. A similar remark applies to the force represented by (2) except in so far as the movement of electrons from one place to another alters the resulting distribution of material density in the sphere, a consideration absolutely negligible, however, for the case in hand. Similar conclusions apply to the force represented by (3), and consequently, if we obtain three different solutions corresponding respectively to the cases where the electrostatic forces are such as to balance (1) and (2) and (3) separately, the algebraic sum of these solutions will represent the solution of the problem when (1), (2), and (3) are all considered together. It will be remembered in this connection that the sum of any number of solutions of Poisson's equation for given density distributions is itself a solution for a case where the densities are superposed.

In view of the above remarks, it will be convenient to record the solutions for each of the cases (1), (2) and (3) separately, and in order that the reader need not go through the mathematical analysis unless he should feel so disposed, the results will be given

first, in Section I, the detailed analysis pertaining thereto in Section II, and the extension of the results to the case of an arbitrary specific inductive capacity, in Section III.

SECTION I.—RESULTS OF MATHEMATICAL ANALYSIS.

C. G. S. electromagnetic units are employed throughout, and the following notation is used:

e and m , the electronic charge and mass, respectively ($e/m = 1.77 \times 10^7$ E.M.U.);

ω , the angular velocity of the sphere of radius a ;

Ω , the magnetic potential at a point outside the sphere;

V , the electrostatic potential;

H , the horizontal, and Z the vertical component of the magnetic field intensity. (In the formulæ, H and Z are measured in the directions of increasing θ and r , respectively; increase of θ corresponds to decrease of northerly latitude, and the direction of rotation of the sphere is from west to east);

Z_p , and H_e , the values of Z at the pole and H at the equator, respectively;

α , the universal constant defined by the condition that αT is the kinetic energy of a gas molecule at a temperature T ;

T_c , and T_s , the temperatures, respectively, of the center and surface of the sphere;

n , the number of free electrons per c.c., a quantity which, following the usual treatment of thermoelectric phenomena, we shall suppose to be related to the temperature by a formula of the type $n = n_0 T^p$, where p varies with the special form of theory of thermoelectric phenomena adopted, and will here be taken as $\frac{1}{2}$;

D , the density of the material of the sphere; and

G , the constant of gravitation. Strictly speaking, we are not justified in assuming that the ratio of the mass to weight for an electron is the same as for ordinary matter. Thus, if gravity does not pull the electrons, the effective value of G is zero. In the calculations made in this paper, the ordinary gravitational constant $G = 6.66 \times 10^{-8}$ C.G.S. units is employed.

It will be convenient to collect in Table 1 the main results of the calculations made in the latter portion of the paper, and for purposes of numerical illustration the values of H_e and Z_p are cal-

¹ O. W. RICHARDSON, The Electron Theory of Contact Electromotive Force and Thermoelectricity, *Phil. Mag.*, S. 6, v. 23., p. 276 (1912).

culated for a sphere of radius 20 cm., and density 8.9 (the density of copper), rotating 100 times per second, and for a sphere of the radius (6.4×10^8 cm.), mean density (5.5), and angular velocity of the Earth.

The magnetic potentials have only been calculated for points outside the sphere, although it is, of course, quite easy to write down the potentials inside the sphere when they are known outside.

The quantity I in the last column denotes the value of $\int \frac{T_c}{T_s} R^3 dT$, where R is the distance from the center to a point in the sphere. I thus depends upon the relation between the temperature and the distance from the center of the sphere. Its greatest value occurs for a case where the whole of the temperature-fall takes place in a shell near the surface, and in this case the value amounts to $a^3 (T_c - T_s)$. The values of H_e and Z_p have been calculated for this case, and for $T_c - T_s = 5000^\circ$ absolute. It will be noted that the values so calculated are independent of the size of the sphere.

When the rotation is from west to east, the magnetic field due to the gravitational effect is of a nature unlike that of the Earth, and corresponds to the existence of a north-seeking pole at the north pole of the sphere. The field due to the Thomson effect is of a nature similar to that of the Earth for the same direction of rotation.

The field resulting from the centrifugal-force effect is interesting in that the horizontal intensity shows a reversal of sign as one travels over the surface of the sphere. It is zero for the values of θ derived from the equation:

$$\frac{1}{5} - \frac{3}{14} (5 \cos^2 \theta - 1) = 0$$

i. e., for $\theta = 51^\circ.5$ and $128^\circ.5$, which correspond to latitudes $38^\circ.5$ north and south, respectively. For latitudes less than $38^\circ.5$, the horizontal intensity is in a direction opposite to that corresponding to the gravitational effect, or in the same direction as that of the Earth. The reversal is, of course, attributable to the portion of the charge which is non-uniformly distributed. The vertical intensity is also peculiar in showing a reversal in sign, not only at the equator, but also at parallels $69^\circ.8$, north and south.

The magnetic fields given in Table 1 are those which would be measured by an observer at rest with respect to the rotating

TABLE 1.²—Magnetic fields supposing observer at rest with respect to rotating sphere.

