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SOME USEFUL GROUPS IN THE TEACHING OF ELEMENTARY TRIGONOMETRY.

THE operation (c) of taking the complement of an angle (a) is of order (period) two, since $90^\circ - (90^\circ - a) = a$. Similarly, the operation (s) of taking the supplement of an angle is of order two. If we perform these two operations in succession there results an operation (cs), which is of order four since it changes a into $a + 90^\circ$. Hence the group $\{c, s\}$ generated by c and s is the dihedral rotation group of order 8. This fact follows directly from the general theorem, *Two operations of order two generate the dihedral rotation group whose order is twice the order of their product.** For the sake of simplicity we give the following independent proof of this fact:

Using rectangular co-ordinates and measuring angles in the ordinary manner, it is evident that the operation c and s are equivalent to the reflections of the terminal line of a on the bisector of the first quadrant and the y -axis respectively. These operations transform into itself the square whose centre is at the origin and whose sides are parallel to the axes. Hence this square is transformed into itself by all the operations which may be obtained by combining c and s in every possible way. That is, it is transformed into itself by $\{c, s\}$. Since it is known (and can readily be verified) that the group of movements of the square is the dihedral rotation group of order 8, it follows that $\{c, s\}$ is contained in this group. It could not be of order 4 since c and s are non-commutative. As the order of a subgroup divides the order of the group, $\{c, s\}$ must be identical with the group of movements of the square.

* *Bulletin of the American Mathematical Society*, vol. 7 (1901), p. 424.

This group is of great historic interest since it is the one which enables us to reduce the solution of the general equation of the fourth degree to that of a cubic. In trigonometry it is perhaps most prominent as the complement-supplement group, while in arithmetic it appears as the group of the operations of subtracting from 2 and dividing by 2.* In elementary geometry it appears as the group of movements of the square, as was noted above. It is commonly called the octic group even if it does not have much greater claims for usefulness than several other groups of the same order—especially the quaternion group. We proceed to consider some fundamental uses of the octic group in elementary trigonometry.

The operations of this group may be written as follows:

$$\begin{array}{cccc} 1 & cs & cscs & sc \\ c & s & scs & csc. \end{array}$$

If these operations are applied to a the corresponding angles are

$$\begin{array}{cccc} a & a+90 & a+180^\circ & a-90 \\ 90^\circ-a & 180^\circ-a & 270^\circ-a & -a. \end{array}$$

It is very interesting to observe that these are the eight angles whose trigonometric functions are expressed in terms of functions of a in the early part of the text-books on elementary trigonometry. The trigonometric functions of all of these angles can be directly obtained from those of any two which correspond to generators of the octic group. For instance, if we know the trigonometric functions of the complement and the supplement of an angle, those of the other angles can be directly obtained by means of the given group symbols. We shall do this for sine and cosine†

$$\begin{aligned} \sin(a+90^\circ) &= \sin cs a = \sin ca = \cos a \\ \cos(a+90^\circ) &= \cos cs a = -\cos ca = -\sin a \\ \sin(a+180^\circ) &= \sin cs . csa = \cos csa = -\sin a \\ \cos(a+180^\circ) &= \cos cs . csa = -\sin csa = -\cos a \\ \sin(a-90^\circ) &= \sin sca = \cos sa = -\cos a \\ \cos(a-90^\circ) &= \cos sca = \sin sa = \sin a \\ \sin(270^\circ-a) &= \sin sc . sa = \sin sca = -\cos a \\ \cos(270^\circ-a) &= \cos sc . sa = -\cos sca = -\sin a \\ \sin(-a) &= \sin cs . ca = \cos csa = -\sin a \\ \cos(-a) &= \cos cs . ca = \sin csa = \cos a. \end{aligned}$$

If we represent the operation of increasing an angle by 90° by p and suppose the functions of $a+90^\circ$ as well as those of

* *Quarterly Journal of Mathematics*, vol. 37 (1905), p. 82.

† Cf. *Quarterly Journal of Mathematics*, vol 37. (1906), p. 228.

