The Addition-Theorem for the Bessel Functions. By H. M. MACDONALD. Read and received April 5th, 1900.

The addition-theorems for $J_0(R)$ and $Y_0(R)$, where

$$R^2 = r^2 + r_1^2 - 2rr_1 \cos \theta,$$

have been given by Neumann, the addition-theorem for $\frac{J_n(R)}{R^n}$ by Gegenbauer and Sonine, and that for $K_1(R)/R^1$ by Heine. The object of the following paper is the development of a method by which the addition-theorems for $\frac{J_n(R)}{R^n}$ and $\frac{K_n(R)}{R^n}$ can be obtained.

In the first section an integral, which is suggested by the integral obtained in a previous paper (*Proceedings*, Vol. xxx., p. 170), for the product $K_n(a)K_n(b)$ is discussed. In the second section this integral is employed to obtain the addition-theorems.

$$1. \text{ The Integral } \int e^{kt - [(a^2 + b^2)/2t]} I_n \left(\frac{ab}{t} \right) \frac{dt}{t}.$$

$$\text{Writing } u = \int_{t_0}^{t_1} e^{kt - [(a^2 + b^2)/2t]} I_n \left(\frac{ab}{t} \right) \frac{dt}{t},$$

where the path of integration is subjected to the condition that, if it is not closed, the points at which it ends are independent of a and b , it follows that

$$\begin{aligned} \frac{\partial^2 u}{\partial a^2} + \frac{1}{a} \frac{\partial u}{\partial a} + \left(1 - \frac{n^2}{a^2}\right) u \\ = \int_{t_0}^{t_1} e^{kt - [(a^2 + b^2)/2t]} \frac{dt}{t} \left\{ \left(\frac{a^2}{t^2} - \frac{2}{t} + 1 - \frac{n^2}{a^2} \right) I_n \left(\frac{ab}{t} \right) \right. \\ \left. + \left(\frac{b}{at} - \frac{2ab}{t^2} \right) I_n' \left(\frac{ab}{t} \right) + \frac{b^2}{t^2} I_n'' \left(\frac{ab}{t} \right) \right\}, \end{aligned}$$

provided, as will subsequently appear, that this integral is convergent. Now

$$I_n'' \left(\frac{ab}{t} \right) + \frac{t}{ab} I_n' \left(\frac{ab}{t} \right) - \left(1 + \frac{t^2 n^2}{a^2 b^2} \right) I_n \left(\frac{ab}{t} \right) = 0;$$

therefore

$$\frac{\partial^2 u}{\partial a^2} + \frac{1}{a} \frac{\partial u}{\partial a} + \left(1 - \frac{n^2}{a^2}\right) u = \int_{t_0}^{t_1} e^{it - [(a^2 + b^2)/2t]} \frac{dt}{t} \left\{ \left(\frac{a^2 + b^2}{t^2} - \frac{2}{t} + 1 \right) I_n \left(\frac{ab}{t} \right) - \frac{2ab}{t^2} I_n' \left(\frac{ab}{t} \right) \right\};$$

that is,

$$\frac{\partial^2 u}{\partial a^2} + \frac{1}{a} \frac{\partial u}{\partial a} + \left(1 - \frac{n^2}{a^2}\right) u = 2e^{it - [(a^2 + b^2)/2t]} I_n \left(\frac{ab}{t} \right) \frac{1}{t} \Big|_{t_0}^{t_1}.$$

Now the expression on the right-hand side vanishes, provided that the real parts of $(a \pm b)^2$ are positive, the real part of n is greater than -1 , and t_0, t_1 have for values two of the quantities $0, c + \infty i, c - \infty i$, where c is some real positive quantity; further, all the integrals involved in the above process are then convergent; hence, denoting by u_1, u_2, u_3 the three functions u which correspond to the three sets of values of t_0 and t_1 ,

$$t_0 = c - \infty i, \quad t_1 = c + \infty i,$$

$$t_0 = c - \infty i, \quad t_1 = 0,$$

$$t_0 = 0, \quad t_1 = c + \infty i,$$

any one of them satisfies the differential equation

$$\frac{d^2 u}{da^2} + \frac{1}{a} \frac{du}{da} + \left(1 - \frac{n^2}{a^2}\right) u = 0,$$

and also, from the symmetry of the expression, the equation

$$\frac{d^2 u}{db^2} + \frac{1}{b} \frac{du}{db} + \left(1 - \frac{n^2}{b^2}\right) u = 0.$$

