Thus, let u, v denote two quadrics, and S any quartic surface passing through 15 fixed points on the curve of intersection of u and v; we have to prove that any other quartic surface which passes through these 15 points must pass through the remaining point of intersection of S with u and v.

Now, through any 8 arbitrary points on the curve of intersection of u and S, and through an arbitrary point P on S, we can draw a quadric, v' say; also, through 8 arbitrary points on the curve of intersection of S and v, we can draw a quadric, u' say, also passing through P (P being supposed not to lie on either u or v); then we have three quartic surfaces, S, uu', vv', each passing through $15+8+8+1 = 32$ fixed points, and every quartic which passes through these points must be of the form $S+\lambda uu'+\mu vv'$, and therefore must pass through the remaining 32 points in which S, uu', and vv' intersect.

Hence every quartic surface which passes through 15 of the points of intersection of S, u, v must pass through the remaining point of intersection.

15. We have at once the theorem, that if three planes A, B, C be drawn cutting a twisted quartic in the points a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4, respectively; the planes $a_1b_1c_1$, $a_2b_2c_2$, $a_3b_3c_3$, $a_4b_4c_4$ will cut the quartic in four coplanar points d_1, d_2, d_3, d_4.

Consequently, all the theorems stated previously may be at once translated so as to apply to twisted quartics.

On the Arithmetical Theory of the Form $x^3+ny^3+n^2z^3-3nxyz$.

By Professor MATHEWS, M.A.

[*Read May 8th*, 1890.]

In the last four papers contained in Vol. I. of Dirichlet's collected works will be found some remarkable propositions relating to certain arithmetical forms of higher degrees.*

* Dirichlet's *Werke*, I., pp. 619, 625, 633, 639. The titles of the papers are—"Sur la Théorie des Nombres" (*Comptes Rendus*, 1840, p. 285, or *Liouville*, Sér. I., t. v., p. 72); "Einige Resultate von Untersuchungen über eine Classe homogener Functionen des dritten und der höheren Grade" (*Berichte über die Verhandlungen d. Königl. Preuss. Akad. d. Wissensch.*, 1841, p. 280); "Verallgemeinerung eines

In the following note I propose to illustrate Dirichlet's method by applying it to a special case, which is otherwise of interest in connexion with an algorithm of Jacobi's (*Crelle*, Vol. LXIX., p. 29).

The form to be considered is

$$F(x, y, z) \equiv x^3+ny^3+n^2z^3-3nxyz,$$

and it will be shown—

(i.) that if an integer m can be represented by the form $F(x, y, z)$ at all, it can be so represented in an infinite number of ways;

(ii.) that the Diophantine equation $F(x, y, z) = 1$ can always be satisfied by integral values of x, y, z;

(iii.) that all the integral solutions of $F(x, y, z) = 1$ can be derived from a single fundamental solution in a manner analogous to that in which the solutions of the Pellian equation are obtained.

1. Let n be a positive integer, t the real cube root of n, θ a complex cube root of unity; then

$$\begin{aligned} F(x, y, z) &= (x+yt+zt^2)(x+y\theta t+z\theta^2t^2)(x+y\theta^2t+z\theta t^2) \\ &= (x+yt+zt^2)\{(x^2-nyz)+(nz^2-xy)\,t+(y^2-zx)\,t^2\} \\ &= x^3+ny^3+n^2z^3-3nxyz. \end{aligned}$$

Dirichlet has shown (*Werke* I., p. 635) that we can choose, in an infinite number of ways, a set of integers x, y, z, so that

$$|\,x+yt+zt^2\,| < 1/\mu^2,$$

where μ is the greater of the quantities $|y|$, $|z|$. (Here, as in what follows, $|z|$ denotes the absolute value or modulus of z.)

His proof is as follows:—Choose for y, z all pairs of values, the same or different, which can be obtained from

$$-p,\ -(p-1),\ \dots\ -1,\ 0,\ 1,\ \dots\ (p-1),\ p$$

where p is any positive integer. Having chosen any pair, let x be determined so that $x+yt+zt^2$ may be less than 1, and not negative;

Satzes aus der Lehre von den Kettenbrüchen ..." (*ibid.*, 1842, p. 93); and "Zur Theorie der Complexen Einheiten" (*ibid.*, 1846, p. 103).