$90^\circ - a$ to be known, we may readily find those of the others in the following manner :

$$\begin{aligned}\sin(a + 180^\circ) &= \sin p^2 a = \cos pa = -\sin a \\ \cos(a + 180^\circ) &= \cos p^2 a = -\sin pa = -\cos a \\ \sin(a - 90^\circ) &= \sin p^3 a = -\sin pa = -\cos a \\ \cos(a - 90^\circ) &= \cos p^3 a = -\cos pa = \sin a \\ \sin(180^\circ - a) &= \sin cpa = \cos ca = \sin a \\ \cos(180^\circ - a) &= \cos cpa = -\sin ca = -\cos a \\ \sin(270^\circ - a) &= \sin cp^2 a = -\sin ca = -\cos a \\ \cos(270^\circ - a) &= \cos cp^2 a = -\cos ca = -\sin a \\ \sin(-a) &= \sin cp^3 a = -\cos ca = -\sin a \\ \cos(-a) &= \cos cp^3 a = \sin ca = \cos a.\end{aligned}$$

The advantages of the present point of view result from the fact that it associates the group and these eight angles in such a way that a knowledge of the one throws light on the other. If we know the octic group, we know directly what two angles may be made fundamental in the study of the others.* The necessary and sufficient condition is that the two angles correspond to generators of the octic group. In other words, they must correspond to two non-commutative operations of this group, since any two non-commutative operations of a non-abelian group of order p^3 , p being a prime number, generate the group and any two generators are non-commutative.

The properties of the octic group suggest a large number of exercises with respect to the eight angles under consideration. For instance, the functions of all of them may be expressed in terms of the functions of any other one of them just as easily as in terms of the functions of a . Since the octic group has only one invariant operation besides the identity the corresponding angle cannot be used as one of the two fundamental angles, while each of the non-invariant operations can be used as one of these two angles. If a is the angle corresponding to the identity, as above, $a + 180^\circ$ corresponds to the invariant operation of order 2.

The use of the octic group makes the work entirely analytic after the trigonometric functions of the fundamental angles are known. The main interest, however, lies in the connections which it exhibits and the comprehensive view which it affords. The teacher should be familiar with this point of view even if he does not consider it wise to present it to the beginner.

Instead of considering the operations of subtracting from 90° and 180° we might consider those of subtracting from any other

* We could not use only one fundamental angle since the octic group is non-cyclic.

angles.* For instance, if t represents the operation of subtracting from 45° while c has the same meaning as before, the operation tc is of order 8, and hence $\{c, t\}$ is the dihedral rotation group of order 16. The operations of this group and the corresponding angles may be written in the same manner as those of the octic group. In fact, it is easy to see that $\{c, t\}$ includes $\{c, s\}$, so that the results obtained above will be appear as a special case under this larger group. We proceed to give a few illustrative examples.

$$\begin{aligned}\sin(a + 45^\circ) &= \sin tca = \cos ta = \frac{1}{2}\sqrt{2}(\cos a + \sin a)^\dagger \\ \cos(a + 45^\circ) &= \cos tca = \sin ta = \frac{1}{2}\sqrt{2}(\cos a - \sin a) \\ \sin(a + 90^\circ) &= \sin(tc)^2a = \frac{1}{2}\sqrt{2}(\cos tca + \sin tca) \\ &= \frac{1}{2}(\cos a - \sin a + \cos a + \sin a) = \cos a \\ \cos(a + 90^\circ) &= \cos(tc)^2a = \frac{1}{2}\sqrt{2}(\cos tca - \sin tca) \\ &= \frac{1}{2}(\cos a - \sin a - \cos a - \sin a) = -\sin a \\ \sin(a + 180^\circ) &= \sin(tc)^4a = \cos(tc)^2a = -\sin a \\ \cos(a + 180^\circ) &= \cos(tc)^4a = -\sin(tc)^2a = -\cos a \\ \sin(a - 90^\circ) &= \sin(tc)^6a = -\sin(tc)^2a = -\cos a \\ \cos(a - 90^\circ) &= \cos(tc)^6a = -\cos(tc)^2a = \sin a \\ \sin(-a) &= \sin(tc)^2ca = \cos(tc)^2a = -\sin a \\ \cos(-a) &= \cos(tc)^2ca = \sin(tc)^2a = \cos a \\ \sin(a + 135^\circ) &= \sin(tc)^3a = \cos tca = \frac{1}{2}\sqrt{2}(\cos a - \sin a) \\ \cos(a + 135^\circ) &= \cos(tc)^3a = -\sin tca = -\frac{1}{2}\sqrt{2}(\cos a + \sin a).\end{aligned}$$