Case	Centrifugal force alone considered	Gravity alone considered	Thermal effects (Thomson effect) alone considered
Nature of charge distribution	Positive volume charge of uniform density, and equal negative surface charge of non-uniform density	Positive volume charge and equal negative surface charge, both of uniform density	Positive volume charge with density a function of distance from center, the form of the function depending on the conditions; the negative surface charge is uniformly distributed and its total amount is equal to that of the volume charge
Ω	Formula (23)	Formula (26)	Formula (29)
H	Formula (23A)	Formula (26A)	Formula (29A)
Z	Formula (23B)	Formula (26B)	Formula (29B)
H_e	Formula (23C)	Formula (26C)	Formula (29C)
Z_p	Formula (23D)	Formula (26D)	Formula (29D)
H_e small sphere ³	-2.6×10^{-18} E.M.U.	$+5.2 \times 10^{-18}$ E.M.U.	-4.9×10^{-14} E.M.U.
Z_p small sphere	$+1.1 \times 10^{-18}$ E.M.U.	$+1.0 \times 10^{-19}$ E.M.U.	-9.8×10^{-11} E.M.U.
H_e for Earth	-4.1×10^{-24} E.M.U.	$+3.8 \times 10^{-22}$ E.M.U.	-5.7×10^{-18} E.M.U.
Z_p for Earth	$+1.7 \times 10^{-24}$ E.M.U.	$+7.3 \times 10^{-22}$ E.M.U.	-1.1×10^{-17} E.M.U.

sphere. The only part which would be affected if the observer were on the sphere is that corresponding to the centrifugal-force effect, for it alone gives rise to an external electrostatic field; this is so since the volume and surface charges corresponding to the other effects are not only equal and opposite, but are distributed in a manner which involves only the distance from the center of the sphere, and not the angular coordinates.

Following the line of argument indicated by Schuster⁴, and independently by the author,⁵ if Y is the vertical component and X the horizontal component of the electric field at the surface of the sphere, the values of H_1 and Z_1 at the surface of the sphere, as measured by a moving observer, are connected with the values H and Z as measured by a fixed observer through the relations:

² When an arbitrary specific inductive capacity K is assigned to the sphere, all of the results in this table become multiplied by $3/(K+2)$, as shown in Section III.

³ Radius $a = 20$ cm.; angular velocity $\omega = 2\pi \times 100$.

⁴ A critical examination of the possible causes of terrestrial magnetism; *Proc. Phys. Soc., London*, vol. 24, p. 132, 1912.

⁵ "The Earth's magnetic field;" *Phil. Mag.*, sec. 6, vol. 24, p. 91, 1912.

$$H_1 = H - \frac{Y \omega a \sin \theta}{c^2} \quad (1)$$

$$Z_1 = Z + \frac{X \omega a \sin \theta}{c^2} \quad (2)$$

where the rotation is supposed to take place from west to east.

The external electrostatic potential for the case where centrifugal force is taken into account as shown on page 162, is equation 24,

$$V = \frac{m \omega^2 a^5}{3 e r^3} (1 - \frac{3}{2} \sin^2 \theta)$$

Hence $Y = -\frac{\partial V}{\partial r} = \frac{m \omega^2 a}{e} (1 - \frac{3}{2} \sin^2 \theta)$ on the surface of the sphere;

and $X = -\frac{1}{r} \left(\frac{\partial V}{\partial \theta} \right) = \frac{m \omega^2 a}{e} \sin \theta \cos \theta$ " " " " .

Hence from the values of H and Z given in table (1), we have, using (1) and (2),

$$H_1 = \frac{3 m \omega^3 a^2}{7 c^2 e} (\frac{1}{2} - \cos^2 \theta) \sin \theta$$

$$Z_1 = -\frac{3 m \omega^3 a^2}{7 c^2 e} (\frac{1}{2} - \sin^2 \theta) \cos \theta$$

The facts with regard to the directions of the magnetic fields are summarized in Table 2. In each case, the component H or Z is classified according as its direction is, or is not, the same as that of the Earth, for the same direction of rotation.

TABLE 2.—Comparison of types of field obtained with that of the Earth.

Influence	Observer fixed in space		Observer moving with sphere	
	H	Z	H_1	Z_1
Centrifugal force	Like Earth, lat. 0° to $38^\circ.5$ (N. and S). Unlike Earth for latitudes greater than $38^\circ.5$ (N. and S).	Like Earth, lat. 0° to $69^\circ.8$ (N. and S). Unlike Earth for latitudes greater than $69^\circ.8$ (N. and S).	Unlike Earth, lat. 0° to $26^\circ.6$ (N. and S). Like Earth for latitudes greater than $26^\circ.6$ (N. and S).	Like Earth, lat. 0° to $50^\circ.8$ (N. and S). Unlike Earth for latitudes greater than $50^\circ.8$ (N. and S).
Gravitational force	Unlike Earth	Unlike Earth	Unlike Earth	Unlike Earth
Thomson effect	Like Earth	Like Earth	Like Earth	Like Earth

It is of interest to observe that the latitudes at which reversals of the fields take place are independent of all the physical constants, w , a , m , e , c , etc.