See also Eisenstein's memoir on cubic forms (*Crelle*, t. XXVIII., p. 289, and XXIX., p. 19); Sylvester, "On the General Solution in certain cases of the Equation $x^3+y^3+Az^3 = Mxyz$," &c., *Phil. Mag.*, XXXI. (1847), pp. 461-7, and "On certain Ternary Cubic Form Equations," *Amer. Math. Jour.*, t. III. (1880), pp. 58-88 and 179-189.

For the last two references I am indebted to Professor Cayley.

we thus obtain for the expression one value zero, and $(2p+1)^2-1$ other values which are all positive proper fractions. Now consider the series of terms

$$0,\ \frac{1}{4p^2},\ \frac{2}{4p^2},\ \ldots\ldots\ \frac{4p^2-1}{4p^2},\ 1;$$

these divide the interval 0 ... 1 into $4p^2$ equal parts; and, since

$$(2p+1)^2-1>4p^2,$$

it follows that, of the proper fractions previously obtained, it will always be possible to find two which are in the same one of the $4p^2$ intervals. Let

$$x_1+y_1t+z_1t^2 \quad \text{and} \quad x_2+y_2t+z_2t^2$$

be two such fractions. They are not equal, for this would imply

$$x_1=x_2,\quad y_1=y_2,\quad z_1=z_2;$$

hence their difference is less than $1/4p^2$ and not zero; further, the differences (y_1-y_2), (z_1-z_2) are both numerically not greater than $2p$; hence, if we put

$$x=x_1-x_2,\quad y=y_1-y_2,\quad z=z_1-z_2,$$

$$|x+yt+zt^2|<1/4p^2,$$

and therefore *a fortiori* $<1/\mu^2$ if μ is the greater of the quantities $|y|$, $|z|$.

Since p was chosen at pleasure, it follows that an infinite number of integers x, y, z can be chosen in the manner stated.

If (x, y, z) be any such set,

$$\begin{aligned}|x+y\theta t+z\theta^2t^2| &= |x+yt+zt^2+(\theta-1)\,yt+(\theta^2-1)\,zt^2|\\ &< |x+yt+zt^2|+|(\theta-1)\,yt|+|(\theta^2-1)\,zt^2|\\ &< |x+yt+zt^2|+\{|yt|+|zt^2|\}\sqrt{3}\\ &\qquad (\text{since } |\theta-1|=|\theta^2-1|=\sqrt{3}),\end{aligned}$$

and *a fortiori* $\qquad < \dfrac{1}{\mu^2}+(t^2+t)\,\mu\sqrt{3}.$

Similarly, $\qquad |x+y\theta^2t+z\theta t^2|<\dfrac{1}{\mu^2}+(t^2+t)\,\mu\sqrt{3};$

and therefore $|F(x,y,z)| < \frac{1}{\mu^2}\{(t^2+t)\,\mu\sqrt{3}+1/\mu^2\}^2$

$$< 3\,(t^2+t)^2+2\sqrt{3}\,(t^2+t)/\mu^3+1/\mu^6.$$

Since μ is not less than 1, it follows *a fortiori* that

$$|F(x,y,z)| < \{\sqrt{3}\,(t^2+t)+1\}^2.$$

If, then, M is the integer next greater than

$$\{\sqrt{3}\,(t^2+t)+1\}^2,$$

there are an infinite number of sets of integers x, y, z such that

$$|F(x,y,z)| < M.$$

But $F(x,y,z)$ is always an integer when x, y, z are integral; therefore at least one value of F must correspond to an infinite number of sets (x,y,z): that is, there must be at least one integer m such that the Diophantine equation $F(x,y,z) = m$ admits of an infinity of integral solutions.

2. Let (x,y,z), (x',y',z') be any two sets such that

$$F(x',y',z') = F(x,y,z) = m.$$

The expression

$$\frac{x'+y't+z't^2}{x+yt+zt^2} = \frac{(x'+y't+z't^2)(x+y\theta t+z\theta^2t^2)(x+y\theta^2t+z\theta t^2)}{F(x,y,z)}$$

$$= \frac{X+Yt+Zt^2}{m},$$

where X, Y, Z are integers, the reduction in the numerator being made by help of $t^3 = n$.

