On a Congruence Theorem relating to an Extensive Class of Coefficients. *By* J. W. L. Glaisher. Communicated June 8th, 1899. Received August 30th, 1899.

1. It is a known theorem enunciated without proof by Sylvester,* in 1861, and proved by Stern,† in 1874, that, if E_n be the n^{th} Eulerian number and if p be any uneven prime, then

$$(-1)^n E_n \equiv (-1)^{n'} E_{n'}, \quad \text{mod } p,$$

if $n-n'$ is a multiple of $\frac{p-1}{2}$. This singular theorem explains why the Eulerian numbers end in 1 and 5 alternately, and gives rise to many other properties of the numbers.

The theorem may be expressed in the form

$$E_n \equiv (-1)^{tj} E_{n-tj}, \quad \text{mod } p,$$

where $j = \frac{1}{2}(p-1)$ and t is any integer such that $n-tj$ is positive; so that, to mod p, any Eulerian number is congruent to one of the first $\frac{1}{2}(p-1)$ Eulerian numbers, $E_1, E_2, \ldots, E_{\frac{1}{2}(p-1)}$.

I have obtained a comparatively simple proof of this theorem by a method which is applicable to expansions of a very general character, and which shows that the property in question is not peculiar to the Eulerian numbers, but is shared by an extensive class of other numbers or coefficients.

As very little simplification is produced by considering the special case of the Eulerian numbers, I proceed at once to prove the general theorems.

* "Sur une propriété des nombres premiers qui se rattache au théorème de Fermat," *Comptes Rendus*, Vol. LII., p. 212.

† "Zur Theorie der Eulerschen Zahlen," *Crelle's Journal*, Vol. LXXIX., p. 67. It should be mentioned that Sylvester and Stern give also more general theorems in which the modulus is p^n and 2^n. In the present paper the modulus is always p or 2.

2. Let $X_0, X_1, X_2, \ldots$ be any quantities connected by the recurring relation

(i.) $$(\lambda+1)\,a^n X_n + (n)_1\,a^{n-1} b X_{n-1} + (n)_2\,a^{n-2} b^2 X_{n-2} + \ldots$$
$$\ldots + (n)_{n-1}\,a b^{n-1} X_1 + (n)_n\,b^n X_0 = c_n,$$

where λ is any constant, a and b any integers, and c_n is any quantity depending upon n, and such that

$$c_r \equiv c_{r-t(p-1)}, \quad \text{mod } p,$$

p being any given uneven prime and t any integer. Thus, for example, c_r might be $A\alpha^r$, if p is not a divisor of A or α, or $A\alpha^r + B\beta^r + C\gamma^r + \ldots$, if p is not a divisor of any of the quantities $A, \alpha, B, \beta, C, \gamma, \ldots$. The notation $(n)_r$ is used to express the number of combinations of n things taken r together, *i.e.*, $(n)_r$ is the coefficient of x^r in the expansion of $(1+x)^n$. The suffix of c is always supposed to be positive.

It will now be assumed that the congruence

$$X_r \equiv X_{r-t(p-1)}, \quad \text{mod } p,$$

p being any uneven prime, and t being any positive integer, holds good for the values $p, p+1, \ldots, n-1$ of r, and by means of the above recurring relation it will be shown that, this being so, it holds good also for $r = n$.

Let $n = kp+q$, where k and q are any positive integers and $q<p$. Then

$$c_n - (\lambda+1)\,a^n X_n$$
$$= b^n X_0 + (n)_1\,a b^{n-1} X_1 + (n)_2\,a^2 b^{n-2} X_2 + \ldots + (n)_{n-1}\,a^{n-1} b X_{n-1}$$
$$= b^n X_0 + (n)_1\,a b^{n-1} X_1 + (n)_2\,a^2 b^{n-2} X_2 + \ldots + (n)_{p-1}\,a^{p-1} b^{n-p+1} X_{p-1}$$
$$+ (n)_p\,a^p b^{n-p} X_p + (n)_{p+1}\,a^{p+1} b^{n-p-1} X_{p+1} + \ldots + (n)_{2p-1}\,a^{2p-1} b^{n-2p+1} X_{2p-1}$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$+ (n)_{(k-1)p}\,a^{(k-1)p} b^{n-(k-1)p} X_{(k-1)p} + n_{(k-1)p+1}\,a^{(k-1)p+1} b^{n-(k-1)p-1} X_{(k-1)p+1}$$
$$+ \ldots + (n)_{kp-1}\,a^{kp-1} b^{n-kp+1} X_{kp-1}$$
$$+ (n)_{kp}\,a^{kp} b^{n-kp} X_{kp} + (n)_{kp+1}\,a^{kp+1} b^{n-kp-1} X_{kp+2} + \ldots$$
$$\ldots + (n)_{kp+q-1}\,a^{kp+q-1} b X_{kp+q-1}.$$

Now it is known* that, if p is prime and q and s are less than p, then

$$(kp+q)_{gp+s} \equiv 0, \text{ mod } p, \qquad \text{if } s > q,$$

and

$$\equiv (k)_g \times (q)_s, \text{ mod } p, \text{ if } s \overline{\overline{<}} q.$$

Reducing the coefficients by this rule, and reducing also the X's by the congruence

$$X_r \equiv X_{r-t(p-1)}, \quad \text{mod } p,$$

and the powers of a and b by the congruences

$$a^r \equiv a^{r-t(p-1)}, \quad \text{mod } p,$$

$$b^r \equiv b^{r-t(p-1)}, \quad \text{mod } p,$$

we find that

$$c_n - (\lambda+1)\, a^n X_n$$

$$\equiv b^{k+q} X_0 + (q)_1\, ab^{k+q-1} X_1 + (q)_2\, a^2 b^{k+q-2} X_2 + \ldots + (q)_q\, a^q b^k X_q$$

$$+ (k)_1 \left\{ ab^{k+q-1} X_1 + (q)_1\, a^2 b^{k+q-2} X_2 + \ldots + (q)_q\, a^{q+1} b^{k-1} X_{q+1} \right\}$$

$$+ (k)_2 \left\{ a^2 b^{k+q-2} X_2 + (q)_1\, a^3 b^{k+q-3} X_3 + \ldots + (q)_q\, a^{q+2} b^{k-2} X_{q+2} \right\}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$+ (k)_{k-1} \left\{ a^{k-1} b^{q+1} X_{k-1} + (q)_1\, a^k b^q X_k + \ldots + (q)_q\, a^{k+q-1} b X_{k+q-1} \right\}$$

$$+ (k)_k \left\{ a^k b^q X_k + (q)_1\, a^{k+1} b^{q-1} X_{k+1} + \ldots + (q)_{q-1} a^{k+q-1} X_{k+q-1} \right\}, \text{ mod } p.$$

Collecting the coefficients of $X_0, X_1, X_2, \ldots$, the right-hand side of this congruence becomes

$$b^{k+q} X_0 + ab^{k+q-1} \left\{ (k)_1 + (q)_1 \right\} X_1 + a^2 b^{k+q-2} \left\{ (k)_2 + (q)_1 (k)_1 + (q)_2 \right\} X_2$$

$$+ a^3 b^{k+q-3} \left\{ (k)_3 + (q)_1 (k)_2 + (q)_2 (k)_1 + (q)_3 \right\} X_3 + \ldots$$

$$\ldots + a^k b^q \left\{ (k)_k + (q)_1 (k)_{k-1} + \ldots + (q)_k\dagger \right\} X_k + \ldots$$

$$\ldots + a^{k+q-1} b \left\{ (q)_{q-1} (k)_k + (q)_q (k)_{k-1} \right\} X_{k+q-1},$$

$$\text{which} = b^{k+q} X_0 + (k+q)_1\, ab^{k+q-1} X_1 + (k+q)_2\, a^2 b^{k+q-2} X_2 + \ldots$$

$$\ldots + (k+q)_{k+q-1} a^{k+q-1} b X_{k+q-1}$$

$$= c_{k+q} - (\lambda+1)\, a^{k+q} X_{k+q}, \text{ by (1).}$$

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* *Quarterly Journal*, Vol. xxx., p. 152.