Therefore

$$u = f_1(b) J_n(a) + f_2(b) J_{-n}(a),$$

where, since u satisfies the second differential equation above,

$$f_1(b) = AJ_n(b) + BJ_{-n}(b),$$

$$f_2(b) = A'J_n(b) + B'J_{-n}(b),$$

and hence

$$u = J_n(a) \{AJ_n(b) + BJ_{-n}(b)\} + J_{-n}(a) \{A'J_n(b) + B'J_{-n}(b)\}.$$

It remains to determine the values of the constants A, B, A', B' belonging to each of the functions u_1, u_2, u_3 . Taking first the integral

$$u_1 = \int_{c - \infty i}^{c + \infty i} e^{it - [(a^2 + b^2)/2t]} I_n \left(\frac{ab}{t} \right) \frac{dt}{t},$$

it appears that $\frac{u_1}{b^n}$ tends to a finite limit as b tends to zero; hence

$$B = B' = 0,$$

and therefore $u_1 = J_n(b) \{AJ_n(a) + A'J_{-n}(a)\}$.

$$\text{Now} \quad \text{Lt}_{b \rightarrow 0} \frac{u_1}{b^n} = \frac{a^n}{2^n \Pi(n)} \int_{-\infty}^{+\infty} e^{it - (a^2/2t)} \frac{dt}{t^{n+1}},$$

$$\text{that is,} \quad \text{Lt}_{b \rightarrow 0} \frac{u_1}{b^n} = \frac{2\pi i}{2^n \Pi(n)} J_n(a),$$

$$\text{but also} \quad \text{Lt}_{b \rightarrow 0} \frac{u_1}{b^n} = \frac{1}{2^n \Pi(n)} \{AJ_n(a) + A'J_{-n}(a)\};$$

$$\text{hence} \quad A' = 0, \quad A = 2\pi i,$$

$$\text{and therefore} \quad u_1 = 2\pi i J_n(a) J_n(b);$$

$$\text{that is,} \quad J_n(a) J_n(b) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{it - [(a^2 + b^2)/2t]} I_n\left(\frac{ab}{t}\right) \frac{dt}{t}.$$

This integral can immediately be transformed into the integral given by Sonine as follows:—

$$\begin{aligned} J_n(a) J_n(b) &= \frac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{it - [(a^2 + b^2)/2t]} \frac{dt}{t^{n+1}} \int_0^\pi \frac{a^n I_n}{2^n \Pi(n - \frac{1}{2}) \sqrt{\pi}} e^{(ab \cos \theta)/t} \sin^{2n} \theta d\theta; \end{aligned}$$

whence, changing the order of integration and integrating,

$$J_n(a) J_n(b) = \frac{a^n b^n}{2^n \sqrt{\pi} \Pi(n - \frac{1}{2})} \int_0^\pi \frac{J_n\{\sqrt{(a^2 + b^2 - 2ab \cos \theta)}\}}{\{\sqrt{(a^2 + b^2 - 2ab \cos \theta)}\}^n} \sin^{2n} \theta d\theta.$$

In the case of u_2 , suppose that $|a| < |b|$; then $\frac{u_2}{a^n}$ tends to a finite limit as a tends to zero, for in the neighbourhood of $t = 0$ the integrand tends to zero, and at every other point of the path of integration to a finite limit; therefore

$$A' = B' = 0;$$

$$\text{and hence} \quad u_2 = J_n(a) \{AJ_n(b) + BJ_{-n}(b)\}.$$

$$\text{Now} \quad \text{Lt}_{a \rightarrow 0} \frac{u_2}{a^n} = \text{Lt}_{a \rightarrow 0} \frac{1}{a^n} \int_{-\infty}^0 e^{it - [(a^2 + b^2)/2t]} I_n\left(\frac{ab}{t}\right) \frac{dt}{t};$$

$$\text{that is,} \quad \text{Lt}_{a \rightarrow 0} \frac{u_2}{a^n} = \frac{b^n}{2^n \Pi(n)} \int_{-\infty}^0 e^{it - b^2/2t} \frac{dt}{t^{n+1}},$$

or
$$*Lt_{a=0} \frac{u_2}{a^n} = -\frac{\pi}{2^n \Pi(n) \sin n\pi} \{J_{-n}(b) - e^{n\pi i} J_n(b)\};$$

but
$$Lt_{a=0} \frac{u_2}{a^n} = \frac{1}{2^n \Pi(n)} \{AJ_n(b) + BJ_{-n}(b)\};$$

therefore
$$u_2 = -\frac{\pi}{\sin n\pi} \{J_{-n}(b) - e^{n\pi i} J_n(b)\} J_n(a).$$