Now we may suppose (x',y',z') congruous (mod. m) to (x,y,z): that is to say,

$$x' \equiv x, \quad y' \equiv y, \quad z' \equiv z \pmod{m};$$

for not more than m^3 sets can be incongruous, and there are ∞ sets.

This being so, we have

$$(X-m)+Yt+Zt^2 = \{(x'-x)+(y'-y)\,t+(z'-z)\,t^2\}$$
$$(x+y\theta t+z\theta^2t^2)(x+y\theta^2t+z\theta t^2)$$
$$= m\,(\alpha+\beta t+\gamma t^2)(x+y\theta t+z\theta^2t^2)(x+y\theta^2t+z\theta t^2)$$
$$= m\,(A+Bt+Ct^2)$$

where A, B, C are integers.

Hence $$X \equiv Y \equiv Z \equiv 0 \pmod{m},$$

so that $$\frac{x'+y't+z't^2}{x+yt+zt^2} = \xi+\eta t+\zeta t^2$$

where ξ, η, ζ are integers; and therefore

$$F(\xi, \eta, \zeta) = \frac{F(x', y', z')}{F(x, y, z)} = 1;$$

that is, (ξ, η, ζ) is an integral solution of

$$F(x, y, z) = 1.$$

3. It is clear that, if (ξ, η, ζ) is any solution, and we put

$$(\xi+\eta t+\zeta t^2)^k = \xi_k+\eta_k t+\zeta_k t^2$$

where k is any integer, then (ξ_k, η_k, ζ_k) will also be a solution. It will now be proved that every integral solution can be derived in this way from a certain fundamental solution.

Write $$u = x+yt+zt^2,$$

$$v = x+y\theta t+z\theta^2 t^2,$$

$$w = x+y\theta^2 t+z\theta t^2,$$

so that, if x, y, z are taken to be ordinary rectangular coordinates, $u = 0$ represents a real plane, and $v = 0$, $w = 0$ two conjugate imaginary planes intersecting in a real line. The surface

$$F(x, y, z)-1 = 0$$

has a real asymptotic plane $u = 0$ and a real asymptotic line $(v = 0, w = 0)$.

It may be conceived to be generated by the intersection of the plane

$$u = c$$

with the elliptic cylinder $$vw = \frac{1}{c},$$

c being a variable parameter.

For real points on the surface vw, and therefore c, is essentially positive, so that the real part of the surface consists of a single sheet on one side of $u = 0$ with a funnel-shaped depression extending to infinity in the direction of the line $(v = 0, w = 0)$.

If x, y, z are whole numbers, let (x, y, z) be called an integral point; and if, moreover,

$$F(x, y, z) = 1,$$

let it be called a radical point.

The plane $$u-1=0$$

goes through the radical point $(1, 0, 0)$, and cuts the surface in an ellipse. Let the plane be moved parallel to itself until it *first* passes through another radical point (ξ_0, η_0, ζ_0). This point is not infinitely near $(1, 0, 0)$, and (on account of the shape of the surface) it is not at an infinite distance from the origin. Hence the plane will have moved through a finite distance; *i.e.*,

$$1 \ldots (\xi_0+\eta_0 t+\zeta_0 t^2)$$

will be a finite interval, and there will be no radical point (ξ, η, ζ) for which

$$1 < \xi+\eta t+\zeta t^2 < \xi_0+\eta_0 t+\zeta_0 t^2$$

(where the signs of inequality must be reversed if $\xi_0+\eta_0 t+\zeta_0 t^2 < 1$).

Now, let (x, y, z) be any other radical point; then, if h is any integer, and we put

$$\frac{(x+yt+zt^2)}{(\xi_0+\eta_0 t+\zeta_0 t^2)^h} = \alpha+\beta t+\gamma t^2,$$

then (α, β, γ) is also a radical point.