† This term is not reached unless $k \overline{\overline{<}} q$; but we may regard all the coefficients as of the form

$$(k)_m + (q)_1 (k)_{m-1} + \ldots + (q)_{m-1} (k)_1 + (q)_m = (k+q)_m,$$

if we suppose that in general $(r)_s$ denotes zero when $s > r$.

Thus, $$c_n-(\lambda+1)\,a^n X_n \equiv c_{k+q}-(\lambda+1)\,a^{k+q}X_{k+q}, \mod p.$$

Now, $$c_{kp+q}\equiv c_{k+q},\quad a_{kp+q}\equiv a^{k+q},\quad \mod p,$$

so that, if a is not divisible by p, this congruence gives

$$X_n \equiv X_{n-k(p-1)}, \quad \mod p;$$

and therefore the congruence

$$X_r \equiv X_{r-t(p-1)}, \quad \mod p,$$

holds good also when $r=n$.

It remains to show that this congruence holds good for $r=p$. Putting $n=p$ in the original recurring equation (i.), we have

$$c_p-(\lambda+1)\,a^pX_p$$
$$= b^pX_0+(p)_1\,ab^{p-1}X_1+(p)_2\,a^2b^{p-2}X_2+\ldots+(p)_{p-1}\,a^{p-1}bX_{p-1}$$

All the coefficients on the right-hand side, except the first, are divisible by p; and therefore

$$c_p-(\lambda+1)\,a^pX_p\equiv b^pX_0, \quad \mod p,$$

i.e., $$c_p-(\lambda+1)\,aX_p\equiv bX_0, \quad \mod p.$$

Also, putting $n=1$ in (i.),

$$c_1-(\lambda+1)\,aX_1 = bX_0,$$

whence, since $$c_p\equiv c_1, \quad \mod p,$$

we have $$X_p\equiv X_1, \mod p;$$

so that the congruence holds good for $r=p$.

3. The preceding investigation fails if a is divisible by p; so that the prime divisors of a must be excluded from the admissible values of p. Also, no divisor of a denominator of any of the X's can be an admissible value of p. If the denominator of X_0 be m, and if the quantities c_n have in their denominators only powers of certain numbers $\alpha, \beta, \gamma, \ldots$, then the denominator of X_n can only contain m and powers of a, $\lambda+1$, and $\alpha, \beta, \gamma, \ldots$. All prime numbers therefore which are not divisors of a, $\lambda+1$, m, $\alpha, \beta, \gamma, \ldots$ are admissible values of p.

It will be noticed that in the recurring relation (i.) we may replace the powers of a and b, a^r and b^r, by a_r and b_r, where a_r and b_r are any quantities which satisfy the same congruence as c_r, *i.e.*, so that

$$a_r\equiv a_{r-t(p-1)},\quad b_r\equiv b_{r-t(p-1)}, \quad \mod p.$$

4. In precisely the same manner, we may show that, if $X_0, X_2, X_4, \ldots$ are quantities connected by the recurring relation

$$\text{(ii.)} \quad (\lambda+1)\, a^{2n} X_{2n} + (2n)_2\, a^{2n-2} b^2 X_{2n-2} + \ldots$$
$$\ldots + (2n)_{2n-2}\, a^2 b^{2n-2} X_2 + (2n)_{2n}\, b^{2n} X_0 = c_{2n},$$

where k, a, b are as before, and

$$c_{2n} \equiv c_{2n-t(p-1)}, \quad \text{mod } p,$$

then
$$X_{2n} \equiv X_{2n-t(p-1)}, \text{ mod } p.$$

It is convenient to introduce the quantities $X_1, X_3, X_5, \ldots$, all of which are supposed to be zero; in the case of these quantities therefore the congruence

$$X_r \equiv X_{r-t(p-1)}, \quad \text{mod } p,$$

holds good.

Thus we may write (ii.),

$$c_{2n} - (\lambda+1)\, a^{2n} X_{2n}$$
$$= b^{2n} X_0 + (2n)_1\, ab^{2n-1} X_1 + (2n)_2\, a^2 b^{2n-2} X_2 + \ldots + (2n)^{2n-1}\, a^{2n-1} b X^{2n-1}.$$

Supposing, now, that the congruence

$$X_r \equiv X_{r-t(p-1)}, \quad \text{mod } p,$$

holds good for $r = p+1, p+2, \ldots, 2n-1$, we find, by putting

$$2n = kp + q$$

and reducing as before the exponents and suffixes, that the right-hand side

$$\equiv c_{k+q} - (\lambda+1)\, a^{k+q} X_{k+q}, \quad \text{mod } p.$$

Therefore
$$X_{2n} \equiv X_{k+q}, \quad \text{mod } p,$$

and the congruence holds good also for $r = 2n$.

Putting $2n = p+1$ in (ii.), we have

$$c_{p+1} - (\lambda+1)\, a^{p+1} X_{p+1} = b^{p+1} X_0 + (p+1)_1\, ab^p X_1 + (p+1)_2\, a^2 b^{p-1} X_2 + \ldots$$
$$\ldots + (p+1)_{p-1}\, a^{p-1} b^2 X_{p-1} + (p+1)_p\, a^p b X_p$$
$$\equiv b^{p+1} X_0 + ab^p X_1 + a^p b X_p, \quad \text{mod } p,$$
$$\equiv b^2 X_0, \quad \text{mod } p,$$

since X_1 and X_p are zero; and, by putting $n = 1$ in (ii.),

$$c_2 - (\lambda+1)\, a^2 X_2 = b^2 X_0,$$

Thus $$c_{p+1}-(\lambda+1)\,a^{p+1}X_{p+1} \equiv c_2-(\lambda+1)\,a^2X_2, \quad \text{mod } p,$$

and therefore $$X_{p+1} \equiv X_2, \quad \text{mod } p;$$

so that the congruence $X_r \equiv X_{r-t(p-1)}$, mod p,

holds good for $r = p+1$, and therefore for all higher values of r.

5. The same reasoning is applicable to the recurring relation

(iii.) $$(\lambda+1)\,a^{2n+1}X_{2n+1}+(2n+1)_2\,a^{2n-1}b^2X_{2n-1}+\dots$$
$$\dots+(2n+1)_{2n-2}a^3b^{2n-2}X_3+(2n+1)_{2n}\,ab^{2n}X_1 = c_{2n+1},$$

where $$c_{2n+1} \equiv c_{2n+1-t(p-1)}, \quad \text{mod } p.$$

For, introducing the zero quantities X_0, X_2, X_4, ..., we have

$$c_{2n+1}-(\lambda+1)\,a^{2n+1}X_{2n+1}$$
$$= b^{2n+1}X_0+(2n+1)_1\,ab^{2n}X_1+\dots+(2n+1)_{2n}\,a^{2n}bX_{2n},$$

and, if we assume, as before, that the congruence

$$X_r \equiv X_{r-t(p-1)}, \quad \text{mod } p,$$

holds good for all values of r from p to $2n$ inclusive, we find, as before, by putting $2n+1 = kp+q$, that the right-hand side

$$\equiv c_{k+q}-(\lambda+1)\,a^{k+q}X_{k+q}, \quad \text{mod } p.$$

Thus $$X_{2n+1} \equiv X_{k+q}, \quad \text{mod } p,$$

and the congruence holds good also for $r = 2n+1$.

Putting $n = p$ in (iii.), we have

$$c_p-(\lambda+1)\,a^pX_p = (p)_0\,b^pX_0+(p)_1\,ab^{p-1}X_1+\dots+(p)_{p-1}a^{p-1}bX_{p-1},$$
$$\equiv 0, \quad \text{mod } p,$$

since $X_0 = 0$; and, putting $2n+1 = 1$ in (iii.),

$$c_1-(\lambda+1)\,aX_1 = 0.$$

Therefore $$X_p \equiv X_1, \quad \text{mod } p,$$

and the congruence $$X_r \equiv X_{r-t(p-1)}, \quad \text{mod } p,$$

holds good for $r = p$, and therefore for all higher values of r.