Of course, all the fields are extremely small, as may be seen from Table 1, and a consideration of the nature of the quantities upon which they depend shows that neither in the laboratory, where large angular velocities may be produced, nor in the solar or stellar systems, where large dimensions are met with, could these fields amount to an order of magnitude sufficiently great to be measurable by laboratory apparatus. Similar remarks apply to the electrostatic field.

THE ELECTROSTATIC FIELD.

As already remarked, the only influence responsible for an external electrostatic field is the centrifugal-force effect, and the potential V arising in this way, as shown on page 162, is equation 24,

$$V = \frac{m \omega^2 a^5}{3 e r^3} (1 - \frac{3}{2} \sin^2 \theta)$$

The maximum electrostatic potential-gradient occurs at the poles, and it is there equal to $m\omega^2 a/e$. Dividing this by 10^8 to convert it to volts per cm., we find, on inserting numerical values, that the potential-gradient here amounts to 4.5×10^{-9} volts per cm. for the case of a sphere of 20 cm. radius rotating 100 times per second. For the case of a sphere of the size and angular velocity of the Earth, it amounts to 2×10^{-15} volt per cm. The field at the poles is directed upwards from the surface; it is interesting to notice, however, that except at the equator and poles the horizontal component of the potential-gradient attains a finite value.

The internal potential-gradients are of course, in the case of each problem, such that when multiplied by e they give the mechanical force to be balanced.

Thus, as far as the centrifugal-force effect is concerned, the internal potential-gradient is perpendicular to the axis of rotation, and, on the surface, at the equator, it is equal to $m\omega^2 a/(e \times 10^8)$ volts per cm., i. e., 4.5×10^{-9} volts per cm. in the case of the smaller and 2×10^{-15} volts per cm. in that of the larger sphere. It is directed outwards from the axis.

For the case where gravity alone is considered, the internal potential-gradient at the surface is $4\pi G D m a / (3 e \times 10^8)$ volts per cm. This amounts to about 2.8×10^{-20} volts per cm. for the smaller, and 5.5×10^{-13} volts per cm. for the larger sphere. The force is in such a direction as to draw positive electricity towards the center of the sphere.

In the case of the Thomson effect, the maximum internal potential-gradient, for a given difference in temperature between center and surface is determined by the way in which the temperature is distributed. On the free electron theory of thermoelectric phenomena, if P is the pressure of the electrons, the equilibrium of a volume element comprised within the radii R and $R + dR$ and within the solid angle $d\phi$ requires that

$$R^2 n e \frac{\partial V}{\partial R} dR d\phi = \frac{\partial}{\partial R} (R^2 P) dR d\phi - P \frac{\partial R^2}{\partial R} dR d\phi$$

where n is the number of electrons per c.c. The first term on the right-hand side of the above equation represents the resultant of the hydrostatic pressure on the spherical surface of the volume element, and the second term represents the component force which arises parallel to dR , as a result of the pressure on the remaining surface of the volume element. Since $P = 2anT/3$, the above equation becomes

$$\frac{\partial V}{\partial R} = \frac{2a}{3en} \frac{d(nT)}{dR} \quad (3)$$

For a case where n varies as $T^{\frac{3}{2}}$, this gives

$$\frac{\partial V}{\partial R} = \frac{5a}{3e} \frac{dT}{dR}$$

For a case where $\frac{dT}{dR}$ is constant

$$\frac{\partial V}{\partial R} = - \frac{5a(T_c - T_s)}{3ea} = - 2 \times 10^4 \frac{(T_c - T_s)}{a}$$

or $- 2 \times 10^{-4} \frac{(T_c - T_s)}{a}$ Volts per cm.

Thus the field amounts to only 2×10^{-4} volts per cm. per gradient of 1° C. per cm. The field is, of course radial, and is in a direction such as to drive positive electricity outwards from the center.

SECTION II.—MATHEMATICAL ANALYSIS PERTAINING TO
SECTION I.

We shall first quote one or two well-known results which we shall have occasion to use.

Let there be a circular ring carrying a current u , the ring being of such a size that its diameter subtends an angle 2β at the center of a sphere of radius a . Then, the magnetic potential Ω at a point outside the sphere is given by⁶

$$\Omega = 2\pi u \sin^2 \beta \sum_{n=1}^{\infty} \left(\frac{1}{n+1} \right) \frac{a^{n+1}}{r^{n+1}} P_n'(\beta) P_n(\theta) \quad (4)$$

where, if μ is written for $\cos \beta$, $P_n'(\beta) = \frac{dP_n(\mu)}{d\mu}$

and the P^s refer to the ordinary Legendre coefficients.