Hence it follows that

$$-\frac{\pi}{\sin n\pi} \{J_{-n}(b) - e^{n\pi i} J_n(b)\} J_n(a) = \int_{c-\infty i}^0 e^{kt - [(a^2+b^2)/2t]} I_n\left(\frac{ab}{t}\right) \frac{dt}{t},$$

that is,
$$J_n(a) K_n(b) = -\frac{e^{-n\pi i}}{2} \int_{c-\infty i}^0 e^{kt - [(a^2+b^2)/2t]} I_n\left(\frac{ab}{t}\right) \frac{dt}{t},$$

where $|a| < |b|$. Similarly it can be shown that

$$\frac{\pi}{\sin n\pi} \{J_{-n}(b) - e^{-n\pi i} J_n(b)\} J_n(a) = \int_0^{c+\infty i} e^{kt - [(a^2+b^2)/2t]} I_n\left(\frac{ab}{t}\right) \frac{dt}{t},$$

where $|a| < |b|$. The relation given in the *Proceedings*, Vol. xxx., p. 170, and other similar relations, can be obtained in an analogous manner.

2. The Addition-Theorems.

The addition-theorem for $\frac{J_n(R)}{R^n}$ can be at once deduced from the integral relation first established as follows:—

The function $\frac{J_n(R)}{R^n}$ is given by the relation

$$\frac{J_n(R)}{R^n} = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} e^{kt - R^2/2t} \frac{dt}{t^{n+1}},$$

that is, writing $R^2 = a^2 + b^2 - 2ab \cos \theta$,

$$\frac{J_n \{ \sqrt{a^2 + b^2 - 2ab \cos \theta} \}}{\{ \sqrt{a^2 + b^2 - 2ab \cos \theta} \}^n} = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} e^{kt - [(a^2+b^2)/2t] + (nb/t) \cos \theta} \frac{dt}{t^{n+1}};$$

* *Proceedings*, Vol. xxix., p. 110.

$$\text{Now* } e^{(ab/t)\cos\theta} = 2^{n-1} \Pi(n-1) \sum_{k=0}^{k=\infty} (n+k) C_k^n(\cos\theta) \frac{J_{n+k}\left(\frac{ab}{t}\right)}{\left(\frac{ab}{t}\right)^n t^{n+k}},$$

$$\text{or } e^{(ab/t)\cos\theta} = 2^{n-1} \Pi(n-1) \sum_{k=0}^{k=\infty} (n+k) C_k^n(\cos\theta) \frac{I_{n+k}\left(\frac{ab}{t}\right)}{\left(\frac{ab}{t}\right)^n},$$

where n is not zero and $C_k^n(\cos\theta)$ is the coefficient of h^k in the expansion of $(1-2h\cos\theta+h^2)^{-n}$ in powers of h . Hence it follows that

$$\frac{J_n\{\sqrt{a^2+b^2-2ab\cos\theta}\}}{\{\sqrt{a^2+b^2-2ab\cos\theta}\}^n} = 2^{n-1} \Pi(n-1) \sum_{k=0}^{k=\infty} (n+k) \frac{C_k^n(\cos\theta)}{a^n b^n} \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} e^{kt - [(a^2+b^2)/2t]} I_{n+k}\left(\frac{ab}{t}\right) \frac{dt}{t},$$

$n \neq 0$, that is, since

$$\begin{aligned} \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} e^{kt - [(a^2+b^2)/2t]} I_{n+k}\left(\frac{ab}{t}\right) \frac{dt}{t} &= J_{n+k}(a) J_{n+k}(b), \\ \frac{J_n\{\sqrt{a^2+b^2-2ab\cos\theta}\}}{\{\sqrt{a^2+b^2-2ab\cos\theta}\}^n} &= 2^n \Pi(n-1) \sum_{k=0}^{k=\infty} (n+k) \frac{J_{n+k}(a)}{a^n} \frac{J_{n+k}(b)}{b^n} C_k^n(\cos\theta), \end{aligned}$$