The remarks in § 3 apply also to the recurring formulæ (ii.) and (iii.).

6. It has thus been shown that, if the X's are defined by any one of the recurring relations

(i.) $(\lambda+1)\, a^n X_n + (n)_1 a^{n-1} b X_{n-1} + \dots + (n)_{n-1} ab^{n-1} X_1 + (n)_n b^n X_0 = c_n,$

(ii.) $(\lambda+1)\, a^{2n} X_{2n} + (2n)_2 a^{2n-2} b^2 X_{2n-2} + \dots$

$$\dots + (2n)_{2n-2} a^2 b^{2n-2} X_2 + (2n)_{2n} b^{2n} X_0 = c_{2n},$$

(iii.) $(\lambda+1)\, a^{2n+1} X_{2n+1} + (2n+1)_2 a^{2n-1} b^2 X_{2n-1} + \dots$

$$\dots + (2n+1)_{2n-2} a^3 b^{2n-2} X_3 + (2n+1)_{2n} ab^{2n} X_1 = c_{2n+1},$$

where, in (i.), $c_n \equiv c_{n-t(p-1)}, \mod p,$

in (ii.), $c_{2n} \equiv c_{2n-t(p-1)}, \mod p,$

in (iii.), $c_{2n+1} \equiv c_{2n+1-t(p-1)}, \mod p,$

then $X_n \equiv X_{n-t(p-1)}, \mod p.$

No number which is a divisor of the denominator of any X is an admissible value of p, and there are also the other restrictions mentioned in § 3.

In the recurring formulæ (i.), (ii.), (iii.), the powers of a and b, viz., a^r and b^r, may be replaced by a_r and b_r, where a_r and b_r satisfy congruences similar to that satisfied by c_r in the same relation.

7. The recurring relation connecting the Eulerian numbers, viz.,

$$E_n - (2n)_2 E_{n-1} + (2n)_4 E_{n-2} - \dots + (-1)^{n-1} (2n)_{2n-2} E_1 + (-1)_n (2n)_{2n} E_0 = 0,$$

is a particular case of (ii.), corresponding to

$$\lambda = 0, \quad a = 1, \quad b = 1, \quad c_0 = 1, \quad c_2, c_4, \dots = 0,$$

$$X_{2n} = (-1)^n E_n.$$

In this case, putting $j = \frac{1}{2}(p-1)$ as before, the general congruence theorem becomes

$$(-1)^n E_n \equiv (-1)^{n-tj} E_{n-tj}, \mod p,$$

which is the Sylvester-Stern relation (§ 1). The Eulerian numbers are integers, and therefore all uneven primes are admissible values of p.

8. The recurring equations (i.), (ii.), (iii.) arise respectively from the expansions

$$\text{(i.)} \quad \frac{c_0+c_1\dfrac{x}{a}+\dfrac{c_2}{2!}\dfrac{x^2}{a^2}+\dfrac{c_3}{3!}\dfrac{x^3}{a^3}+\&\text{c.}}{\lambda+e^{\frac{b}{a}x}} = X_0+X_1x+\frac{X_2}{2!}x^2+\frac{X_3}{3!}x^3+\&\text{c.},$$

$$\text{(ii.)} \quad \frac{c_0+\dfrac{c_2}{2!}\dfrac{x^2}{a^2}+\dfrac{c_4}{4!}\dfrac{x^4}{a^4}+\&\text{c.}}{\lambda+\cosh\dfrac{b}{a}x} = X_0+\frac{X_2}{2!}x^2+\frac{X_4}{4!}x^4+\&\text{c.},$$

$$\text{(iii.)} \quad \frac{c_1\dfrac{x}{a}+\dfrac{c_3}{3!}\dfrac{x^3}{a^3}+\dfrac{c_5}{5!}\dfrac{x^5}{a^5}+\&\text{c.}}{\lambda+\cosh\dfrac{b}{a}x} = X_1x+\frac{X_3}{3!}x^3+\frac{X_5}{5!}x^5+\&\text{c.};$$

or, putting ax for x, from the expansions

$$\text{(i.)} \quad \frac{c_0+c_1x+\dfrac{c_2}{2!}x^2+\dfrac{c_3}{3!}x^3+\&\text{c.}}{\lambda+e^{bx}} = X_0+X_1ax+\frac{X_2}{2!}a^2x^2+\frac{X_3}{3!}a^3x^3+\&\text{c.},$$

$$\text{(ii.)} \quad \frac{c_0+\dfrac{c_2}{2!}x^2+\dfrac{c_4}{4!}x^4+\&\text{c.}}{\lambda+\cosh bx} = X_0+\frac{X_2}{2!}a^2x^2+\frac{X_4}{4!}a^4x^4+\&\text{c.},$$

$$\text{(iii.)} \quad \frac{c_1x+\dfrac{c_3}{3!}x^3+\dfrac{c_5}{5!}x^5+\&\text{c.}}{\lambda+\cosh bx} = X_1ax+\frac{X_3}{3!}a^3x^3+\frac{X_5}{5!}a^5x^5+\&\text{c.}$$

9. If we put $\lambda = 0$, $a = 1$, and replace b^r by b_r, these expansions become

$$\text{(i.)} \quad \frac{\Sigma_0^\infty \dfrac{c_n}{n!}x^n}{\Sigma_0^\infty \dfrac{b_n}{n!}x^n} = \Sigma_0^\infty \frac{X_n}{n!}x^n,$$

$$\text{(ii.)} \quad \frac{\Sigma_0^\infty \dfrac{c_{2n}}{(2n)!}x^{2n}}{\Sigma_0^\infty \dfrac{b_{2n}}{(2n)!}x^{2n}} = \Sigma_0^\infty \frac{X_{2n}}{(2n)!}x^{2n},$$

$$\text{(iii.)} \quad \frac{\Sigma_0^\infty \dfrac{c_{2n+1}}{(2n+1)!}x^{2n+1}}{\Sigma_0^\infty \dfrac{b_{2n}}{(2n)!}x^{2n}} = \Sigma_0^\infty \frac{X_{2n+1}}{(2n+1)!}x^{2n+1}.$$

The sole condition in (1) is that b_n and c_n should satisfy the congruence

$$u_n \equiv u_{n-t(p-1)}, \quad \text{mod } p.$$

If this condition is fulfilled, X_n also satisfies the same congruence. Similarly, in (ii.), if b_{2n} and c_{2n} satisfy the congruence

$$u_{2n} \equiv u_{2n-t(p-1)}, \quad \text{mod } p,$$

then X_{2n} satisfies the same congruence. In (iii.), if

$$c_{2n+1} \equiv c_{2n+1-t(p-1)}, \quad \text{mod } p,$$

and $$b_{2n} \equiv b_{2n-t(p-1)}, \quad \text{mod } p,$$

then $$X_{2n+1} \equiv X_{2n+1-t(p-1)}, \quad \text{mod } p.$$

The expansion (i.) shows that, if we have two series of the form

$$\Sigma_0^\infty \frac{a_n}{n!} x^n,$$

in which the coefficient a_n satisfies the congruence

$$u_n \equiv u_{n-t(p-1)}, \quad \text{mod } p,$$

then X_n, the coefficient of $\frac{x^n}{n!}$ in their quotient, satisfies the same congruence.