These examples seem sufficient to illustrate the method of work. The general theory of these subtraction groups is given in the articles published in the *Annals of Mathematics* to which reference is made above. The object of the method is to exhibit relations which are apt to be overlooked. It aims to clarify and intensify the knowledge of fundamental facts rather than to extend this knowledge. Some of the relations between the functions of the same angle afford another very interesting application of the group concept. The two operations of subtracting from unity and dividing unity may be represented by s_1, s_2 respectively. It is evident that each of these operations is of order 2 and that their product s_1s_2 is of order 3. Hence they generate the dihedral rotation group of order 6. This is well illustrated by the six values of an anharmonic ratio and the group of movements of the equilateral triangle.

* Cf. *Annals of Mathematics*, vol. 6 (1905), p. 41.

† Since t is one of the assumed generators of our group of order 16 the functions of the angle $(45^\circ - a)$ which corresponds to t are supposed to be known.

The six operations of $\{s_1, s_2\}$ may be represented as follows:

$$\begin{array}{ccc} 1 & s_1 s_2 & s_2 s_1 \\ s_1 & s_2 & s_1 s_2 s_1. \end{array}$$

If we perform these operations on $\sin^2 \alpha$ the resulting functions are, in order,

$$\begin{array}{ccc} \sin^2 \alpha & \sec^2 \alpha & -\cot^2 \alpha \\ \cos^2 \alpha & \csc^2 \alpha & -\tan^2 \alpha. \end{array}$$

Since the six operations in question constitute a group the same six functions are obtained if the operations are performed on any one of them. This method exhibits not only the close relations between these functions and the values of an anharmonic ratio, but it may also be regarded as a special case of a group of subtraction and division. The fundamental properties of these groups have been developed in an article published in volume 37 of the *Quarterly Journal of Mathematics*. The same volume contains other interesting examples of the application of group theory to trigonometry.

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ON HIGHER TRIGONOMETRY.

III. THE ELEMENTARY TRANSCENDENTAL FUNCTIONS.

Algebraic Lemma.

The following system of algebraic inequalities is of great value and of wide application:

$$(i) \quad 1 - a < \frac{1}{1 + a}, \quad 1 + a < \frac{1}{1 - a}$$

for all positive proper fractional values of a ;

$$(ii) \quad \prod_{r=1}^{r=n} (1 - a_r) > 1 - \Sigma_1 + \Sigma_2 - \dots - \Sigma_{2k-1}$$

$$\text{and} \quad < 1 - \Sigma_1 + \Sigma_2 - \dots + \Sigma_{2k}$$

for positive proper fractional values of $a_1, a_2, \dots, a_n$, and values of k such that $2k < n$; Σ_r denoting the sum of the r -products of the a 's.

These inequalities (ii) may be given otherwise in the form

$$\Pi(1 - a) = 1 - \Sigma_1 + \Sigma_2 \dots + (-)^{k-1} \cdot \Sigma_{k-1} + (-)^k \cdot \phi_k \cdot \Sigma_k,$$

where ϕ_k is a function of $a_1, a_2, \dots, a_n$, whose value is a positive proper fraction when positive proper fractional values are given to the a 's. In particular

$$1 - na < (1 - a)^n < 1 - na + \frac{n(n-1)}{1 \cdot 2} a^2,$$