If M units of electric charge are distributed uniformly on the above ring, the electrostatic potential V due to this charge is, at a point outside the sphere of radius a ,

$$V = \frac{Mc^2}{r} \sum_{n=0}^{\infty} \frac{a^n}{r^n} P_n(\beta) P_n(\theta) \quad (5)$$

and, at a point inside the sphere,

$$V = \frac{Mc^2}{a} \sum_{n=0}^{\infty} \frac{r^n}{a^n} P_n(\beta) P_n(\theta) \quad (6)$$

The magnetic potential due to a sphere of radius a rotating with angular velocity ω , and charged to a surface density s is, at a point outside the sphere⁷,

$$\Omega = \frac{4\pi\omega s a^4}{3r^2} \cos \theta = \frac{E\omega a^2}{3r^2} \cos \theta \quad (7)$$

where E is the total charge on the sphere.

 PROBLEM 1.—DETERMINATION OF Ω , WHEN CENTRIFUGAL FORCE
ALONE IS CONSIDERED.

The charge distribution within the rotating sphere must satisfy Poisson's equation, which for the present case is

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) = -4\pi\rho c^2 r^2 \quad (8)$$

⁶ See MAXWELL, *Treatise on Electricity and Magnetism*, second edition, vol. 2, p. 333.

⁷ See, for example, W. SUTHERLAND, *Terr. Mag.*, v. 5, p. 74, 1900.

Since the electrical force on an electron must balance the centrifugal force, we have

$$-e \frac{\partial V}{\partial r} = m \omega^2 r \sin^2 \theta, \quad -\frac{e}{r} \left(\frac{\partial V}{\partial \theta} \right) = m \omega^2 r \sin \theta \cos \theta \quad (9)$$

where it is to be understood that in the above formulæ, *as in all formulæ in this paper*, a positive number is to be inserted for e , although the electron itself is, of course, negatively charged.

From (9) we readily find that V must be given by,

$$V = -\frac{m \omega^2 r^2}{2e} \sin^2 \theta + V_0 \quad (10)$$

where V_0 is a constant. On substituting in (8) the value of V given by (10), we obtain,

$$\rho = \frac{m \omega^2}{2 \pi c^2 e} \quad (11)$$

The charge density is thus constant throughout the sphere, and is, of course, positive.

The distribution ρ , as given by (11), will not by itself produce the value of V given by (10); indeed, it corresponds only to a part U given by

$$\begin{aligned} U &= \frac{4 \pi a^3 \rho c^2}{3a} + \frac{4}{3} \pi \rho c^2 \int_r^a \frac{r^3}{r^2} dr \\ &= \frac{m \omega^2}{3e} (3a^2 - r^2) \end{aligned} \quad (12)$$

where it will be observed that the first and second members on the right-hand side of (12) represent, respectively, the parts of U corresponding to the paths from infinity to the surface of the sphere, and from the surface to the distance r from the center. We have

$$V - U = \frac{m \omega^2 r^2}{3e} (1 - \frac{2}{3} \sin^2 \theta) - \frac{m \omega^2 a^2}{e} + V_0 \quad (13)$$

The part of V represented by $V-U$ is accounted for by the appearance, on the surface of the sphere, of a negative charge, which is the complement of the positive distribution ρ . $V-U$ must, of course, be a solution of (8) with ρ zero, i. e., of La Place's equation; and that it is so may easily be verified. The total amount of negative charge on the surface of the sphere is, of course,

$$Q_s = -\frac{4}{3} \pi a^3 \rho = -\frac{2 m \omega^2 a^3}{3 c^2 e} \quad (14)$$

The distribution of negative charge on the surface of the sphere may be looked upon as made up of two parts, a part distributed with variable surface density σ in such a way as to produce the

term $\frac{m \omega^2 r^2}{3 e} (1 - \frac{2}{3} \sin^2 \theta)$ in the expression for $V-U$, and a part uniformly distributed, which accounts for the contribution $-\frac{m \omega^2 a^2}{e} + V_0$ in (13). The fact that the total surface charge

has to be equal to the value given by (14) is, of course, the determining factor in fixing V_0 .

Now using (6) with M replaced by $2 \pi a^2 \sin \beta d\beta$ to correspond to the range β to $\beta + d\beta$, and remembering that $1 - \frac{2}{3} \sin^2 \theta = P_2(\theta)$, we see that σ must be such that

$$\begin{aligned} \frac{m \omega^2 r^2}{3 e} P_2(\theta) &= 2 \pi a c^2 \int_0^\pi \sigma \sin \beta \sum_{n=0}^{\infty} \frac{r^n}{a^n} P_n(\beta) P_n(\theta) d\beta \\ &= 2 \pi a c^2 \sum_{n=0}^{\infty} \frac{r^n}{a^n} P_n(\theta) \int_{-1}^{+1} \sigma P_n(\mu) d\mu \end{aligned} \quad (15)$$

If this equation is to hold, the integral must be zero for all values of n except $n = 2$, and this restricts σ to be proportional to $P_2(\mu)$. Writing $\sigma = k P_2(\mu)$, k readily becomes determined from (15), and we find

$$\sigma = \frac{5 m \omega^2 a}{12 \pi c^2 e} P_2(\mu) \quad (16)$$

Since, as may readily be verified, the integration of $P_2(\mu)$ over the surface of the sphere leads to a zero result, we see that σ contributes nothing to the total charge on the surface, so that the uniformly distributed portion must account for the whole of the value of Q_s given by (14), and so must be equal, but, of course, opposite in sign, to the total volume charge.