10. The formulæ (i.), (ii.), (iii.) of § 9 include some very general expansions. Thus (i.) includes the expansion of any quantity of the form

$$\frac{Ae^{ax}+Be^{\beta x}+Ce^{\gamma x}+\dots}{A'e^{a'x}+B'e^{\beta' x}+C'e^{\gamma' x}+\dots},$$

where $a, \beta, \gamma, \dots, a', \beta', \gamma', \dots$ are integers; (ii.) includes the expansion of any quantity of the form

$$\frac{\Sigma A \cosh ax}{\Sigma A' \cosh a'x};$$

and (iii.) of the form $$\frac{\Sigma A \sinh ax}{\Sigma A' \cosh a'x}.$$

If in (ii.) and (iii.) we replace the hyperbolic by circular functions, which merely requires the substitution of xi for x on the right-hand side, we obtain the expansions

$$\text{(ii.)}\quad \frac{\Sigma A\cos ax}{\Sigma A'\cos a'x}=\Sigma_0^\infty \frac{Y_{2n}}{(2n)!}x^{2n},$$

$$\text{(iii.)}\quad \frac{\Sigma A\sin ax}{\Sigma A'\cos a'x}=\Sigma_0^\infty \frac{Y_{2n+1}}{(2n+1)!}x^{2n+1},$$

where, in (ii.), $\qquad Y_{2n}=(-1)^n X_{2n},$

and in (iii.), $\qquad Y_{2n+1}=(-1)^n X_{2n+1}.$

The Y-coefficients therefore satisfy the respective congruences

$$Y_{2n}\equiv(-1)^{t\cdot\frac{1}{2}(p-1)}\,Y_{2n+t(p-1)},\qquad \text{mod } p,$$

$$Y_{2n+1}\equiv(-1)^{t\cdot\frac{1}{2}(p-1)}\,Y_{2n+1-t(p-1)},\qquad \text{mod } p.$$

11. These expansions include, besides the Eulerian numbers, several similar sets of coefficients which have been considered in some recent papers in the *Quarterly Journal of Mathematics** and *Messenger of Mathematics*.†

The Eulerian numbers may be regarded as defined by the expansion

$$\text{(i.)}\quad \frac{1}{\cos x}=1+\frac{E_1}{2!}x^2+\frac{E_2}{4!}x^4+\frac{E_3}{6!}x^6+\&c.,$$

and the other coefficients I_n, H_n, J_n, ... as defined by

$$\text{(ii.)}\quad \frac{1}{2\cos x+1}=\tfrac{2}{3}\left\{I_0+\frac{I_1}{2!}x^2+\frac{I_2}{4!}x^4+\frac{I_3}{6!}x^6+\&c.\right\},$$

$$\text{(iii.)}\quad \frac{1}{2\cos x-1}=\tfrac{2}{3}\left\{H_0+\frac{H_1}{2!}x^2+\frac{H_2}{4!}x^4+\frac{H_3}{6!}x^6+\&c.\right\},$$

$$\text{(iv.)}\quad \frac{2\cos x}{2\cos 2x+1}=\tfrac{1}{3}\left\{J_0+\frac{J_1}{2!}x^2+\frac{J_2}{4!}x^4+\frac{J_3}{6!}x^6+\&c.\right\},$$

$$\text{(v.)}\quad \frac{\cos x}{\cos 2x}=P_0+\frac{P_1}{2!}x^2+\frac{P_2}{4!}x^4+\frac{P_3}{6!}x^6+\&c.,$$

$$\text{(vi.)}\quad \frac{\sin x}{\cos 2x}=Q_1x+\frac{Q_2}{3!}x^3+\frac{Q_3}{5!}x^5+\&c.,$$

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* "On the Bernoullian Function," Vol. xxix., pp. 1–168.

† "On the Definite Integrals connected with the Bernoullian Function," Vol. xxvi., pp. 151–182, and Vol. xxvii., pp. 20–98.

(vii.) $$\frac{\cos^2 x}{\cos 3x} = R_0 + \frac{R_1}{2!}x^2 + \frac{R_2}{4!}x^4 + \frac{R_3}{6!}x^6 + \&c.,$$

(viii.) $$\frac{\sin x \cos x}{\cos 3x} = T_1 x + \frac{T_2}{3!}x^3 + \frac{T_3}{5!}x^5 + \&c.^*$$

All these coefficients therefore satisfy a congruence of exactly the same form; *ex. gr.*, taking I_n, we have

$$(-1)^n I_n \equiv (-1)^{n-\theta} I_{n-\theta}, \quad \bmod p,$$

that is,

$$I_n \equiv (-1)^{\theta} I_{n-\theta}, \quad \bmod p.$$

The coefficients are all integers, except the I's and J's, and the I's and J's contain only powers of 3 in the denominator (see the next paper). Thus, except in the case of the I's and J's, p may be any uneven prime, and for the I's and J's the value $p = 3$ is alone excluded.

12. The X-coefficients in the expansions of §§ 8 and 9 include the Bernoullian functions $B_n(x)$ and $A'_n(x)$,† which therefore, in general,

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* The coefficients S_n defined by the equation

$$\frac{\cos 2x}{\cos 3x} = S_0 + \frac{S_1}{2!}x^2 + \frac{S_2}{4!}x^4 + \frac{S_3}{6!}x^6 + \&c.$$

were considered in *Messenger*, Vol. XXVIII., p. 49. The quantities R_n and S are connected by the relation

$$2R_n + E_n = 3S_n.$$

Both R_n and S_n can be expressed in terms of E_n, the formula being

$$R_n = \frac{3^{2n+1}+1}{4} E_n, \quad S_n = \frac{3^{2n}+1}{2} E_n.$$

The quantities H_n and J_n may be expressed in terms of I_n by the formulæ

$$H_n = (2^{2n+1}+1)\, H_n, \quad J_n = 2\,(2^{2n}+1)\, I_n$$

(see § 24 of the next paper).

† The functions $B_n(x)$ and $A'_n(x)$ may be defined as follows:—

$$B_n(x) = \frac{1}{n}\left\{x^n - \frac{n}{2}x^{n-1} + (n)_2 B_1 x^{n-2} - (n)_4 B_2 x^{n-4} + \dots\right\},$$

the series being continued up to the term involving x or x^2, so that the last term is

$$(-1)^{\frac{1}{2}(n-1)}(n)_{n-1} B_{\frac{1}{2}(n-1)}\, x \quad \text{or} \quad (-1)^{\frac{1}{2}n}(n)_{n-2} B_{\frac{1}{2}(n-2)}\, x^2,$$

according as n is uneven or even;

$$A'_n(x) = \frac{1}{n}\left\{\frac{n}{2}x^{n-1} - (n)_2 (2^2-1) B_1 x^{n-2} + (n)_4 (2^4-1) B_2 x^{n-4} - \dots\right\},$$

the series being continued up to the term involving x or x^0, so that the last term is

$$(-1)^{\frac{1}{2}(n-1)}(n)_{n-1}(2^{n-1}-1) B_{\frac{1}{2}(n-1)}\, x \quad \text{or} \quad (-1)^{\frac{1}{2}n}(2^n-1) B_{\frac{1}{2}n},$$

according as n is uneven or even. (*Quarterly Journal*, Vol. XXIX., pp. 7, 94). In these formulæ B_r denotes the r^{th} Bernoullian number.

These definitions have been given at full length, as there are several slightly satisfy the congruence

$$u_n \equiv u_{n-t(p-1)}, \quad \text{mod } p.$$

It will be shown that the only inadmissible values of p are the divisors of the denominator of x. The modulus p, as in the preceding sections, is restricted to uneven primes.

Taking first the function $A_n'(x)$, we have

$$\frac{e^{bx}}{e^{ax}+1} = A_1'\left(\frac{b}{a}\right) + aA_2'\left(\frac{b}{a}\right)x + a^2A_3'\left(\frac{b}{a}\right)\frac{x^2}{2!} + a^3A_4'\left(\frac{b}{a}\right)\frac{x^3}{3!} + \&c.,*$$

which is included in formula (i.) of § 8.