Denoting by $\log(x+yt+zt^2)$ the real logarithm of $x+yt+zt^2$, we have

$$\log(\alpha+\beta t+\gamma t^2) = \log(x+yt+zt^2)-h\log(\xi_0+\eta_0 t+\zeta_0 t^2).$$

If $\log(x+yt+zt^2)$ is not a multiple of $\log(\xi_0+\eta_0 t+\zeta_0 t^2)$, it will be possible to determine h so that the expression on the right-hand is intermediate between 0 and $\log(\xi_0+\eta_0 t+\zeta_0 t^2)$, and therefore $\alpha+\beta t+\gamma t^2$ intermediate between 1 and $\xi_0+\eta_0 t+\zeta_0 t^2$: but this has been shown to be impossible; therefore

$$x+yt+zt^2 = (\xi_0+\eta_0 t+\zeta_0 t^2)^k$$

where k is some positive or negative integer.

If $$\frac{1}{\xi_0+\eta_0 t+\zeta_0 t} = \xi_0'+\eta_0' t+\zeta_0' t,$$

then $(\xi_0', \eta_0', \zeta_0')$ has just the same right as (ξ_0, η_0, ζ_0) to be called a fundamental solution; in the geometrical theory we see, correspondingly, that the plane $u-1=0$ may be moved in either direction in order to come to a fundamental point.

4. Various consequences of the preceding theory are immediately evident. Thus all numbers representable by F are cubic residues of n; the product of any number of such integers is representable by F; and if $F(x, y, z)$ be any representation of m, then all the representations are found from

$$x'+y't+z't^2 = (x+yt+zt^2)(\xi_0+\eta_0 t+\zeta_0 t^2)^k.$$

Moreover, this relation between (x', y', z') and (x, y, z) gives an infinity of linear transformations of the form F into itself, the coefficients being integral, and their determinant equal to 1.

The direction cosines of the asymptotic line of the surface

$$F(x, y, z) = m$$

are given by $$\lambda : \mu : \nu = t^2 : t : 1;$$

from which it may be inferred that, among the solutions of

$$F(x, y, z) = m,$$

there are an infinite number for which x, y, z are all positive.

If we take a solution (ξ, η, ζ) of

$$F(x, y, z) = 1,$$

for which $$\xi+\eta t+\zeta t^2 > 1,$$

and put, as before, $(\xi+\eta t+\zeta t^2)^k = \xi_k+\eta_k t+\zeta_k t^2$,

k being a positive integer, then, as k increases indefinitely, we have at last

$$\xi_k : \eta_k : \zeta_k = t^2 : t : 1,$$

that is, $$\xi_k/\zeta_k, \quad \eta_k/\zeta_k$$

are approximations, more and more accurate as k increases, to t^2 and t respectively. By successive squaring, the approximation may be made with considerable rapidity.

For example, if $n = 5$, we may put

$$\xi = 41, \quad \eta = 24, \quad \zeta = 14,$$

and hence $$\xi_2 = 5041, \quad \eta_2 = 2948, \quad \zeta_2 = 1724,$$

and so on; where, in general,

$$\xi_{2k} = \xi_k^2+10\eta_k\zeta_k,$$

$$\eta_{2k} = 5\zeta_k^2+2\xi_k\eta_k,$$

$$\zeta_{2k} = \eta_k^2+2\zeta_k\xi_k.$$

It will be found that $\eta_3/\zeta_3 = 1{\cdot}709976\ldots,$

the true value of $\sqrt[3]{5}$ being $1{\cdot}7099759\ldots$.

For some of the simplest values of n, I have calculated the following solutions by Jacobi's method; I believe they are fundamental, but this remains to be proved.

n	$x, \ y, \ z$
2	1, 1, 1
3	4, 3, 2
4	5, 3, 2
5	41, 24, 14
7	4, 2, 1
11	89, 40, 18

On the Genesis of Binodal Quartic Curves from Conics.

By HENRY M. JEFFERY, F.R.S.

[*Read May 8th*, 1890.]

1. In this memoir I propose to generalise Laguerre's method of generating bicircular quartics for two distinct families of binodal quartics—(1) those which pass through two assigned points, (2) those which touch a given conic in four assigned points. The processes used in (1) and (2) are adapted from those employed for plane bicircular quartics and spherical sphero-quadrics respectively.

In a second memoir I hope to explain the classification of binodal quartics.