$n \neq 0$. When $n=0$, it can be easily shown that

$$e^{(ab/t)\cos\theta} = I_0\left(\frac{ab}{t}\right) + 2 \sum_{k=1}^{k=\infty} I_k\left(\frac{ab}{t}\right) \cos k\theta,$$

and hence, as above, that

$$J_0\{\sqrt{a^2+b^2-2ab\cos\theta}\} = J_0(a) J_0(b) + 2 \sum_1^{\infty} J_k(a) J_k(b) \cos k\theta.$$

These results agree with those given by Neumann, Gegenbauer, and Sonine.

The addition-theorem for $\frac{K_n(iR)}{R^n}$ is obtained as follows:—From the relation

$$\frac{J_{-n}(R) - e^{in\pi} J_n(R)}{2i \sin n\pi R^n} = -\frac{1}{2\pi i} \int_{c-\infty i}^0 e^{kt - [(a^2+b^2-2ab\cos\theta)/2t]} \frac{dt}{t^{n+1}},$$

* Sonine, *Math. Ann.*, Vol. xvi.

where the real part of n is greater than -1 , it follows that

$$-\frac{\pi}{\sin n\pi} \left\{ \frac{J_{-n}(R)}{R^n} - e^{n\pi i} \frac{J_n(R)}{R^n} \right\} \\ = 2^n \Pi(n-1) \sum_{k=0}^{k=\infty} (n+k) \frac{O_k^n(\cos \theta)}{a^n b^n} \int_{c=\infty i}^0 e^{kt - [(n^2 + b^2)/2t]} I_{n+k} \left(\frac{ab}{t} \right) \frac{dt}{t},$$

$n \neq 0$; that is, by § 1,

$$\frac{\pi}{\sin n\pi} \left\{ \frac{J_{-n}(R)}{R^n} - e^{n\pi i} \frac{J_n(R)}{R^n} \right\} \\ = 2^n \Pi(n-1) \sum_{k=0}^{k=\infty} (n+k) \frac{\pi O_k^n(\cos \theta)}{\sin(n+k)\pi} \\ \times \left\{ \frac{J_{-n-k}(b)}{b^n} - e^{(n+k)\pi i} \frac{J_{n+k}(b)}{b^n} \right\} \frac{J_{n+k}(a)}{a^n},$$

where $|a| < |b|$, or, as it may be written,

$$\frac{K_n \{ i \sqrt{(a^2 + b^2 - 2ab \cos \theta)} \}}{\{ \sqrt{(a^2 + b^2 - 2ab \cos \theta)} \}^n} \\ = 2^n \Pi(n-1) \sum_{k=0}^{k=\infty} e^{i(k\pi)} (n+k) \frac{K_{n+k}(ib)}{b^n} \frac{J_{n+k}(a)}{a^n} C_k^n(\cos \theta).$$

This includes, as a particular case ($n = \frac{1}{2}$), Heine's result

$$\frac{1}{\sqrt{(2\pi)}} \frac{K_{\frac{1}{2}} \{ i \sqrt{(a^2 + b^2 - 2ab \cos \theta)} \}}{\{ \sqrt{(a^2 + b^2 - 2ab \cos \theta)} \}^{\frac{1}{2}}} \\ \equiv \frac{e^{-i\sqrt{(a^2 + b^2 - 2ab \cos \theta)}}}{\sqrt{(a^2 + b^2 - 2ab \cos \theta)}} = \sum_{k=0}^{k=\infty} e^{i(k\pi)} \left(k + \frac{1}{2} \right) \frac{K_{k+\frac{1}{2}}(ib)}{b^{\frac{1}{2}}} \frac{J_{k+\frac{1}{2}}(a)}{a^{\frac{1}{2}}} P_n(\cos \theta),$$

and, when $n = 0$, it takes the form

$$K_0 \{ i \sqrt{(a^2 + b^2 - 2ab \cos \theta)} \} \\ = K_0(ib) J_0(a) + 2 \sum_{k=1}^{\infty} e^{i(k\pi)} K_k(ib) J_k(a) \cos k\theta,$$

which is equivalent to Neumann's result. Addition-theorems for any other solution of Bessel's equation can be obtained in a similar way.