It follows therefore that, if a and b be any positive integers, which we may take to be prime to one another, then

$$a^{n-1}A_n'\left(\frac{b}{a}\right) \equiv a^{n-1-t(p-1)}A'_{n-t(p-1)}\left(\frac{b}{a}\right), \quad \text{mod } p.$$

Since
$$a^n \equiv a^{n-t(p-1)}, \quad \text{mod } p,$$
we find therefore that, if p is not a divisor of a, then

$$A_n'\left(\frac{b}{a}\right) \equiv A'_{n-t(p-1)}\left(\frac{b}{a}\right), \quad \text{mod } p.$$

It may be remarked that, if a and b are integers, the quantity $a^{n-1}A_n'\left(\frac{b}{a}\right)$ is necessarily an integer, except for a denominator containing powers of 2; for, putting

$$\alpha_{n-1} = a^{n-1}A_n'\left(\frac{b}{a}\right),$$

the expansion is

$$\frac{e^{bx}}{e^{ax}+1} = \alpha_0 + \alpha_1 x + \frac{\alpha_2}{2!}x^2 + \frac{\alpha_3}{2!}x^3 + \&c.,$$

---

differing forms of the Bernoullian function, each of which is specially adapted to some of its applications. Thus for very many purposes it is convenient to use a function $A_n(x)$ in place of $B_n(x)$ as just defined, where $A_{2n+1}(x)$ is the same as $B_{2n+1}(x)$, but

$$A_{2n}(x) = B_{2n}(x) + (-1)^{n-1}\frac{B_n}{2n}.$$

It is also frequently convenient to make a further modification, and use the functions $V_n(x)$ and $U_n(x)$, where

$$V_n(x) = nA_n(x) \quad \text{and} \quad U_n(x) = nA_n'(x)$$

(*Quarterly Journal, loc. cit.*, p. 115).

* This formula may be derived from p. 94 of Vol. XXIX. of the *Quarterly Journal*.

which gives the recurring relation

$$2a_n + (n)_1 a a_{n-1} + (n)_2 a^2 a_{n-2} + \ldots + (n)_n a^n a_0 = b^n,$$

where $a_0 = \frac{1}{2}$. Thus a_n must be of the form $\dfrac{\text{integer}}{\text{power of } 2}$.

This recurring relation shows also that, if a be prime to b, the numerator of a_n cannot be divisible by a.

Since the congruence

$$a^{n-1} A_n' \left(\frac{b}{a}\right) \equiv a^{m-1} A_m' \left(\frac{b}{a}\right), \quad \text{mod } p,$$

where $$n - m = t\,(p-1),$$

holds good for all (uneven) values of p, and since $a^n A_n' \left(\frac{b}{a}\right)$ contains only a power of 2 in the denominator, and cannot contain a as a factor in the numerator, we see that nothing exceptional occurs when p is a divisor of a, for in this case the congruence does not in general reduce to $0 \equiv 0$, mod p.

As an example, take the formula

$$3^{2n} A_{2n+1}' \left(\tfrac{1}{3}\right) = (-1)^n \frac{H_n}{3},\text{*}$$

where H_n is the same as in the second expansion in § 11.

It follows therefore that $\frac{H_n}{3}$ is an integer, except for powers of 2 in the denominator (which, as a fact, do not occur), and that the numerator of $\frac{H_n}{3}$ $\left(\text{that is } \frac{H_n}{3} \text{ itself}\right)$ is not divisible by 3, and we have, taking $p = 3$,

$$\frac{H_n}{3} \equiv (-1)^t \frac{H_{n-t}}{3}, \quad \text{mod } 3.$$

13. The proof just given of the congruence

$$a^{n-1} A_n' \left(\frac{b}{a}\right) \equiv a^{m-1} A_m' \left(\frac{b}{a}\right), \quad \text{mod } p,$$

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* *Quarterly Journal*, Vol. XXIX., p. 107, or *Messenger*, Vol. XXVI., p. 178.

applies to all values of the suffix n, even or uneven; it is, however, interesting to give the expansion formulæ in which the suffixes are all uneven or all even, and from which the congruence-theorem may be derived separately for uneven and for even suffixes. These expansion formulæ are

$$\tfrac{1}{2}\frac{\cosh(2b-a)x}{\cosh ax} = A_1'\left(\frac{b}{a}\right)+a^2A_3'\left(\frac{b}{a}\right)\frac{(2x)^2}{2!}+a^4A_5'\left(\frac{b}{a}\right)\frac{(2x)^4}{4!}+\&c.,$$

$$\tfrac{1}{2}\frac{\sinh(2b-a)x}{\cosh ax} = aA_2'\left(\frac{b}{a}\right)2x+a^3A_4'\left(\frac{b}{a}\right)\frac{(2x)^3}{3!}$$

$$+a^5A_6'\left(\frac{b}{a}\right)\frac{(2x)^5}{5!}+\&c.,^*$$

which are included respectively in (ii.) and (iii.) of § 8.

14. Passing now to the function $B_n(x)$, we have

$$\frac{e^{bx}-1}{e^{ax}-1} = \frac{b}{a}+aB_2\left(\frac{b}{a}\right)x+a^2B_3\left(\frac{b}{a}\right)\frac{x^2}{2!}+a^3B_4\left(\frac{b}{a}\right)\frac{x^3}{3!}+\&c.,^\dagger$$

and, by dividing both numerator and denominator by e^x-1, the left-hand side becomes

$$\frac{e^{(b-1)x}+e^{(b-2)x}+\ldots+e^x+1}{e^{(a-1)x}+e^{(a-2)x}+\ldots+e^x+1}.$$

This form is included in (i.) of § 9, being a special case of the first form noticed in § 10; so that

$$a^{n-1}B_n\left(\frac{b}{a}\right) \equiv a^{m-1}B_m\left(\frac{b}{a}\right), \quad \text{mod } p,$$

if $$n-m = t(p-1);$$

and therefore, a and b being prime to each other, and p not being a divisor of a,

$$B_n\left(\frac{b}{a}\right) \equiv B_{n-t(p-1)}\left(\frac{b}{a}\right), \quad \text{mod } p.$$

It is easy to see that $B_n\left(\frac{b}{a}\right)$ can contain only powers of a in the denominator, for, putting

$$\mu_r = 1^r+2^r+\ldots+(a-1)^r,$$

$$\nu_r = 1^r+2^r+\ldots+(b-1)^r,$$

---

* *Quarterly Journal*, Vol. xxix., p. 107.

† *Ib.*, Vol. xxix., p. 7.

and $$\beta_{r-1} = a^{r-1}B_r\left(\frac{b}{a}\right),$$

the expansion formula gives

$$\frac{b+\nu_1 x+\frac{\nu_2}{2!}x^2+\frac{\nu_3}{3!}x^3+\&c.}{a+\mu_1 x+\frac{\mu_2}{2!}x^2+\frac{\mu_3}{3!}x^3+\&c.} = \frac{b}{a}+\beta_1 x+\frac{\beta_2}{2!}x^2+\frac{\beta_3}{3!}x^3+\&c.,$$

which leads to the recurring relation

$$a\beta_n+(n)_1\mu_1\beta_{n-1}+(n)_2\mu_2\beta_{n-2}+\ldots+(n)_n\mu_n\frac{b}{a} = \nu_n.$$

Since the μ's and ν's are necessarily integers, this relation shows that β_n can contain only powers of a in the denominator.

15. The expansion formulæ in which the suffixes of the B's are all uneven or all even are

$$\tfrac{1}{2}\frac{\sinh(2b-a)x}{\sinh ax.} = \frac{2b-a}{2a}+a^2B_3\left(\frac{b}{a}\right)\frac{(2x)^2}{2!}+a^4B_5\left(\frac{b}{a}\right)\frac{(2x)^4}{4!}+\&c.,$$

$$\tfrac{1}{2}\frac{\cosh(2b-a)x-\cosh ax}{\sinh ax} = aB_2\left(\frac{b}{a}\right)2x+a^3B_4\left(\frac{b}{a}\right)\frac{(2x)^3}{3!}+\&c.*$$

If r is a positive integer, $\sinh rx$ contains $\sinh x$ as a factor, the other factor being

$$1+2\cosh 2x+2\cosh 4x+\ldots+2\cosh(r-1)x, \text{ if } r \text{ is uneven,}$$

and $$2\cosh x+2\cosh 3x+\ldots+2\cosh(r-1)x, \quad \text{if } r \text{ is even.}$$

Thus the left-hand side of the first equation is of the form

$$\frac{\Sigma A\cosh ax}{\Sigma A'\cosh a'x};$$

and, since $\cosh(2b-a)x-\cosh ax = 2\sinh bx\sinh(b-a)x$,

the left-hand side of the second equation is of the form

$$\frac{\Sigma A\sinh ax}{\Sigma A'\sinh a'x}.$$

The two expansion formulæ are therefore included respectively in (ii.) and (iii.) of § 9.