Thus, as far as the effect of centrifugal force is concerned, equilibrium is attained by the appearance of⁸

(A) A volume charge of constant density $\rho = \frac{m \omega^2}{2 \pi c^2 e}$;

(B) A surface charge of uniform density and total amount $-\frac{2 m \omega^2 a^3}{3 c^2 e}$ plus a surface charge of density $\sigma = \frac{5 m \omega^2 a}{12 \pi c^2 e} P_2(\mu)$, and consequently of total amount zero.

Using (7), we find for the magnetic potential $d\Omega_1$ due to the rotation of the portion of the volume charge comprised between the radii R and $R + dR$

$$d\Omega_1 = \frac{4 \pi \omega \rho R^4}{3 r^2} \cos \theta dR \quad (17)$$

Hence, the potential Ω_1 due to the whole volume distribution, is

$$\Omega_1 = \frac{4 \pi \rho \omega a^5}{15 r^2} \cos \theta$$

or in terms of the total volume charge Q

$$\Omega_1 = \frac{Q \omega a^2}{5 r^2} \cos \theta \quad (18)$$

Substituting the value of Q as determined above, we have

$$\Omega_1 = \frac{2 m \omega^3 a^5}{15 c^2 e r^2} \cos \theta \quad (19)$$

The potential Ω_2 due to the portion of the surface charge which is uniformly distributed and of total amount $-2 m \omega^2 a^3 / 3 c^2 e$ is, using (7)

$$\Omega_2 = -\frac{2 m \omega^3 a^5}{9 c^2 e r^2} \cos \theta \quad (20)$$

In order to obtain the potential Ω_3 due to the portion σ of the surface distribution, we must use (4), replacing u by $\omega a^2 \sigma \sin \beta d\beta$, and integrating from $\beta = 0$ to π . Thus, using (16)

$$\Omega_3 = \frac{5 m \omega^3 a^3}{6 c^2 e} \sum_{n=1}^{\infty} \left(\frac{1}{n+1} \right) \frac{a^{n+1}}{r^{n+1}} \int_0^\pi P_n(\beta) P_n'(\beta) \sin^3 \beta d\beta \quad (21)$$

⁸ It may be remarked that, although an electron in the body of the sphere would be in equilibrium under the above forces, the surface charge *itself* would not be in equilibrium under these forces alone. This does not affect our calculations, however; for a little consideration will show that the force which, as far as the above influences are concerned, is unbalanced, is directed along the outwardly drawn normal to the surface, and it becomes balanced by the attraction which the matter exerts in preventing the electricity from leaving the sphere. The point is, in fact, exactly analogous to that met with in the case of a stationary and uniformly charged conducting sphere, where, although the force inside the sphere, due to the surface charge, is zero, there is nevertheless a force on the surface charge itself of amount $2\pi \sigma^2$ per unit of area.

On making use of the relations

$$\int_{-1}^{+1} P_m(\mu) P_n(\mu) d\mu = 0 \text{ for } n \neq m$$

$$= \frac{2}{2n+1} \text{ for } n = m$$

it may readily be shown that the integral in (21) is zero except for $n = 1$ and $n = 3$, in which cases it amounts to $-4/15$ and $+24/35$, respectively. Hence, on substituting the values,

$$P_1(\theta) = \cos \theta \text{ and } P_3(\theta) = \frac{1}{2} (5 \cos^3 \theta - 3 \cos \theta)$$

in (21), we readily obtain

$$\Omega_s = -\frac{m \omega^3 a^3}{3 c^2 e} \left\{ \frac{a^2}{3 r^2} \cos \theta - \frac{3 a^4}{14 r^4} (5 \cos^3 \theta - 3 \cos \theta) \right\} \quad (22)$$

Hence, from (19), (20), and (22), we obtain for the total magnetic potential:

$$\Omega = -\frac{m \omega^3 a^3}{c^2 e} \left\{ \frac{a^2}{5 r^2} \cos \theta - \frac{a^4}{14 r^4} (5 \cos^3 \theta - 3 \cos \theta) \right\} \quad (23)$$

The corresponding horizontal and vertical components (H and Z) of the field are:

$$H = -\frac{m \omega^3 a^3}{c^2 e} \left\{ \frac{a^2}{5 r^3} - \frac{3 a^4}{14 r^5} (5 \cos^2 \theta - 1) \right\} \sin \theta \quad (23A)$$

$$Z = -\frac{m \omega^3 a^3}{c^2 e} \left\{ \frac{2 a^2}{5 r^3} - \frac{2 a^4}{7 r^5} (5 \cos^2 \theta - 3) \right\} \cos \theta \quad (23B)$$

The value of H at the equator, and on the surface, and the value of Z at the pole, and on the surface are respectively:

$$H_e = -\frac{29 m \omega^3 a^2}{70 c^2 e} \quad (23C)$$

$$Z_p = \frac{6 m \omega^3 a^2}{35 c^2 e} \quad (23D)$$

PROBLEM 2.—DETERMINATION OF THE EXTERNAL ELECTROSTATIC
POTENTIAL WHEN CENTRIFUGAL FORCE ALONE
IS CONSIDERED.

