180. Note on Simson's Line

Author(s): E. P. Rouse

Source: *The Mathematical Gazette*, Vol. 3, No. 54 (Dec., 1905), pp. 261-262

Published by: Mathematical Association

Stable URL: <http://www.jstor.org/stable/3603878>

Accessed: 16-01-2016 16:04 UTC

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at <http://www.jstor.org/page/info/about/policies/terms.jsp>

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

Mathematical Association is collaborating with JSTOR to digitize, preserve and extend access to *The Mathematical Gazette*.

<http://www.jstor.org>

180. [K. 2. a.] *Note on Simson's Line.*

A, B, C, D are points on a circle, centre O , radius R ; K, k orthocentres of the triangles ABC, DBC .

Then AK, Dk are parallel and each equal to twice the perpendicular from O on BC , hence $AKkD$ is a parallelogram, and its diagonals bisect each other in G .

The rectangular hyperbola through A, B, C, D must also pass through K and k ; so that G is its centre, and the nine-point circles of the triangles ABC, DBC, ABD, ACD intersect in G .

If LMN, lmn be the Simson lines of D and A with respect to the triangles ABC, DBC respectively, we may assume as known that LMN bisects DK , i.e. it passes through G .

Similarly the pedal lines of A, B, C with respect to the triangles DBC, ACD, ABD also pass through G .

Let BA, CD meet in E ; and BC, AD meet in F .

Now $DNA = AnD = 90^\circ$, $\therefore ANnD$ is cyclic, and Nn, AD are antiparallel with respect to EAB, EDC . Hence Nn is parallel to BC , and in the same way Mm is parallel to BC .

The perpendicular from G on BC bisects LL ; $\therefore GL = GL, GM = Gm$, and $GN = Gn$.

Thus the four Simson lines not only co-intersect in G , but are all of the same length, and have their segments measured from G to the sides of the respective triangles all equal to one another.

Let Bb, Cc be perpendiculars to AD ; then

$$Gb = Gc = Gl = GL.$$

Thus G is the circum-centre of the triangle bcb , which is similar to ABC (for $\angle bcb = \angle lba$ and $\angle ccb = \angle bca$).

Hence G is the centroid of masses $\sin 2A, \sin 2B, \sin 2C$ at l, b, c respectively.

But

$$\angle Abm = \angle ABD = \angle ACD = \angle Acn.$$

$$\therefore bm \text{ is parallel to } cn.$$

Thus G is also the centroid of masses $\sin 2A, \sin 2B, \sin 2C$ at l, m, n respectively, or (as the respective distances from G are equal) of those masses at L, M, N . Similar results hold for the other similar lines.

Lastly, let us take the proposition for which Mr. R. F. Davis (*Math. Gazette*, No. 48, p. 116) desires a geometrical meaning.

Suppose A, B, C fixed and D variable.

$$\begin{aligned}
\Sigma OL^2 \sin 2A &= OG^2 \Sigma \sin 2A + \Sigma GL^2 \sin 2A \\
&= OG^2 \Sigma \sin 2A + \Sigma GL^2 \sin 2A \\
&= \Sigma OL^2 \sin 2A \\
&= R^2 \Sigma \sin 2A - BL \cdot LC \sin 2A - Bm \cdot mD \sin 2B \\
&\quad + Cn \cdot nD \sin 2C.
\end{aligned}$$

$$\begin{aligned}
\text{But } BL \cdot LC \sin 2A &= 2AB \cdot \cos B \cdot AC \cdot \cos C \sin A \cos A = 8h^2 \cdot \Pi \sin A \cdot \Pi \cos A, \\
Bm \cdot mD \sin 2B &= 2AB \cos ABD \cdot AD \cos ADB \cdot \sin B \cos B \\
&= 2AC \cdot \cos ACD \cdot AD \cos ADN \cdot \sin C \cdot \cos C \\
&= Cn \cdot nD \sin 2C.
\end{aligned}$$

$$\text{Thus } \Sigma OL^2 \sin 2A = 4R^2 \Pi \sin A (1 - 2\Pi \cos A). \quad \text{E. P. ROUSE.}$$

181. [1. 2. b.] *A solution, not by elliptic functions, is wanted of the following:*

Given 5 lines $abcde$ in a plane, it is known that the pairs of points

$$ab, ce; bc, da; cd, eb; de, ac; ea, bd$$

are in a collineation. Prove that the fixed triangle of this collineation is self-polar as to both the conic on ab, bc, cd, de, ea and the conic on ac, ce, eb, bd, da . J. MORLEY.

182. [K.] *Proofs of Euler's Theorem, etc.*

Lemma:—The resultant of successive reflexions in two planes intersecting at an angle α and in a line l is a rotation through 2α about l .

To find the resultant of a rotation through α about an axis OA followed by a rotation through β about an axis OB .

Take OC such that the angle between the planes OAB and $OAC = \frac{1}{2}\alpha$ and the angle between OBC , $OBA = \frac{1}{2}\beta$. Then the required resultant $\equiv$ the resultant of successive reflexions in OAC , OAB , OBA , $OBC \equiv$ the resultant of successive reflexions in OAC , $OBC \equiv$ a rotation about OC through twice the angle between OCB , OCA .

To reduce any displacement of a rigid body to its simplest form.

(1) Let $A, B, \dots$ be the initial and $\alpha, \beta, \dots$ the final positions of a set of parallel plaues. Then it is readily shown that A and α , B and $\beta, \dots$ meet on a fixed plane p . Let $A', B', \dots$ and $\alpha', \beta', \dots$ be the initial and final positions of a second set of parallel planes, and let p' be the locus of the intersections of A' and α' , B' and $\beta', \dots$. Let S be any point on the intersection l of p and p' and let M, μ, M', μ' be the planes through S parallel to A, α , and A', α' . Let l and S be moved by the displacement to λ and σ . Since S lies on M and M' , σ lies on μ and μ' . Hence every point of l is moved parallel to the intersection of α and α' , and therefore l and S are moved to λ and σ either (i) by a translation, or (ii) by a rotation through some point O on l . In either case the displacement of the body is completed by a rotation about λ . Hence the total displacement is equivalent (i) to a translation followed by a rotation, i.e. in general to a screw; (ii) to two rotations about lines through O , i.e. to a rotation.

(2) Let two points of the body whose final positions are A' and B' be brought to A and B by a translation which brings any one point to its final position O . Let β be the reflexion of B in the plane p bisecting AA' at right angles, and let the plane q bisect $B'\beta$ at right angles. Then p passes through O since $OA = OA'$, and q passes through O and A' since $OB = OB' = OB'$ and $A'\beta = AB = A'B'$. Now the points O, A, B are brought to O, A', B' by successive reflexions in p and q , i.e. by a rotation about the intersection of p and q . But three points determine the position of a body, and therefore the displacement is equivalent to a translation followed by a rotation.

These proofs can be modified to suit the case in which the initial and final positions of a figure are enantiomorphous. HAROLD HILTON.