* *Quarterly Journal*, Vol. xxix., pp. 5 and 6, or p. 119.

16. In connexion with these formulæ it may be remarked that the expansion

$$\frac{\sinh cx}{\sinh bx} = X_0 + \frac{X_2}{2!}a^2x^2 + \frac{X_4}{4!}a^4x^4 + \frac{X_6}{6!}a^6x^6 + \&c.$$

gives rise to the recurring relation

$$(2n+1)\,a^{2n}bX_{2n} + (2n+1)_3\,a^{2n-2}b^3X_{2n-2} + \ldots + (2n+1)_{2n-1}\,a^2b^{2n-1}X_2$$
$$+ (2n+1)_{2n+1}\,b^{2n+1}X_0 = c^{2n+1}.$$

This relation is of the same kind as (i.), (ii.), (iii.) of § 6, and it is easy to see that the reasoning employed in §§ 2 and 4 holds good also in the case of this formula, and shows that X_{2n} satisfies the congruence

$$X_{2n} \equiv X_{2n-t(p-1)}, \quad \text{mod } p.$$

It would seem, however, that this result could not be of any practical value, since X_{2n}, as calculated from the above recurring relation, might contain in the denominator any uneven numbers up to $2n+1$, so that there might be no admissible value of p; but this is not the case, for the left-hand side, on dividing both numerator and denominator by $\sinh x$, becomes the quotient of expansions which are of the forms $2\Sigma \cosh(2r+1)x$ or $1+2\Sigma \cosh 2rx$; and therefore the expansion is included in (ii.) of § 9. We see, too, by forming the recurring equation corresponding to this form of the left-hand side, that $a^{2n}X_{2n}$ can contain only powers of b in the denominator. Thus the only values of p to be excluded are those which are divisors of a and b.

If we put $a = 1$, so that

$$\frac{\sinh cx}{\sinh bx} = X_0 + \frac{X_2}{2!}x^2 + \frac{X_4}{4!}x^4 + \frac{X_6}{6!}x^6 + \&c.,$$

then
$$X_{2n} \equiv X_{2n-t(p-1)}, \quad \text{mod } p,$$

for all uneven values of p that are not divisors of b.

17. We may obtain this result also in another manner; for, from the first formula in § 15, we have

$$\tfrac{1}{2}\frac{\sinh cx}{\sinh bx} = \frac{c}{2b} + b^2B_2\left(\frac{b+c}{2b}\right)\frac{(2x)^2}{2!} + b^4B_5\left(\frac{b+c}{2b}\right)\frac{(2x)^4}{4!} + \&c.$$

Comparing this expansion with that just written, we have

$$X_{2n} = 2^{2n+1} b^{2n} B_{2n+1}\left(\frac{b+c}{2b}\right);$$

and therefore, by § 14, if p is not a divisor of b,

$$X_{2n} \equiv X_{2n-t(p-1)}, \quad \text{mod } p.$$

18. In the preceding sections p has always been supposed to be an uneven prime, and it now remains to consider the case of $p = 2$. The residues of the X-coefficients with respect to modulus 2 may be easily determined, in the case of any of the expansions, by means of the recurring formulæ.

Consider first the recurring formula (i.) of § 6, in which, writing λ' for $\lambda+1$, and putting $n = 1, 2, 3, \ldots$, we have

$$\lambda' a X_1 + b X_0 = c_1,$$
$$\lambda' a^2 X_2 + 2_1 ab X_1 + b^2 X_0 = c_2,$$
$$\lambda' a^3 X_3 + 3_1 a^2 b X_2 + 3_2 ab^2 X_1 + b^3 X_0 = c_3,$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

Suppose $\lambda' \equiv 1$, mod 2, and let a and b be uneven integers. The assumption $\lambda' \equiv 1$, mod 2, excludes the case of λ' having 2 as a divisor in the denominator. It is supposed that $c_1, c_2, c_3, \ldots$ are all $\equiv 0$ or all $\equiv 1$, mod 2; so that none of them can have 2 as a divisor in the denominator. The quantity X_0 (which is the value of the function expanded, when x is put $= 0$) is also supposed not to have a denominator divisible by 2.*

It will now be shown that $X_1, X_2, X_3, \ldots$ are all $\equiv 0$, mod 2, if X_0, $c_1, c_2, c_3, \ldots$ are all $\equiv 1$, or all $\equiv 0$, mod 2; but that $X_1, X_2, X_3, \ldots$ are all $\equiv 1$, mod 2, if $X_0 \equiv 1$ and $c_1, c_2, c_3, \ldots$ are all $\equiv 0$, mod 2, or if $X_0 \equiv 0$ and $c_1, c_2, c_3, \ldots$ are all $\equiv 1$, mod 2.

I. Let $X_0 \equiv 1$, and $c_1, c_2, c_3, \ldots \equiv 1$, mod 2.

The recurring formulæ give

$\lambda' a X_1 \equiv 1-1 \equiv 0$, mod 2, so that $X_1 \equiv 0$, mod 2;

$\lambda' a^2 X_2 \equiv 1-1-0 \equiv 0$, mod 2, so that $X_2 \equiv 0$, mod 2;

$\lambda' a^3 X_3 \equiv 1-1-0-0 \equiv 0$, mod 2, so that $X_3 \equiv 0$, mod 2;

and so on.

* Since $c_0 = \lambda' X_0 \equiv X_0$, mod 2, we may use c_0 in place of X_0 throughout.

II. Let $X_0 \equiv 1$, and $c_1, c_2, c_3, \ldots \equiv 0$, mod 2.

In this case

$\lambda' a X_1 \equiv 0-1 \equiv 1$, mod 2; so that $X_1 \equiv 1$, mod 2,

$\lambda' a^2 X_2 \equiv 0-(b^2+2_1 ab) \equiv a^2-(a+b)^2 \equiv 1$, mod 2;

so that $X_2 \equiv 1$, mod 2,

$\lambda' a^3 X_3 \equiv a^3-(a+b)^3 \equiv 1$, mod 2; so that $X_3 \equiv 1$, mod 2,

and so on; since, a and b being uneven, $a^n-(a+b)^n$ is necessarily uneven.

III. Let $X_0 \equiv 0$, and $c_1, c_2, c_3, \ldots \equiv 1$, mod 2.

In this case

$\lambda' a X_1 \equiv 1-0 \equiv 1$, mod 2; so that $X_1 \equiv 1$, mod 2,

$\lambda' a^2 X_2 \equiv 1-0-2_1 ab \equiv 1+a^2+b^2-(a+b)^2 \equiv 1$, mod 2;

so that $X_2 \equiv 1$, mod 2,

$\lambda' a^3 X_3 \equiv 1+a^3+b^3-(a+b)^3 \equiv 1$, mod 2; so that $X_3 \equiv 1$, mod 2,

and so on.

IV. Let $X_0 \equiv 0$ and $c_1, c_2, c_3, \ldots \equiv 0$, mod 2.

In this case

$\lambda' a X_1 \equiv 0-0 \equiv 0$, mod 2; so that $X_1 \equiv 0$, mod 2,

$\lambda' a^2 X_2 \equiv 0$, mod 2; so that $X_2 \equiv 0$, mod 2,

and so on.

If b is even, the general recurring formula (i.) shows that

$$X_n \equiv c_n, \text{ mod } 2;$$

so that, whether the residue of X_0 be 1 or 0, mod 2, the X's $\equiv$ the c's, mod 2. We may always regard a as uneven, and, in fact, there is no loss of generality in the expansion-formula (§ 8) by putting $a = 1$.