The external potential arises entirely from the non-uniformly distributed portion of the surface charge represented by σ , since the uniformly distributed part is just equal in total amount to the uniformly distributed volume charge. In order to obtain the external potential V arising from the distribution σ , we use (5), replacing M by $2 \pi a^2 \sigma \sin \beta d\beta$, and integrate from $\beta = 0$ to π . Thus, using (16)

$$V = \frac{5 m \omega^2 a^3}{6 e r} \sum_{n=0}^{\infty} \frac{a^n}{r^n} P_n(\theta) \int_0^{\pi} \sin \beta P_2(\beta) P_n(\beta) d\beta$$

Putting $\cos \beta = \mu$, we see that the integral becomes

$$\int_{-1}^{+1} P_2(\mu) P_n(\mu) d\mu,$$

which is zero, unless $n = 2$, in which case it is $2/5$. Hence remembering that $P_2(\theta) = 1 - 3/2 \sin^2 \theta$, we have

$$V = \frac{m \omega^2 a^5}{3 e r^3} (1 - \frac{3}{2} \sin^2 \theta) \quad (24)$$

PROBLEM 3.—DETERMINATION OF Ω , WHEN GRAVITY ALONE
IS CONSIDERED.

Using the notation defined on page 151, and subject to the restriction there cited with regard to G , the gravitational force on an electron at a distance R from the center of the sphere is $4 \pi R D G m / 3$. It is thus possible to balance this force by a uniform distribution of negative charge throughout the sphere, the density ρ_0 of the charge being given by,

$$\frac{4}{3} \pi R D G m = - \frac{4}{3} \pi R \rho_0 C^2 e$$

Thus

$$\rho_0 = - \frac{D G m}{C^2 e}$$

The total negative volume-charge corresponding to this distribution is

$$Q_0 = - \frac{4 \pi D a^3 G m}{3 C^2 e} \quad (25)$$

and the condition for neutrality of the sphere as a whole is satisfied by placing an equal positive charge on the surface. If this charge is uniformly distributed, it will produce no force inside the sphere, and so will not alter the conclusions recorded above with regard to the equilibrium of the gravitational and electrical forces.⁹

From (18) we see that the magnetic potential arising from the charge Q_0 is

$$\bar{\Omega}_1 = - \frac{4\pi D G m \omega a^5}{15 c^2 e r^2} \cos \theta$$

and from (7) the magnetic potential arising from the surface charge is

$$\bar{\Omega}_2 = \frac{4\pi D G m \omega a^5}{9 c^2 e r^2} \cos \theta$$

and the resultant potential is

$$\bar{\Omega} = \frac{8\pi D G m \omega a^5}{45 c^2 e r^2} \cos \theta \quad (26)$$

Of course, the volume and surface charges together give rise to no electrostatic potential at a point outside the sphere.

The corresponding values of H , Z , H_e , and Z_p are:

$$\bar{H} = \frac{8\pi D G m \omega a^5}{45 c^2 e r^3} \sin \theta \quad (26A)$$

$$\bar{Z} = \frac{16\pi D G m \omega a^5}{45 c^2 e r^3} \cos \theta \quad (26B)$$

$$\bar{H}_e = \frac{8\pi D G m \omega a^2}{45 c^2 e} \quad (26C)$$

$$\bar{Z}_p = \frac{16\pi D G m \omega a}{45 c^2 e} \quad (26D)$$

PROBLEM 4.—DETERMINATION OF $\bar{\Omega}$ WHEN EFFECTS OF THE NATURE OF THE "THOMSON EFFECT" ALONE ARE CONSIDERED.

As shown on page 156, the equilibrium of the electrons in the volume element $R^2 dR d\psi$ requires that the potential-gradient shall satisfy the equation (3). Since $\frac{\partial V}{\partial R}$ is independent of the

⁹See footnote 8, page 160.

angular coordinates, it will be provided for if, within the radius R , there is a charge Q_R given by $\frac{c^2 Q_R}{R^2} = -\frac{\partial V}{\partial R}$. Thus, using (3),

$$Q_R = -\frac{2\alpha R^2}{3c^2 e n} \frac{d(nT)}{dR} \quad (27)$$

The total surface-charge will, of course, be equal and opposite to the volume charge, so that since both are *symmetrically* distributed with respect to the center (although the volume charge is not *uniformly* distributed), there will be no external electric field.