19. Exactly the same reasoning holds good with respect to the recurring formula (ii.) of § 6, viz., we have

$$\lambda' a^2 X_2 + b^2 X_0 = c_2,$$

$$\lambda' a^4 X_4 + 4_2 a^2 b^2 X_2 + b^4 X_0 = c_4,$$

$$\lambda' a^6 X_6 + 6_2 a^4 b^2 X_4 + 6_4 a^2 b^4 X_2 + b^6 X_0 = c_6,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots,$$

and, under the same conditions as those expressed at the beginning of the preceding section, viz., $\lambda' \equiv 1$, mod 2, a and b uneven, and

$c_2, c_4, c_6, \ldots$ all $\equiv 1$ or all $\equiv 0$, mod 2, we find that X_2, X_4, X_6, are all $\equiv 0$, mod 2, if X_0, c_2, c_4, c_6, are all $\equiv 1$, or all $\equiv 0$, mod 2, but that $X_2, X_4, X_6, \ldots$ are all $\equiv 1$, mod 2, if $X_0 \equiv 1$, and $c_2, c_4, c_6, \ldots$ are all $\equiv 0$, mod 2, or if $X_0 \equiv 0$, and $c_2, c_4, c_6, \ldots$ are all $\equiv 1$, mod 2.

In proving these results we have, in Case II.,

$$\lambda' a^{2n} X_{2n} \equiv a^{2n} + b^{2n} - \frac{(a+b)^{2n} + (a-b)^{2n}}{2}, \quad \text{mod } 2,$$

the right-hand side of which is necessarily even.

If b is even, we have evidently

$$X_{2n} \equiv c_{2n}, \quad \text{mod } 2.$$

20. In the case of the recurring formula (iii.) of § 6, we have

$$\lambda' a^3 X_3 + 3_2 a b^2 X_1 = c_3,$$

$$\lambda' a^5 X_5 + 5_2 a^3 b^2 X_3 + 5_4 a b^4 X_1 = c_5,$$

$$\lambda' a^7 X_7 + 7_2 a^5 b^2 X_5 + 7_4 a^3 b^4 X_3 + 7_6 a b^6 X_1 = c_7,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

and supposing $\lambda' \equiv 1$, mod 2, a and b being uneven integers, and $c_3, c_5, c_7, \ldots$ all $\equiv 1$ or $\equiv 0$, mod 2, and separating the four cases

I. *$X_1 \equiv 1$, and $c_3, c_5, c_7, \ldots$ all $\equiv 1$, mod 2;

II. $X_1 \equiv 1$, and $c_3, c_5, c_7, \ldots$ all $\equiv 0$, mod 2;

III. $X_1 \equiv 0$, and $c_3, c_5, c_7, \ldots$ all $\equiv 1$, mod 2;

IV. $X_1 \equiv 0$, and $c_3, c_5, c_7, \ldots$ all $\equiv 0$, mod 2;

we find that $X_3, X_5, X_7, \ldots$ are all $\equiv 0$, mod 2, if $X_1, c_3, c_5, c_7, \ldots$ are all $\equiv 1$, or all $\equiv 0$, mod 2, but that $X_3, X_5, X_7, \ldots$ are all $\equiv 1$, mod 2, if $X_1 \equiv 1$ and $c_3, c_5, c_7, \ldots$ are all $\equiv 0$, mod 2, or if $X_1 \equiv 0$ and $c_3, c_5, c_7, \ldots$ are all $\equiv 1$, mod 2.

In proving these results, we have, in Case II.,

$$\lambda' a^{2n+1} X_{2n+1} \equiv \frac{(a+b)^{2n+1} + (a-b)^{2n+1}}{2} - a^{2n+1}, \quad \text{mod } 2,$$

the right-hand side of which is necessarily even; in Case III. the extra term 1 occurs, but the term in X_1 which $\equiv 1$, mod 2, is omitted, so that in this case also

$$X_{2n+1} \equiv 1, \quad \text{mod } 2.$$

If b is even, we have $\quad X_{2n+1} \equiv c^{2n+1}, \quad$ mod 2.

* Since $ac = \lambda' X_1$, we have $c_1 \equiv X_1$, mod 2, and we may therefore use c_1 in place of X_1 throughout.

21. In the eight expansions of § 11, we have respectively

(i.) $\lambda' = 1$; $a = 1$; $b = 1$; $c_2, c_4, \ldots = 0$; $X_0 = 1$ $[X_{2n} = (-1)^n E_n]$.

(ii.) $\lambda' = \frac{3}{2}$; $a = 1$; $b = 1$; $c_2, c_4, \ldots = 0$; $X_0 = \frac{1}{2}$ $[X_{2n} = (-1)^n I_n]$.

(iii.) $\lambda' = \frac{1}{2}$; $a = 1$; $b = 1$; $c_2, c_4, \ldots = 0$; $X_0 = \frac{3}{2}$ $[X_{2n} = (-1)^n H_n]$.

(iv.) $\lambda' = \frac{3}{2}$; $a = 1$; $b = 2$; $c_2, c_4, \ldots = 1$; $X_0 = 2$ $[X_{2n} = (-1)^n J_n]$.

(v.) $\lambda' = 1$; $a = 1$; $b = 2$; $c_2, c_4, \ldots = 1$; $X_0 = 1$ $[X_{2n} = (-1)^n P_n]$.

(vi.) $\lambda' = 1$; $a = 1$; $b = 2$; $c_3, c_5, \ldots = 1$; $X_1 = 1$

$$[X_{2n-1} = (-1)^{n-1} Q_n].$$

(vii.) $\lambda' = 1$; $a = 1$; $b = 3$; $c_2, c_4, \ldots \equiv 0$, mod 2; $X_0 = 1$

$$[X_{2n} = (-1)^n R_n].$$

(viii.) $\lambda' = 1$; $a = 1$; $b = 3$; $c_3, c_5, \ldots \equiv 0$, mod 2; $X_1 = 1$

$$[X_{2n-1} = (-1)^{n-1} T_n].$$

These expansions are included in (i.) of § 8, except the sixth and eighth, which belong to (iii.) of § 8.

The theorems of the three preceding sections are applicable to the first, and to the last four, of these expansions, and show that in these cases the coefficients $\equiv 1$, mod 2. Since these coefficients are all integers, we thus see that E_n, P_n, Q_n, R_n, T_n are all uneven numbers.

22. When λ' and X_0 contain powers of 2 in the denominator, as in (ii.), (iii.), (iv.), it is easy to determine in each case the residues of the X's with respect to modulus 2 by means of the recurring relation. For example, in (ii.), where $\lambda' = \frac{3}{2}$, $X_0 = \frac{1}{2}$, and $c_2, c_4, \ldots = 0$, we see at once, from the recurring relation, that

$$3X_{2n}+1 \equiv 0, \text{ mod } 2, \quad \text{so that } X_{2n} \equiv 1, \text{ mod } 2;$$

and similarly, in (iii.),

$$X_{2n}+3 \equiv 0, \text{ mod } 2, \quad \text{so that } X_{2n} \equiv 1, \text{ mod } 2;$$

in (iv.),

$$3X_{2n} \equiv 0, \text{ mod } 2, \quad \text{so that } X_{2n} \equiv 0, \text{ mod } 2.$$

23. The fact that in each of the expansions the coefficients when integral (such as the E's, the P's, &c.) end in one or other of two

digits, or all end in the same digit,* is explained by the consideration of their residues with respect to the modulus 2 and the modulus 5; for, if all the coefficients have the same residue to modulus 2, and if all the alternate coefficients have the same residue to modulus 5, it is evident that all the alternate coefficients must have the same residue to modulus 10; so that the difference between any coefficient and the coefficient next but one to it must, if integral, be a multiple of 10.

Putting $p = 5$ in the general congruence formulæ of § 6, we have,

when X_n is defined by (i.) of § 8, $X_n \equiv X_{n-4t}$, mod 5,

„ X_{2n} „ „ (ii.) „ $X_{2n} \equiv X_{2n-4t}$, „

„ X_{2n+1} „ „ (iii.) „ $X_{2n+1} \equiv X_{2n+1-4t}$, „

Therefore, putting $X_{2n} = (-1)^n Z_n$ in (ii.), so that the expansion is

$$\text{(ii.)}\qquad \frac{c_0 - \frac{c_2}{2!}x^2 + \frac{c_4}{4!}x^4 - \&c.}{\lambda + \cos bx} = Z_0 + \frac{Z_1}{2!}x^2 + \frac{Z_2}{4!}x^4 + \&c.,$$

we have $Z_n \equiv Z_{n-2t}$, mod 5;

and, putting $X_{2n-1} = (-1)^{n-1} Z_n$ in (iii.), so that the expansion is

$$\text{(iii.)}\qquad \frac{c_1 x - \frac{c_3}{3!}x^3 + \frac{c_5}{5!}x^5 - \&c.}{\lambda + \cos bx} = Z_1 x + \frac{Z_2}{3!}x^3 + \frac{Z_4}{5!}x^5 + \&c.,$$

we have $Z_n \equiv Z_{n-2t}$, mod 5.

Thus, in both (ii.) and (iii.), the difference between two alternate Z's $\equiv 0$, mod 5, and when the Z's are all congruent to one another, mod 2, this difference must be $\equiv 0$, mod 10.

It will be noticed that, by putting $p = 3$, we see that the sum of any two consecutive Z's, both in (ii.) and (iii.), $\equiv 0$, mod 3, except, of course, when 3 occurs as a factor in the denominator of any of the Z's.

* The Eulerian numbers end in 1 and 5 alternately, the H's all end in 3, the P's end in 3 and 7 alternately, the Q's all end in 1, the R's end in 7 and 5 alternately, and the T's in 1 and 3 alternately (*Quarterly Journal*, Vol. xxix., pp. 63, 66, 71, 76; or *Messenger*, Vol. xxviii., p. 51). The endings of the I's and J's are considered in the next paper (§§ 17, 18, and 30). A table of all the coefficients up to $n = 5$ was given in the *Messenger*, Vol. xxviii., p. 51. More extensive tables of I_n, H_n, J_n (up to $n = 13$) are contained in the next paper (§§ 14, 25).

24. In Vol. XXVIII., pp. 75, 76, of the *Messenger*, it was shown by means of recurring formula that, if $2n-1$ is prime,

$$\left.\begin{array}{r} E_n \equiv Q_n \equiv T_n \equiv (-1)^{n-1} \\ H_n \equiv P_n \equiv (-1)^{n-1}3 \end{array}\right\}, \quad \text{mod } 2n-1,$$

with similar congruences relating to other coefficients, and it was stated that these results might be extended to the modulus $2n-3$, if prime, to the modulus $2n-5$, if prime, &c., and indeed to any prime modulus. It is this extension which has formed the subject of the present paper.

25. It may be remarked that, by putting $n = p-1$, where p is any uneven prime, in the recurring equation (i.) of § 6, we obtain a congruence connecting $X_0, X_1, X_2, \ldots, X_{p-1}$, mod p, viz., we have

$$(\lambda+1)\, a^{p-1} X_{p-1} + (p-1)_1\, a^{p-2} b X_{p-2} + (p-1)_2 a^{p-3} b^2 X_{p-3} + \ldots$$
$$\ldots + (p-1)_{p-1} b^{p-1} X_0 = c_{p-1},$$

giving

$$(\lambda+1)\, a^{p-1} X_{p-1} - a^{p-2} b X_{p-2} + a^{p-3} b^2 X_{p-3} - \ldots + b^{p-1} X_0 \equiv c_{p-1}, \quad \text{mod } p.$$

Similarly, by putting $2n = p-1$ in the relation (ii.) of § 6, we find

$$(\lambda+1)\, a^{p-1} X_{p-1} + a^{p-3} b^2 X_{p-3} + a^{p-5} b^4 X_{p-5} + \ldots\ b^{p-1} X_0 \equiv c_{p-1}, \quad \text{mod } p.$$

In the case of the Eulerian numbers this congruence gives, since $E_0 = E_1$,

$$E_{\frac{1}{2}(p-1)} - E_{\frac{1}{2}(p-3)} + E_{\frac{1}{2}(p-5)} - \ldots + (-1)^{\frac{1}{2}(p-5)} E_2 \equiv 0, \quad \text{mod } p\,;$$

for the I-numbers, since $\quad I_0 - I_1 = \frac{1}{6}$,

it gives

$$\tfrac{3}{2} I_{\frac{1}{2}(p-1)} - I_{\frac{1}{2}(p-3)} + I_{\frac{1}{2}(p-5)} - \ldots + (-1)^{\frac{1}{2}(p-5)} I_2 \equiv (-1)^{\frac{1}{2}(p-3)} \tfrac{1}{6}, \quad \text{mod } p,$$

and so on.

26. In the formula (iii.) of § 6 we cannot put $2n+1 = p-1$, but, by putting $2n+1 = p-2$, we find

$$(\lambda+1)\, a^{p-2} X_{p-2} + 3a^{p-4} b^2 X_{p-4} + 5a^{p-6} b^4 X_{p-6} + \ldots$$
$$\ldots + (p-2)\, ab^{p-3} X_1 \equiv c_{p-2}, \text{ mod } p.$$

By putting $n = p-2$ in (i.), we obtain a formula of the same character connecting $X_0, X_1, X_2, \ldots, X_{p-2}$, viz.,

$$(\lambda+1)\, a^{p-2} X_{p-2} - 2a^{p-3} b X_{p-3} + 3a^{p-4} b^2 X_{p-4} - \ldots$$
$$\ldots - (p-1)\, b^{p-2} X_0 \equiv c_{p-2}, \quad \text{mod } p.$$

Other relations may be derived from the recurring formulæ in the same manner by putting n or $2n = p-2$, &c., but the numerical coefficients are less simple.

[27. Since this paper was communicated to the Society, my attention has been called to a paper by Lucas in Vol. VI.* of the *Bulletin de la Société Mathématique de France*, in which the use of a recurring series to prove the congruence property of the Eulerian numbers is indicated. Denoting the r^{th} Eulerian number by $(-1)^r E_{2r}$, Lucas shows, by putting $2n = p-1$ in the recurring relation connecting the first n Eulerian numbers, that

$$E_{p-1} + E_{p-3} + E_{p-5} + \ldots + E_2 + E_0 \equiv 0, \quad \text{mod } p,$$

which is the formula obtained in § 25, and, by putting $2n = p+1$, $p+3$, $p+5$, ..., he shows that $E_{p+1} \equiv E_2$, $E_{p+3} \equiv E_4$, $E_{p+5} \equiv E_6$, ..., mod p, whence it is inferred that generally $E_{2n} \equiv E_{2n+k(p-1)}$, mod p. The extension, however, to the general theorem seems to me to require a definite investigation of the same kind as that given in §§ 2–4 of the present paper. Lucas points out that a similar congruence property would also hold good with respect to the coefficients in the expansion of $\left(\dfrac{2}{e^x+e^{-x}}\right)^a$, and, with restrictions, to any function of e^x. These forms are included in the general expressions of § 10.

I may mention that, since writing this paper, I have proved by means of the theorem in § 5 that, B_n denoting the n^{th} Bernoullian number,

$$\frac{B_n}{n} \equiv (-1)^{tj} \frac{B_{n-tj}}{n-tj}, \quad \text{mod } p,$$

where, as in § 1, $j = \frac{1}{2}(p-1)$, and p is any uneven prime, such that $p-1$ is not a divisor of $2n$. This theorem and its consequences have been considered in two papers in the *Messenger*† and one in the *Quarterly Journal*.‡ In this last paper the theorem in § 5 is proved separately in detail.]

* "Sur les congruences des nombres eulériens et des coefficients différentiels des fonctions trigonométriques, suivant un module premier," pp. 49–54.

† "Fundamental Theorems relating to the Bernoullian Numbers," Vol. XXIX., p. 49 and p. 128.

‡ "A Congruence Theorem relating to the Bernoullian Numbers," Vol. XXXI., p. 253.