Since the charge contained within the shell $4\pi R^2 dR$ is $\frac{dQ}{dR}$ the magnetic potential $d\bar{\Omega}_1$, resulting from the rotation of this shell is, using (7):

$$d\bar{\Omega}_1 = \frac{\omega R^2}{3r^2} \frac{dQ_R}{dR} \cos\theta dR$$

and the magnetic potential due to the rotation of the whole volume charge is, after integrating by parts:

$$\bar{\Omega}_1 = \frac{\omega a^2 Q_a}{3r^2} \cos\theta - \frac{2\omega \cos\theta}{3r^2} \int_0^a R Q_R dR \quad (28)$$

where Q_a is the total volume charge.

Again, using (7), the magnetic potential due to the surface charge $-Q_a$ is seen to be equal in magnitude but opposite in sign to the first term of (28), so that we have for the total potential $\bar{\Omega}$ resulting from volume and surface charges,

$$\bar{\Omega} = -\frac{2\omega \cos\theta}{3r^2} \int_0^a R Q_R dR$$

Using (27), we obtain, after changing the variable from R to T ,

$$\bar{\Omega} = -\frac{4\alpha \omega \cos\theta}{9c^2 e r^2} \int_{T_s}^{T_c} \frac{R^3}{n} \frac{d(nT)}{dT} dT$$

where T_c and T_s are respectively the temperatures of the center and surface of the sphere. We cannot proceed further without assigning some manner in which T is to vary with R , and assuming a relation between n and the temperature. If, following certain considerations resulting from the free electron theory of thermo-

electric phenomena,¹⁰ we assume n proportional to T^p , where p is a constant which varies somewhat with the form of theory adopted, we obtain,

$$\bar{\Omega} = - \frac{4(p+1)\alpha\omega\cos\theta}{9c^2er^2} \int_{T_s}^{T_c} R^3 dT \quad (29)$$

Writing I for the integral, the corresponding values of H , Z , H_e , and H_p are:

$$\bar{H} = - \frac{4(p+1)\alpha\omega}{9c^2er^3} I \sin\theta \quad (29A)$$

$$\bar{Z} = - \frac{8(p+1)\alpha\omega}{9c^2er^3} I \cos\theta \quad (29B)$$

$$\bar{H}_e = - \frac{4(p+1)\alpha\omega}{9c^2ea^3} I \quad (29C)$$

$$\bar{Z}_p = - \frac{8(p+1)\alpha\omega}{9c^2ea^3} I \quad (29D)$$

SECTION III.—EXTENSION OF THE RESULTS TO THE CASE WHERE THE SPECIFIC INDUCTIVE CAPACITY IS NOT UNITY.

Let K be the specific inductive capacity of the sphere. The actual force on an electron in the body of the sphere is now no longer the force derivable from the free volume and surface charges. If E is the electric intensity in the dielectric, and P the polarization (moment per unit volume), the force on unit charge is

$$F = E + 4\pi\gamma P$$

where γ is a constant, less than unity, depending on the nature of the medium. For isotropic bodies we may take γ as roughly $\frac{1}{3}$,¹¹ so that since $KE = E + 4\pi P$, we have

$$F = E + \frac{(K-1)E}{3} = \frac{(K+2)E}{3}$$

Hence, in order to balance the mechanical forces, it is now necessary that the value of E at each point on the sphere shall be $3/(K+2)$ times the value appropriate for the case $K=1$ already discussed. Poisson's equation for the present case becomes

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial V}{\partial\theta} \right) = - \frac{4\pi r^2 c^2 \rho_0}{K}$$

¹⁰ See O. W. RICHARDSON, The electron theory of contact electromotive force and thermoelectricity, *Phil. Mag.*, ser. 6, vol. 23, p. 276, 1912.

¹¹ See H. A. LORENTZ, *Theory of Electrons*, p. 139.

where ρ_0 is the true volume density at a point in the sphere. Hence, since, at each point of the sphere, V/r and V/θ must now have values equal to $3/(K+2)$ times the values for $K=1$, we see that if ρ is the volume density for $K=1$, Poisson's equation will be satisfied for any value of K by taking

$$\frac{\rho_0}{K} = \frac{3\rho}{K+2}$$

Now according to the well-known theory of dielectric action, the portion of a dielectric medium enclosed within a boundary S acts, as regards the potential at external points, or at points in a hole within the boundary, as though it comprised a volume charge of density $-\left(\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}\right)$ at each point of its volume, and a surface charge of surface density Pn at each point of its surface, including the surface of the hole, if there be one. Pn is here the component of P in the direction of the outwardly drawn normal. At a point within the dielectric, E is defined as the field obtained in a small cavity surrounding the point, after subtracting the part contributed by the "fictitious" charge on the surface of the cavity.

If ρ_1 refers to the fictitious volume density, we have

$$\rho_1 = -\left(\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}\right) = -\frac{(K-1)}{4\pi K} \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right)$$

where B_x, B_y, B_z are the components of the electric induction.

$$\text{But, } \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 4\pi\rho_0$$

$$\text{Hence, } \rho_1 = \frac{(1-K)}{K}\rho_0$$

$$\text{and, } \rho_1 + \rho_0 = \frac{\rho_0}{K} = \frac{3\rho}{(K+2)} \quad (29)$$

Hence, if σ refers to the surface density for the case $K=1$, and if σ_0 and σ_1 refer respectively to the true and "fictitious" surface densities for the present value of K , we readily see that by writing

$$\sigma_1 + \sigma_0 = \frac{3\sigma}{(K+2)} \quad (30)$$

the value of $\sigma_1 + \sigma_0$ so assigned will be just of the right type to provide for the necessary forces not provided for by the volume

charge. In fact, calling the sum of the true and fictitious charge densities the "virtual" charge density, we shall now have a virtual volume charge density of amount $3\rho/(K+2)$, and a virtual surface charge density of amount $3\sigma/(K+2)$, and these will provide for fields $3/(K+2)$ times those provided for by the distributions ρ and σ in the case $K=1$, which is just what is required.

Further, our choice of $\sigma_1 + \sigma_0$ as given by (30) provides for the compensation of total volume and surface charges. For the equality of the volume integral of ρ and the negative of the surface integral of σ has been provided for in the problems for the case $K=1$, and equations (29) and (30) show that this equality extends to the volume integral of $\rho_1 + \rho_0$ and the negative of the surface integral of $\sigma_1 + \sigma_0$. Again, the volume integral of ρ_1 is equal to the negative of the surface integral of σ_1 , since

$$\iiint \left(\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z} \right) dx dy dz = \iint P_n dS$$

Hence, the volume integral of ρ_0 is equal to the negative of the surface integral of σ_0 ¹².

We have thus seen that the virtual volume and surface distribution are $3/(K+2)$ times those found for the case $K=1$. Hence, the external and internal electric fields are $3/(K+2)$ times those found for the case $K=1$.

The magnetic field is produced partly by the motion of the true charges and partly by the motion of the polarized dielectric. It is not immediately obvious that the latter part is accounted for by the rotation of the fictitious charges, but that it is so accounted for may be seen from the following analysis:

The magnetic field may be expressed as the curl of the vector-potential corresponding to the motion of all the electricity in the system. If r is the distance from a moving charge of strength e , to a point O , the vector-potential at O , in any direction specified by the subscript h , and due to the moving charge is, to a first approximation, $\frac{eV_h}{r}$, where the vector V_h is the component velocity of the charge parallel to the assigned direction. If now in company with the charge e there moves another charge $-e$, at a small distance l from it, the combined contribution to the vector-potential

¹² Although $\sigma_1 + \sigma_0$ is proportional to σ , it is interesting to note that, in general, for example in the case of the problem where centrifugal force is considered, σ_1 is not proportional to σ , although ρ_1 is always proportional to ρ . The dielectric alters the types of distribution of the true surface charges, but compensates for such alteration in type by means of the fictitious charge.

is $le \frac{d}{ds} \left(\frac{V_h}{r} \right)$, where the differentiation is performed in the direction of the length of the doublet, from the negative to the positive charge.

$$\begin{aligned} \text{Now } le \frac{\partial}{\partial s} \left(\frac{V_h}{r} \right) &= le \left\{ \frac{\partial}{\partial x} \left(\frac{V_h}{r} \right) \frac{\partial x}{\partial s} + \frac{\partial}{\partial y} \left(\frac{V_h}{r} \right) \frac{\partial y}{\partial s} + \frac{\partial}{\partial z} \left(\frac{V_h}{r} \right) \frac{\partial z}{\partial s} \right\} \\ &= \mu_x \frac{\partial}{\partial x} \left(\frac{V_h}{r} \right) + \mu_y \frac{\partial}{\partial y} \left(\frac{V_h}{r} \right) + \mu_z \frac{\partial}{\partial z} \left(\frac{V_h}{r} \right) \end{aligned} \quad (31)$$

where μ_x, μ_y, μ_z , are the components of polarization of the doublet. The contribution to the vector-potential by unit volume of the polarized medium is obtained by replacing μ_x, μ_y, μ_z , by P_x, P_y, P_z , and the total values of the vector-potential in the direction specified by h , and resulting from the whole dielectric inclosed within the surface S is

$$\begin{aligned} &\iiint \left\{ P_x \frac{\partial}{\partial x} \left(\frac{V_h}{r} \right) + P_y \frac{\partial}{\partial y} \left(\frac{V_h}{r} \right) + P_z \frac{\partial}{\partial z} \left(\frac{V_h}{r} \right) \right\} dx dy dz \\ &= - \iiint \frac{V_h}{r} \left(\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z} \right) dx dy dz + \iint \left(\frac{P_n V_h}{r} \right) ds \end{aligned}$$

But this is just the contribution which would be supplied by the fictitious volume and surface charges. Hence, the rotation of the sphere is represented as regards its magnetic and electrical effects by the rotation of the true and fictitious volume and surface charges, and therefore, the magnetic and electric forces may be obtained for any value of K , from the corresponding results for $K = 1$, by simply multiplying those results by $3/(K + 2)$.

The modifications introduced when the observing instruments participate in the motion of the sphere may be carried out in a manner exactly analogous to that adopted on pages 153, 154.

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